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TABLES AND
FORMULAE

30TH
EDITION

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DANIEL ZWILLINGER

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Preface

It has long been the established policy of CRC Press to publish, in handbook form, the most up-to-date, authoritative, logically arranged, and readily usable reference material available. Prior to the preparation of this 30th Edition of the CRC Standard Mathematical Tables and Formulae, the content of such a book was reconsidered. Previous editions were carefully reviewed, and input obtained from practitioners in the many branches of mathematics, engineering, and the physical sciences. The content selected for this Handbook provides the basic mathematical reference materials required for each of these disciplines.

While much material was retained, several topics were completely reworked, and many new topics were added. New and completely revised topics include: partial differential equations, scientific computing, integral equations, group theory, and graph theory. For each topic, old and new, the contents have been completely rewritten and retypeset. A more comprehensive index has been added.

The same successful format which has characterized earlier editions of the Handbook is retained, while its presentation is updated and more consistent from page to page. Material is presented in a multi-sectional format, with each section containing a valuable collection of fundamental reference material—tabular and expository.

In line with the established policy of CRC Press, the Handbook will be kept as current and timely as is possible. Revisions and anticipated uses of newer materials and tables will be introduced as the need arises. Suggestions for the inclusion of new material in subsequent editions and comments concerning the accuracy of stated information are welcomed.

No book is created in a vacuum, and this one is no exception. Not only did we start with an excellent previous edition, but our editorial staff was superb, and the contributors did an amazingly good job. I wholeheartedly thank them all. There were also many proofreaders, too many to name individually; again, thank you for your efforts.

Lastly, this book would not have been possible without the support of my loving wife, Janet Taylor.

Daniel Zwillinger
zwilling@world.std.com
Contributors

Karen Bolinger
Clarion University
Clarion, Pennsylvania

Patrick J. Driscoll
U.S. Military Academy
West Point, New York

M. Lawrence Glasser
Clarkson University
Potsdam, New York

Jeff Goldberg
University of Arizona
Tucson, Arizona

Rob Gross
Boston College
Chestnut Hill, Massachusetts

Melvin Hausner
Courant Institute (NYU)
New York, New York

Christopher Heil
Georgia Tech
Atlanta, Georgia

Paul Jameson
BBN
Cambridge, Massachusetts

Victor J. Katz
MAA
Washington, DC

Silvio Levy
Geometry Center
University of Minnesota
Minneapolis, Minnesota

Michael Mascagni
Supercomputing Research Center
Bowie, Maryland

Ray McLenaghan
University of Waterloo
Waterloo, Ontario

John Michaels
SUNY Brockport
Brockport, New York

William C. Rinaman
LeMoyne College
Syracuse, New York

Catherine Roberts
Northern Arizona University
Flagstaff, Arizona

John S. Robertson
Georgia College
Milledgeville, Georgia

Joseph J. Rushanan
MITRE Corporation
Burlington, Massachusetts

Neil J. A. Sloane
AT&T Bell Labs
Murray Hill, New Jersey

Mike Sousa
MITRE Corporation
Burlington, Massachusetts

Gary L. Stanek
Youngstown State University
Youngstown, Ohio

Michael T. Strauss
Zwillinger & Associates
Newton, Massachusetts

Nico M. Temme
CWI
Amsterdam, The Netherlands

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REFERENCES
1.1 CONSTANTS

1.1.1 TYPES OF NUMBERS

Natural numbers
The natural numbers are customarily denoted by \( \mathbb{N} \). They are the set \( \{0, 1, 2, \ldots\} \). Many authors do not consider 0 to be a natural number.

Integers
The integers are customarily denoted by \( \mathbb{Z} \). They are the set \( \{0, \pm 1, \pm 2, \ldots\} \).

Rational numbers
The rational numbers are customarily denoted by \( \mathbb{Q} \). They are the set \( \{p/q | p, q \in \mathbb{Z}, q \neq 0\} \). Two fractions \( \frac{p}{q} \) and \( \frac{r}{s} \) are equal if \( ps = qr \).

Addition of fractions is defined by \( \frac{p}{q} + \frac{r}{s} = \frac{ps + qr}{qs} \). Multiplication of fractions is defined by \( \frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs} \).

Real numbers
The real numbers are customarily denoted by \( \mathbb{R} \). Real numbers are defined to be converging sequences of rational numbers or as decimals that might or might not repeat.

Real numbers are often divided into two subsets. One subset, the algebraic numbers, are real numbers which solve a polynomial equation in one variable with integer coefficients. For example; \( \frac{1}{\sqrt{2}} \) is an algebraic number because it solves the polynomial equation \( 2x^2 - 1 = 0 \), and rational numbers are algebraic. Real numbers that are not algebraic numbers are called transcendental numbers. Examples of transcendental numbers include \( \pi \) and \( e \).

Complex numbers
The complex numbers are customarily denoted by \( \mathbb{C} \). They are numbers of the form \( a + bi \), where \( i^2 = -1 \), and \( a \) and \( b \) are real numbers. See Section 1.5.

The sum of two complex numbers \( a + bi \) and \( c + di \) is \( a + c + (b + d)i \). The product of two complex numbers \( a + bi \) and \( c + di \) is \( ac - bd + (ad + bc)i \). The reciprocal of the complex number \( a + bi \) is \( \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i \). If \( z = a + bi \), the complex conjugate of \( z \) is \( \bar{z} = a - bi \). Properties include: \( \overline{z + w} = \overline{z} + \overline{w} \) and \( \overline{zw} = \overline{z} \overline{w} \).
1.1.2 REPRESENTATION OF NUMBERS

Numerals as usually written have radix or base 10, because the numeral $a_na_{n-1}\ldots a_2a_1a_0$ represents the number $a_n10^n+a_{n-1}10^{n-1}+\ldots+a_210^2+a_110+a_0$. However, other bases can be used, particularly bases 2, 8, and 16 (called binary, octal, and hexadecimal, respectively). When another base is used, it is indicated by a subscript:

$$543_7 = 5 \times 7^2 + 4 \times 7 + 3 = 276,$$
$$10111_2 = 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 1 \times 2 + 1 = 23,$$
$$A3_{16} = 10 \times 16 + 3 = 163.$$  

When writing a number in base $b$, the digits used can range from 0 to $b-1$. If $b > 10$, then the digit A stands for 10, B for 11, etc.

The above algorithm can be used to convert a numeral from base $b$ to base 10. To convert a numeral from base 10 to base $b$, divide the numeral by $b$, and the remainder will be the last digit. Then divide the quotient by $b$, using the remainder as the previous digit. Continue dividing the quotient by $b$ until a quotient of 0 is arrived at.

For example, to convert 574 to base 12, divide, yielding a remainder of 10 and a quotient of 47. Hence, the last digit of the answer is A. Divide 47 by 12, giving a remainder of 11 again and a quotient of 3. Divide 3 by 12, giving a remainder of 3 and a quotient of 0. Therefore, $574_{10} = 3BA_{12}$.

In general, to convert from base $b$ to base $r$, it is simplest to convert to base 10 as an intermediate step. However, it is simple to convert from base $b$ to base $b^n$. For example, to convert 1101111012 to base 16, group the digits in fours (because 16 is $2^4$), yielding 1 1011 1101, and then convert each group of 4 to base 16 directly, yielding $1BD_{16}$.
1.1.3 DECIMAL MULTIPLES AND PREFIXES

The prefix and symbols below are taken from Conference Générale des Poids et Mesures, 1991. The common names are for the U.S.

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<th>Symbol</th>
<th>Common name</th>
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<td>googolplex</td>
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<td>$10^{100}$</td>
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<td>$10^{-6}$</td>
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<td>yocto</td>
<td>y</td>
<td>heptillionth</td>
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</table>

1.1.4 ROMAN NUMERALS

The major symbols in Roman numerals are I = 1, V = 5, X = 10, L = 50, C = 100, D = 500, and M = 1,000. The rules for constructing Roman numerals are

1. A symbol following one of equal or greater value adds its value (for example, II = 2, XI = 11, DV = 505).
2. A symbol following one of lesser value has the lesser value subtracted from the larger value (for example, IV = 4, IX = 9, VM = 995).
3. When a symbol stands between two of greater value, its value is subtracted from the second and the result is added to the first (for example, XIV = 10 + (5 − 1) = 14, CIX = 100 + (10 − 1) = 109, DVL = 500 + (50 − 5) = 545).
4. When two ways exist for representing a number, the one in which the symbol of larger value occurs earlier in the string is preferred (for example, 14 is represented as XIV, not as VIX).
### 1.1.5 DECIMAL EQUIVALENTS OF COMMON FRACTIONS

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### 1.1.6 Hexadecimal Addition and Subtraction Table

A = 10, B = 11, C = 12, D = 13, E = 14, F = 15.
Example: 6 + 2 = 8; hence 8 − 6 = 2 and 8 − 2 = 6.
Example: 4 + E = 12; hence 12 − 4 = E and 12 − E = 4.

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1.2.5 SPECIAL CONSTANTS

The constant $\pi$

The transcedental number $\pi$ is defined as the ratio of the circumference of a circle to the diameter. It is also the ratio of the area of a circle to the square of the radius ($r$) and appears in several other formulas from elementary geometry (see Section 4.6)

\[
C(\text{circle}) = 2\pi r, \quad V(\text{sphere}) = \frac{4}{3} \pi r^3, \\
A(\text{circle}) = \pi r^2, \quad SA(\text{sphere}) = 4\pi r^2.
\]

One method of computing $\pi$ is to use the infinite series for the function $\tan^{-1} x$ and one of the identities

\[
\pi = 4 \tan^{-1} 1 = 6 \tan^{-1} \frac{1}{\sqrt{3}} \\
= 2 \tan^{-1} \frac{1}{2} + 2 \tan^{-1} \frac{1}{3} + 8 \tan^{-1} \frac{1}{5} - 2 \tan^{-1} \frac{1}{239} \\
= 24 \tan^{-1} \frac{1}{8} + 8 \tan^{-1} \frac{1}{57} + 4 \tan^{-1} \frac{1}{239} \\
= 48 \tan^{-1} \frac{1}{18} + 32 \tan^{-1} \frac{1}{57} - 20 \tan^{-1} \frac{1}{239}
\]

There are many other identities involving $\pi$. See Section 1.4.3. For example:

\[
\frac{\pi}{32} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = 1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{5^3} + \ldots
\]

To 200 decimal places:

\[
\pi \approx 3.14159 26535 89793 23846 26433 83279 50288 41971 69399 37510 58209 74944 59230 78164 06286 20899 86280 34825 34211 70679 82148 08651 32823 06647 09384 46095 50582 23172 53594 08128 48111 74502 84102 70193 85211 05559 64462 29899 54930 38196
\]

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To 50 decimal places:

\[
\begin{align*}
\pi/20 &\approx 0.15707 96326 79489 66192 31321 69163 97514 42098 58469 96876 \\
\pi/15 &\approx 0.20943 95102 39319 54923 08428 92218 63352 56131 44626 62501 \\
\pi/12 &\approx 0.26179 93877 99149 43653 85536 15273 29190 70164 30783 28126 \\
\pi/11 &\approx 0.28559 93321 44526 65804 20584 89389 04571 67451 97218 12501 \\
\pi/10 &\approx 0.31415 92653 58979 32384 62643 38327 95028 84197 16939 93751 \\
\pi/9 &\approx 0.34906 58503 98865 91538 47381 53697 72254 26885 74377 70835 \\
\pi/8 &\approx 0.39269 90816 98724 15480 78304 22909 93786 05246 46174 92189 \\
\pi/7 &\approx 0.44879 89505 12827 60549 46633 40468 50041 20281 67057 05359 \\
\pi/6 &\approx 0.52359 87755 98298 87307 71072 30546 58381 40328 61566 56252 \\
\pi/5 &\approx 0.62831 85307 17958 64769 25286 67665 90057 68394 33789 87502 \\
\pi/4 &\approx 0.78539 81633 97448 30961 56608 45819 87572 10492 92439 84378 \\
\pi/3 &\approx 1.04719 75511 96597 74615 42144 61093 16762 80657 23133 12504 \\
\pi/2 &\approx 1.57079 63267 94896 61923 13216 91639 75144 20985 84699 68755 \\
\end{align*}
\]

\[
\begin{align*}
2\pi/3 &\approx 2.09439 51023 93195 49230 84289 22186 33525 61314 46266 25007 \\
3\pi/2 &\approx 4.71238 89803 46898 58769 39650 74919 25432 62957 54099 06266 \\
5\pi/2 &\approx 7.85398 16339 74483 09615 46838 58198 57621 04929 23498 43776 \\
\sqrt{\pi} &\approx 1.77254 38509 05516 02729 81674 83341 14518 27975 49456 12239
\end{align*}
\]

The constant \(e\)

The transcendental number \(e\) is the base of natural logarithms. It is given by

\[
e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!}.
\]

To 200 decimal places:

\[
e \approx 2. 71828 18284 59045 23536 02874 71352 66249 77572 40963 74423 42330 12441 47548 24456 30746 88008 91628 11747 41670 95517 02902 76880 73390 50022 09123 97949 16324 15018 88235 37653 78979 37156 28782 65309 41951 75232 76507 70119 54613 14754 78419 41892 07548 20581 28222 40417 11270 70174 46564 44102 18172 80388 23470 99362 28056 72036 05265 13514 32356 29282 73644 82854 81964 64763 28627 95623 44728 96730 29172 10425 33324 75090 71692 82340 46049 06646 89568 69629 48648 51877 56551 80458 79199 40927 65780 12212 32399 25622 12094 02804 01216 83564
\]

To 50 decimal places:

\[
\begin{align*}
e/8 &\approx 0.33978 52285 57380 54456 00359 33919 08281 22196 55886 71249 \\
e/7 &\approx 0.38832 59754 94149 31933 71839 24478 95178 53938 92441 95714 \\
e/6 &\approx 0.45304 69714 09840 87256 00479 11892 11041 62928 74955 61666 \\
e/5 &\approx 0.54365 63656 91809 04707 20574 94270 53249 95514 49418 73999 \\
e/4 &\approx 0.67957 04714 17461 30884 00718 67838 16562 44393 11773 42499 \\
e/3 &\approx 0.90609 39428 19681 74512 00958 23784 22083 25857 40931 2332 \\
e/2 &\approx 1.35914 09142 95252 61768 01437 35676 33124 88786 23546 84998 \\
2e/3 &\approx 1.81218 78856 39363 49024 01916 47568 44166 51714 98062 46664
\end{align*}
\]

The function \(e^x\) is defined by \(e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}\). The numbers \(e\) and \(\pi\) are related by the formula \(e^{\pi i} = -1\).
To 50 decimal places:

\[ e^\pi \approx 23.14069263277926900572908636794854738026610624260021 \]

\[ \pi^e \approx 22.45915771836104547342715220454373502758931513399669 \]

**The constant \( \gamma \)**

Euler’s constant \( \gamma \) is defined by

\[ \gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right). \]

To 200 decimal places:

\[ \gamma \approx 0.577215664901532860605933593992359880576723488486772677766467093694706329174674951463144724980708248096050401448654283622417399764492353625350033374293733773767394279259525824709492 \]

It is not known whether \( \gamma \) is rational or irrational.

**The constant \( \phi \)**

The golden ratio \( \phi \) is defined as the positive root of the equation \( \frac{\phi}{1} = \frac{1 + \sqrt{5}}{2} \); that is \( \phi = \frac{1 + \sqrt{5}}{2} \).

To 200 decimal places:

\[ \phi \approx 1.61803398874989448820458683436563811772030917980576286213544862270526046281890244970720720418939113748475408807538689175212663386222353693179318006076672635443338908659593958290563832266131992829026788 \]

### 1.2.6 Factorials

The factorial of \( n \), denoted \( n! \), is the product of all integers less than or equal to \( n \). \( n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 \). The double factorial of \( n \), denoted \( n!! \), is the product of every other integer: \( n!! = n \cdot (n-2) \cdot (n-4) \cdots \), where the last element in the product is either 2 or 1, depending on whether \( n \) is even or odd. The generalization of the factorial function is the gamma function (see Section 6.11). When \( n \) is an integer, \( \Gamma(n) = (n-1)! \).

The shifted factorial (also called the falling factorial and Pochhammer’s symbol) is denoted by \( (a)_n \) (sometimes \( a^\underline{n} \)) and is defined as

\[ (a)_n = \frac{a \cdot (a+1) \cdot (a+2) \cdots (a+n-1)!}{(a-1)!} = \frac{\Gamma(a+n)}{\Gamma(a)} \quad (1.2.1) \]
The \( q \)-shifted factorial is defined as

\[
(a; q)_0 = 1, \quad (a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}).
\]

(1.2.2)

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1.2.7 IMPORTANT NUMBERS IN DIFFERENT BASES

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| Base 8 | π = 3.110375524210264302151423063050… |
| Base 12 | π = 3.243F6A8885A308D313198A2E037073… |
| Base 16 | π = 3.243F6A8885A308D313198A2E037073… |

| Base 2 | e = 10.1011011111000010100011011100… |
| Base 8 | e = 2.575605213050535512465277342542… |
| Base 12 | e = 2.8752360698219BA71971009B388AA8… |
| Base 16 | e = 2.8752360698219BA71971009B388AA8… |

| Base 2 | γ = 0.10010111110001100110111110100… |
| Base 8 | γ = 0.10010111110001100110111110100… |
| Base 12 | γ = 0.10010111110001100110111110100… |
| Base 16 | γ = 0.10010111110001100110111110100… |

| Base 2 | √2 = 1.01101000001001111011001101001… |
| Base 8 | √2 = 1.132404763177167462204262766115… |
| Base 12 | √2 = 1.14B79170A7B8537704B085468535… |
| Base 16 | √2 = 1.14B79170A7B8537704B085468535… |

| Base 2 | log(2) = 0.1011000111110010011111101111100… |
| Base 8 | log(2) = 0.1011000111110010011111101111100… |
| Base 12 | log(2) = 0.1011000111110010011111101111100… |
| Base 16 | log(2) = 0.1011000111110010011111101111100… |

1.2.8 BERNOULLI POLYNOMIALS AND NUMBERS

The Bernoulli polynomials \( B_n(x) \) are defined by the generating function

\[
\frac{te^t}{e^t-1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.
\]  

These polynomials can also be defined recursively by means of \( B_0(x) = 1, \ B'_n(x) = nB_n(x) \), and \( \int_0^1 B_n(x) \, dx = 0 \) for \( n \geq 1 \). The identity \( B_{k+1}(x + 1) - B_{k+1}(x) = (k+1)x^k \) means that

\[
1^k + 2^k + \cdots + n^k = \frac{B_{k+1}(n+1) - B_{k+1}(0)}{k+1}.
\]  

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The Bernoulli numbers are the Bernoulli polynomials evaluated at 0: \( B_n = B_n(0) \). A generating function for the Bernoulli numbers is

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\[ \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1}. \] In the following table each Bernoulli number is written as a fraction of integers: \( B_n = N_n/D_n. \) Note that \( B_{2m+1} = 0 \) for \( m \geq 1. \)

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### 1.2.9 Euler Polynomials and Numbers

The Euler polynomials \( E_n(x) \) are defined by the generating function

\[
\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.
\]

\[ (1.2.5) \]

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Alternating sums of powers can be computed in terms of Euler polynomials

\[ \sum_{i=1}^{n} (-1)^{n-i} i^k = n^k - (n-1)^k + \cdots \mp 2^k \pm 1^k = \frac{E_k(n+1) + (-1)^n E_k(0)}{2}. \] (1.2.6)

The Euler numbers are the Euler polynomials evaluated at \(1/2\), and scaled: \(E_n = 2^n E_n(\frac{1}{2})\). A generating function for the Euler numbers is

\[ \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} = \frac{2e^t}{e^{2t}+1}. \]

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### 1.2.10 FIBONACCI NUMBERS

The Fibonacci numbers \(\{F_n\}\) are defined by the recurrence:

\[ F_1 = 1, \quad F_2 = 1, \quad F_{n+2} = F_n + F_{n+1}. \]
An exact formula is available: \[ F_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( \frac{1-\sqrt{5}}{2} \right)^n \right) \]. Note that \( \lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \phi \), the golden ratio. Also, \( F_n \sim \phi^n / \sqrt{5} \) as \( n \to \infty \).

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### 1.2.11 POWERS OF INTEGERS

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### 1.2.12 SUMS OF POWERS OF INTEGERS

Define \( s_k(n) = 1^k + 2^k + \cdots + n^k = \sum_{m=1}^{n} m^k \). Properties include:

- \( s_k(n) = (k+1)^{-1} \left[ B_{k+1}(n+1) - B_{k+1}(0) \right] \)

(Where the \( B_k \) are Bernoulli polynomials, see Section 1.2.8.)
Writing $s_k(n)$ as $\sum_{m=1}^{k+1} a_m n^{k-m+2}$ there is the recursion formula:

$$s_{k+1}(n) = \left( \frac{k+1}{k+2} \right) a_1 n^{k+2} + \cdots + \left( \frac{k+1}{k} \right) a_3 n^k$$

$$+ \cdots + \left( \frac{k+1}{2} \right) a_{k+1} n^2 + \left[ 1 - (k + 1) \sum_{m=1}^{k+1} \frac{a_m}{k + 3 - m} \right] n.$$

(1.2.7)

$s_1(n) = 1 + 2 + 3 + \cdots + n = \frac{1}{2} n(n + 1)$.

$s_2(n) = 1^2 + 2^2 + 3^2 + \cdots + n^2$

$$= \frac{1}{6} n(n + 1)(2n + 1).$$

$s_3(n) = 1^3 + 2^3 + 3^3 + \cdots + n^3$

$$= \frac{1}{4} (n^2(n + 1)^2) = [s_1(n)]^2.$$

$s_4(n) = 1^4 + 2^4 + 3^4 + \cdots + n^4$

$$= \frac{1}{5} (3n^2 + 3n - 1)s_2(n).$$

$s_5(n) = 1^5 + 2^5 + 3^5 + \cdots + n^5$

$$= \frac{1}{12} n^2(n + 1)^2(2n^2 + 2n - 1).$$

$s_6(n) = 1^6 + 2^6 + 3^6 + \cdots + n^6$

$$= \frac{n}{42} (n + 1)(2n + 1)(3n^4 + 6n^3 - 3n + 1).$$

$s_7(n) = 1^7 + 2^7 + 3^7 + \cdots + n^7$

$$= \frac{n^2}{24} (n + 1)^2(3n^4 + 6n^3 - n^2 - 4n + 2).$$

$s_8(n) = 1^8 + 2^8 + 3^8 + \cdots + n^8$

$$= \frac{n}{90} (n + 1)(2n + 1)(5n^6 + 15n^5 + 5n^4 - 15n^3 - n^2 + 9n - 3).$$

$s_9(n) = 1^9 + 2^9 + 3^9 + \cdots + n^9$

$$= \frac{n^2}{20} (n + 1)^2(2n^6 + 6n^5 + n^4 - 8n^3 + n^2 + 6n - 3).$$

$s_{10}(n) = 1^{10} + 2^{10} + 3^{10} + \cdots + n^{10}$

$$= \frac{n}{66} (n + 1)(2n + 1)(3n^8 + 12n^7 + 8n^6 - 18n^5$$

$$- 10n^4 + 24n^3 + 2n^2 - 15n + 5).$$
1.2.13 NEGATIVE INTEGER POWERS

Riemann’s zeta function is $\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$. Related functions are

\[
\alpha(n) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^n}, \quad \beta(n) = \sum_{k=0}^{\infty} \frac{(-1)^{k}}{(2k+1)^n}, \quad \gamma(n) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^n}.
\]

Properties include:

\[
\alpha(n) = (1 - 2^{1-n})\zeta(n), \quad \gamma(n) = (1 - 2^{-n})\zeta(n),
\]

\[
\zeta(2k) = \frac{(2\pi)^{2k}}{2(2k)!} |B_{2k}|, \quad \beta(2k+1) = \frac{(\pi/2)^{2k+1}}{2(2k)!} |E_{2k}|.
\]

(1.2.8)

The series $\beta(1) = 1 - \frac{1}{3} + \frac{1}{5} - \cdots = \pi/4$ is known as Gregory’s series. Catalan’s constant is $G = \beta(2)$.
1.2.14 INTEGER SEQUENCES

These sequences are arranged in numerical order (disregarding any leading zeros or ones). Note that \( C(n, k) = \binom{n}{k} \).

1. \( 1, -1, -1, 0, -1, 1, -1, 0, -1, 0, -1, 0, -1, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, -1, -1, 0, 1, 0, 1, 0, -1, 0, 0, -1, -1, 0, 1, 1, 0, -1, 1, 0, 0, 0, 1, 0, -1, 0, 0, 0, 1, 0, 1, 0, 0 \),

\[ \zeta(2) = \pi^2/6 \quad \beta(1) = \pi/4 \]
\[ \zeta(4) = \pi^4/90 \quad \beta(3) = \pi^3/32 \]
\[ \zeta(6) = \pi^6/945 \quad \beta(5) = 5\pi^5/1536 \]
\[ \zeta(8) = \pi^8/9450 \quad \beta(7) = 61\pi^7/184320 \]
\[ \zeta(10) = \pi^{10}/93555 \quad \beta(9) = 277\pi^9/8257536 \]

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2. \( \frac{n}{2} \) for \( n \geq 1 \),

3. \( \frac{n}{2} \) for \( n \geq 1 \),

4. \( \frac{n}{2} \) for \( n \geq 1 \),

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18. \( \frac{n}{2} \) for \( n \geq 1 \),

19. \( \frac{n}{2} \) for \( n \geq 1 \),

20. \( \frac{n}{2} \) for \( n \geq 1 \).
1. $1, 2, 3, 4, 6, 8, 10, 12, 16, 18, 20, 24, 30, 36, 42, 48, 60, 72, 84, 90, 100, 120, 144, 168, 180, 210, 216, 240, 288, 300, 336, 360, 420, 480, 504, 540, 600, 630, 660$

Highly abundant numbers: where sum-of-divisors function increases

2. $1, 2, 3, 4, 6, 8, 10, 12, 16, 20, 24, 30, 36, 42, 48, 60, 72, 84, 90, 100, 120, 144, 168, 180, 210, 216, 240, 288, 300, 336, 360, 420, 480, 504, 540, 600, 630, 660$

Prime powers

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21. 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, 1002, 1255, 1575, 1958, 2436, 3010, 3718, 4565, 5604, 6842, 8349, 10143, 12310, 14883
   Number of partitions of \( n \), \( n \geq 1 \)
22. 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 251, 607, 1279, 2203, 2281, 3127, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, 110503, 132049, 216091, 756839, 859433
   Mersenne primes: \( p \) such that \( 2^p - 1 \) is prime
23. 1, 1, 1, 2, 3, 6, 11, 23, 47, 106, 235, 551, 1301, 3159, 7741, 19320, 48629, 123867, 317955, 823065, 2144505, 5623756, 14828074, 39299897, 104636890, 279793450
   Number of trees with \( n \) unlabeled nodes, \( n \geq 1 \)
24. 2, 3, 6, 10, 20, 35, 70, 126, 252, 462, 924, 1944, 3888, 8451, 17904, 39808, 92378, 184756, 352716, 705432, 1430864, 2461408, 4922816, 9745632, 19491264, 38982528, 77965056, 155930112, 311860224, 623720448
   Central binomial coefficients: \( C(n, [n/2]) \), \( n \geq 1 \)
25. 1, 2, 3, 6, 11, 22, 44, 84, 165, 330, 654, 1308, 2605, 5210, 10398, 20796, 41550, 83100, 166116, 332232, 664464, 1328928, 2657856, 5315712, 10626819, 21243029
   Stern’s sequence: \( a(n+1) \) is sum of \( 1 + [n/2] \) preceding terms, \( n \geq 1 \)
26. 1, 2, 3, 6, 11, 22, 44, 84, 165, 330, 654, 1308, 2605, 5210, 10398, 20798, 41550, 83100, 166116, 332232, 664464, 1328928, 2657856, 5315712, 10626819
   Narayana–Zidek–Capell numbers: \( a(2n) = 2a(2n-1), a(2n+1) = 2a(2n) - a(n) \)
27. 1, 1, 1, 2, 3, 6, 11, 22, 44, 88, 177, 355, 711, 1422, 2845, 5690, 11381, 22762, 45524, 91048, 182096, 364192, 728384, 1456768, 2913536, 5827072, 11654144, 23308288, 46616576, 93233152, 186466304, 372932608, 745865216
   Wedderburn–Etherington numbers: interpretations of \( X^n, n \geq 1 \)
28. 1, 1, 1, 3, 6, 11, 22, 44, 88, 177, 355, 711, 1422, 2845, 5690, 11381, 22762, 45524, 91048, 182096, 364192, 728384, 1456768, 2913536, 5827072, 11654144, 23308288, 46616576, 93233152, 186466304, 372932608, 745865216
   Number of Euler graphs or 2-graphs with \( n \) crossings, \( n \geq 1 \)
29. 0, 0, 1, 1, 2, 3, 7, 18, 41, 123, 367, 1288, 4878
   Number of alternating prime knots with \( n \) crossings, \( n \geq 1 \)
30. 0, 0, 1, 1, 2, 3, 7, 18, 41, 123, 367, 1288, 4878
   Number of prime knots with \( n \) crossings, \( n \geq 1 \)
31. 1, 2, 3, 8, 14, 42, 81, 262, 538, 1828, 3926, 13820, 30694, 110954, 252939, 933458, 2172830, 8152860, 19304190, 73424650, 176343390, 678390116, 1649008456
   Meandric numbers: ways a river can cross a road \( n \) times, \( n \geq 1 \)
32. 1, 2, 3, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25, 26, 29, 32, 34, 36, 37, 40, 41, 45, 49, 50, 52, 53, 58, 61, 64, 65, 68, 72, 73, 74, 80, 81, 82, 85, 89, 90, 97, 98, 100, 101, 104, 106
   Numbers that are sums of 2 squares
33. 1, 2, 4, 5, 8, 10, 14, 15, 16, 21, 22, 25, 26, 28, 33, 34, 35, 36, 38, 40, 42, 46, 48, 49, 50, 53, 57, 60, 62, 64, 65, 70, 77, 80, 81, 83, 85, 86, 90, 91, 92, 100, 104, 107
   MacMahon’s prime numbers of measurement, or segmented numbers
34. 1, 2, 4, 6, 10, 14, 20, 26, 36, 46, 60, 74, 94, 114, 140, 166, 202, 238, 284, 330, 390, 450, 524, 598, 692, 786, 900, 1014, 1154, 1294, 1460, 1626, 1828, 2030, 2268, 2506
   Binary partitions (partitions of \( 2n \) into powers of 2), \( n \geq 0 \)
35. 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, 16384, 32768, 65536, 131072, 262144, 524288, 1048576, 2097152, 4194304, 8388608, 16777216, 33554432, 67108864
   Powers of 2

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38. 1, 1, 2, 4, 9, 20, 48, 115, 286, 719, 1842, 4766, 12486, 32973, 87811, 235381, 634847, 1721159, 4688676, 12826228, 35221832, 97055181, 268282855, 743724984, 2067174645
   Number of rooted trees with \( n \) unlabeled nodes, \( n \geq 1 \)
39. 1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, 41835, 113634, 310572, 853467, 2356779, 6536382, 18199284, 50852019, 14257559, 400763223, 1129760415
   Motzkin numbers: ways to join \( n \) points on a circle by chords
40. 1, 1, 2, 4, 9, 22, 59, 167, 490, 1486, 4639, 14805, 48107, 158808, 531469, 1799659, 6157068, 21258104, 73996100, 259451116, 951695102, 3251073303
   Number of different scores in \( n \)-team round-robin tournament, \( n \geq 1 \)
41. 1, 1, 2, 4, 11, 34, 156, 1044, 12346, 274668, 12005168, 1018997864, 165091172592, 50502031367952, 29054155657235488, 31426485969804308768
   Number of graphs with \( n \) unlabeled nodes, \( n \geq 0 \)
42. 0, 1, 2, 5, 12, 29, 76, 169, 408, 985, 2378, 5741, 13860, 33461, 80782, 195025, 470832, 1136689, 2744210, 6625109, 15994428, 38613965, 93222358, 225058681, 543339720
   Pell numbers: \( a(n) = 2a(n - 1) + a(n - 2) \)
43. 1, 1, 2, 5, 12, 35, 108, 369, 1285, 7936, 50521, 353792, 2700671950, 11123060678, 43191857688
   Polyominoes with \( n \) cells, \( n \geq 1 \)
44. 1, 1, 2, 4, 12, 56, 456, 6880, 191536, 9733056, 903753248, 154108311168, 48542114686912, 28401423719122304, 31201002160355166848
   Number of outcomes of \( n \)-team round-robin tournament, \( n \geq 1 \)
45. 1, 1, 2, 5, 14, 38, 120, 353, 1148, 3527, 11622, 36627, 121622, 389560, 1301140, 4215748, 13976335, 46235800, 155741571, 512559185, 1732007938, 5732533570
   Number of ways to fold a strip of \( n \) blank stamps, \( n \geq 1 \)
46. 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420, 24466267020
   Catalan numbers: \( C(2n, n)/(n + 1), n \geq 0 \)
47. 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, 678570, 4213597, 27644437, 190899322, 1382958545, 10480142147, 82864689804, 682076806159, 5832742205057
   Bell or exponential numbers: expansion of \( e^{x(x-1)} \)
48. 1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, 353792, 2702765, 22368256, 199360981, 1903757312, 19391512145, 209865342976, 2404879675441, 29088885112832
   Euler numbers: expansion of \( \sec x + \tan x \)
49. 0, 2, 6, 12, 20, 30, 42, 56, 72, 90, 110, 132, 156, 182, 210, 240, 272, 302, 342, 380, 420, 462, 506, 552, 600, 650, 702, 756, 812, 870, 930, 992, 1056, 1122, 1190, 1260, 1332
   Pronic numbers: \( n(n + 1), n \geq 0 \)
50. 1, 2, 6, 20, 70, 252, 924, 3432, 12870, 48620, 184756, 705432, 2704156, 10400600, 40116600, 155117520, 601080390, 2333606220, 9075135300, 35345263800
   Central binomial coefficients: \( C(2n, n), n \geq 0 \)
51. 1, 1, 2, 6, 21, 112, 853, 11117, 261080, 17116571, 1006700565, 164059830476, 50335907869219, 29003487462848061, 3139783114276214960
   Number of connected graphs with \( n \) unlabeled nodes, \( n \geq 0 \)
52. 1, 2, 6, 22, 101, 573, 3836, 29228, 250749, 2409581, 25598186, 296643390, 3727542188, 50626553988, 738680521142
   Kendall–Mann numbers: maximal inversions in permutation of \( n \) letters, \( n \geq 1 \)
53. 1, 1, 2, 4, 24, 120, 720, 5040, 40320, 362880, 3628800, 39916800, 4790016000, 6227020800, 87178291200, 1307674368000, 20922789888000, 355674282096000, 6402373705728000

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54. Robbins numbers: \( \prod_{k=0}^{n-1} (3k + 1)!/(n + k)! , n \geq 1 \)

55. Closed meandric numbers: ways a loop can cross a road \( 2n \) times, \( n \geq 1 \)

56. Double factorial numbers: \( (2n)!! = 2^n n! , n \geq 0 \)

57. Derangements: permutations of \( n \) elements with no fixed points, \( n \geq 1 \)

58. Tangent numbers: expansion of \( \tan x \)

59. Numbers of the form \( x^2 + xy + y^2 \)

60. Lucas numbers: \( L(n) = L(n-1) + L(n-2) \)

61. Number of ways to cut an \( n \)-sided polygon into triangles, \( n \geq 1 \)

62. Shortest Golomb ruler with \( n \) marks, \( n \geq 2 \)

63. Number of mappings from \( n \) unlabeled points to themselves, \( n \geq 1 \)

64. Cullen numbers: \( n \cdot 2^n + 1 , n \geq 1 \)

65. Powers of 3
Number of topologies or transitive-directed graphs with \( n \) unlabeled nodes, \( n \geq 1 \)

70. 1, 3, 9, 33, 139, 718, 4535

Number of trees with \( n \) unlabeled nodes, \( n \geq 1 \)

71. 1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, 518859, 2646723, 13648869, 71039373, 372693519, 1968801519, 10463578353, 55909013009, 300159426963

Schroeder’s second problem: ways to interpret \( X_1X_2 \ldots X_n \), \( n \geq 1 \)

72. 1, 3, 11, 50, 274, 1764, 13068, 109584, 1026256, 10254380, 148642880, 19802759040, 283465647360, 4339163001600, 70734282393600, 1223405590579200

73. 1, 3, 13, 75, 541, 4683, 47293, 545835, 7087261, 102247563, 1622632573, 28091567595, 526858348381, 10641342970443, 23028319077853, 5315654681981355

Preferential arrangements of \( n \) things, \( n \geq 1 \)

74. 1, 3, 15, 105, 945, 10395, 135135, 2027025, 34459425, 654729075, 13749310575, 316234143225, 7905853580625, 213458046667875, 6190283353629375

75. 1, 3, 16, 125, 1296, 16807, 26144, 4782969, 100000000, 2357947691, 61917364224, 1792160394037, 56693912375296, 1946195068359375, 72057594037927936

Number of trees with \( n \) labeled nodes: \( n^{n-2} \), \( n \geq 2 \)

76. 1, 3, 16, 218, 9608, 1540944, 882033440, 1793359192848, 13027956824399552, 341260431952972580352, 3252290938505588611197440

80. 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 782, 841, 900, 961, 1024, 1089, 1156, 1225, 1296

The squares

81. 1, 1, 3, 7, 15, 31, 63, 126, 252, 504, 1008, 2016, 4032, 8064, 16128, 32256, 64512, 129024, 258048, 516096, 1032192, 2064384, 4128768, 8257536, 16515072

82. 1, 2, 5, 11, 23, 47, 95, 191, 383, 767, 1535, 3070, 6139, 12278, 24556, 49112, 98224, 196448, 392896, 785792, 1571584, 3143168, 6286336

83. 1, 4, 10, 20, 35, 56, 84, 120, 165, 220, 286, 364, 455, 560, 680, 816, 969, 1140, 1330, 1540, 1771, 2024, 2300, 2600, 2925, 3276, 3654, 4060, 4495, 4960, 5456, 5984

84. 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 782, 841, 900, 961, 1024, 1089, 1156, 1225, 1296

85. 1, 5, 12, 22, 35, 51, 70, 92, 117, 145, 176, 210, 247, 287, 330, 376, 425, 477, 532, 590, 651, 715, 782, 832, 925, 1001, 1080, 1162, 1247, 1335, 1426, 1520, 1617, 1717, 1820

Pentagonal numbers: \( n(3n - 1)/2 \), \( n \geq 1 \)

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86. 1, 5, 13, 25, 41, 61, 85, 113, 145, 181, 221, 265, 313, 365, 421, 481, 545, 613, 685, 761, 841, 925, 1013, 1105, 1201, 1301, 1405, 1513, 1625, 1741, 1861, 1985, 2113, 2245

Centered square numbers: \( n^2 + (n - 1)^2, n \geq 1 \)

87. 1, 5, 14, 30, 55, 91, 140, 204, 285, 385, 506, 650, 819, 1015, 1240, 1496, 1785, 2113, 2470, 2870, 3311, 3795, 4324, 4900, 5525, 6201, 6930, 7714, 8555, 9455, 10416

Square pyramidal numbers: \( n(n + 1)(2n + 1)/6, n \geq 1 \)

88. 1, 5, 25, 125, 625, 3125, 15625, 78125, 390625, 1953125, 9765625, 48828125, 244140625, 1220703125, 6103515625, 30517578125, 152587890625, 762939453125, 3814697265625

Powers of 5

89. 1, 5, 52, 1522, 145984, 48464496, 174695272746749919580928

Number of possible relations on \( n \) unlabeled points, \( n \geq 1 \)

90. 1, 1, 5, 61, 1385, 50521, 2702765, 199360981, 152587890625, 762939453125, 3814697265625

Euler numbers: expansion of \( \sec x \)

91. 1, 5, 109, 32297, 2147321017, 9223372023970362989, 170141183460469231667123699502996689125

Number of ways to cover an \( n \) set, \( n \geq 1 \)

92. 1, 6, 15, 28, 45, 66, 91, 120, 150, 190, 231, 276, 325, 378, 435, 496, 561, 630, 703, 780, 861, 946, 1035, 1128, 1225, 1326, 1431, 1540, 1653, 1770, 1891, 2016, 2145, 2278

Hexagonal numbers: \( n(2n - 1), n \geq 1 \)

93. 1, 6, 25, 90, 301, 966, 3025, 9330, 28501, 86526, 261625, 788970, 237438691328, 1742343008139952128, 2658455991568635501

Stirling numbers of second kind: \( \{ \text{n choose 3} \}, n \geq 3 \)

94. 6, 28, 496, 8128, 33550336, 8589869056, 3505843008139952128, 2658455991568635501

Perfect numbers: equal to the sum of their proper divisors

95. 1, 8, 21, 40, 65, 96, 137, 176, 225, 280, 341, 408, 481, 560, 645, 736, 833, 936, 1045, 1160, 1281, 1408, 1541, 1680, 1825, 1976, 2133, 2296, 2465, 2640, 2821, 3008, 3201

Octagonal numbers: \( n(3n - 2), n \geq 1 \)

96. 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, 1728, 2197, 2744, 3375, 4096, 4913, 5832, 6859, 8000, 9261, 10648, 12167, 13824, 15625, 17576, 19683, 21952, 24389

The cubes

97. 1, 24, 252, 512, 1472, 4830, 6048, 16744, 84480, −113643, −115920, 534612, −370944, −577738, 401856, 1217160, 987136, −6905934, 2727432, 10661420

Ramanujan \( \tau \) function

98. 341, 561, 645, 1105, 1387, 1729, 1905, 2047, 2465, 2701, 2821, 3277, 4033, 4369, 4371, 4681, 5461, 6601, 7957, 8321, 8481, 8911, 10261, 10585, 11305, 12801, 13741, 13747

Sarrus numbers: pseudo-primes to base 2

99. 561, 1105, 1729, 2465, 2821, 6601, 8911, 10585, 15841, 29341, 41041, 46657, 52633, 62745, 63973, 75361, 101101, 115921, 126217, 162401, 172081, 188461, 252601, 278545

Carmichael numbers

100. 744, 196884, 21493760, 864299970, 20245856256, 333202640600, 4252023300996, 44656994071935, 40149088665600, 3176440229784420, 22567393309593600

Coefficients of the modular function \( j \)

For more information about all of these sequences including formulae and references, see N.J.A. Sloane and S. Plouffe, *Encyclopedia of Integer Sequences*, Academic Press, 1995, where over 5000 other sequences are also described.

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1.2.15 DE BRUIJN SEQUENCES

A sequence of length \( q^n \) over an alphabet of size \( q \) is a de Bruijn sequence if every possible \( n \)-tuple occurs in the sequence (allowing wraparound to the start of the sequence). There are de Bruijn sequences for any \( q \) and \( n \). The table below gives some small examples.

<table>
<thead>
<tr>
<th>( q )</th>
<th>( n )</th>
<th>Length</th>
<th>Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
<td>0110</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>8</td>
<td>01110100</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>9</td>
<td>00122021</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>16</td>
<td>0011310221203323</td>
</tr>
</tbody>
</table>

1.3 SERIES AND PRODUCTS

1.3.1 DEFINITIONS

If \( \{a_n\} \) is a sequence of numbers or functions, then

- \( S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \ldots + a_N \).
- \( S_N \) is the \( N \)th partial sum of \( S \).
- The series is said to converge if the limit exists and diverge if it does not.
- For an infinite series: \( S = \lim_{N \to \infty} S_N = \sum_{n=1}^{\infty} a_n \) (when the limit exists).
- If \( a_n = b_n x^n \), where \( b_n \) is independent of \( x \), then \( S \) is called a power series.
- If \( a_n = (-1)^n |a_n| \), then \( S \) is called an alternating series.
- If \( \sum |a_n| \) converges, then the series converges absolutely.
- If \( S \) converges, but not absolutely, then it converges conditionally.

For example, the harmonic series \( S = 1 + \frac{1}{2} + \frac{1}{3} + \ldots \) diverges. The corresponding alternating series (called the alternating harmonic series) \( S = 1 - \frac{1}{2} + \frac{1}{3} + \cdots + (-1)^{n-1} \frac{1}{n} + \ldots \) converges (conditionally) to \( \log 2 \).

1.3.2 GENERAL PROPERTIES

1. Adding or removing a finite number of terms does not affect the convergence or divergence of an infinite series.
2. The terms of an absolutely convergent series may be rearranged in any manner without affecting its value.
3. A conditionally convergent series can be made to converge to any value by suitably rearranging its terms.

4. If the component series are convergent, then \( \sum (\alpha a_n + \beta b_n) = \alpha \sum a_n + \beta \sum b_n \).

5. \( \sum_{n=0}^{\infty} a_n \left( \sum_{n=0}^{\infty} b_n \right) = \sum_{n=0}^{\infty} c_n \) where \( c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 \).

6. Summation by parts: let \( \sum a_n \) and \( \sum b_n \) converge. Then
\[
\sum a_n b_n = \sum S_n (b_n - b_{n+1})
\]
where \( S_n \) is the \( n \)th partial sum of \( \sum a_n \).

7. A power series may be integrated and differentiated term by term within its interval of convergence.

8. Schwarz inequality:
\[
\sum |a_n| |b_n| \leq \left( \sum a_n^2 \right)^{1/2} \left( \sum b_n^2 \right)^{1/2}
\]

9. Holder’s inequality:
\[
\sum |a_n b_n| \leq \left( \sum |a_n|^{1/p} \right) \left( \sum |b_n|^{1/q} \right)
\]
when \( 1/p + 1/q = 1 \) and \( p, q > 1 \)

10. Minkowski’s inequality:
\[
\left( \sum |a_n + b_n|^p \right)^{1/p} \leq \left( \sum |a_n|^p \right)^{1/p} + \left( \sum |b_n|^p \right)^{1/p}
\]
when \( p \geq 1 \)

For example:

1. Let \( T \) be the alternating harmonic series \( S \) rearranged so that each positive term is followed by the next two negative terms. By combining each positive term of \( T \) with the succeeding negative term, we find that \( T_{3N} = \frac{1}{2} S_{2N} \). Hence, \( T = \frac{1}{2} \log 2 \).

2. The series \( 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots \) diverges, whereas
\[
1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots + \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} + \cdots
\]
converges to \( \log(2\sqrt{2}) \).
1.3.3 CONVERGENCE TESTS

1. **Comparison test**: If \(|a_n| \leq b_n\) and \(\sum b_n\) converges, then \(\sum a_n\) converges.

2. **Limit test**: If \(\lim_{n \to \infty} a_n \neq 0\), then \(\sum a_n\) is divergent.

3. **Ratio test**: Let \(\rho = \lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right|\). If \(\rho < 1\), the series converges absolutely. If \(\rho > 1\), the series diverges.

4. **Cauchy root test**: Let \(\sigma = \lim_{n \to \infty} \sqrt[n]{|a_n|}\). If \(\sigma < 1\), the series converges. If \(\sigma > 1\), it diverges.

5. **Integral test**: Let \(\mid a_n\mid = f(n)\) with \(f(x)\) being monotone decreasing, and \(\lim_{x \to \infty} f(x) = 0\). Then \(\int_1^\infty f(x)\,dx\) and \(\sum a_n\) both converge or both diverge for any \(A > 0\).

6. **Gauss’s test**: If \(\frac{|a_{n+1}|}{a_n} = 1 - \frac{p}{n} + \frac{A_n}{n^q}\) where \(q > 1\) and the sequence \(\{A_n\}\) is bounded, then the series is absolutely convergent if and only if \(p > 1\).

7. **Alternating series test**: If \(|a_n|\) tends monotonically to 0, then \(\sum (-1)^n|a_n|\) converges.

For example:

1. For \(S = \sum_{n=1}^{\infty} x^n, \rho = \lim_{n \to \infty} (1 + \frac{1}{n}) x = x\). Hence, using the ratio test, \(S\) converges for \(0 < x < 1\) and any value of \(c\).

2. For \(S = \sum_{n=1}^{\infty} \frac{5^n}{n^n}, \sigma = \lim_{n \to \infty} (\frac{5^n}{n^n})^{1/n} = 5\). Therefore the series diverges.

3. For \(S = \sum_{n=1}^{\infty} n^{-s}\), let \(f(x) = x^{-s}\). Then
   \[
   \int_1^\infty f(x)\,dx = \int_1^\infty \frac{dx}{x^s} = \frac{1}{s-1}
   \]
   for \(s > 1\), and the integral diverges for \(s \leq 1\). Hence, \(S\) converges for \(s > 1\).

4. The sum \(\sum_{n=2}^{\infty} \frac{1}{n^{s-1} \log n}\) converges for \(s > 1\) by the integral test.

5. Let \(a_n = \frac{c^1}{n!} = \frac{c(c+1)...(c+n-1)}{n^n}\) where \(c\) is not 0 or a negative integer. Then \(|a_{n+1}/a_n| = 1 - (c+1)/n + (c+1)/n^2(1+1/n)\). By Gauss’s test, the series converges absolutely if and only if \(c > 0\).

1.3.4 TYPES OF SERIES

**Bessel series**

1. **Fourier–Bessel series**:
   \[
   \sum_{n=0}^{\infty} a_n J_v(j,vz)
   \]

2. **Neumann series**:
   \[
   \sum_{n=0}^{\infty} a_n J_{v+n}(z)
   \]
3. **Kapteyn series:**
\[ \sum_{n=0}^{\infty} a_n J_{\nu+n}[(\nu+n)z] \]

4. **Schlömilch series:**
\[ \sum_{n=1}^{\infty} a_n J_{\nu}(nz) \]

For example:
- \[ \sum_{n=0}^{\infty} \frac{1}{n!} J_{\nu+n}(2) = \frac{1}{\Gamma(\nu+1)} \]
- \[ \sum_{n=1}^{\infty} J_{\nu}(nz) = \frac{1}{2} \frac{z}{1-z} \text{ for } 0 < z < 1 \]
- \[ \sum_{n=1}^{\infty} (-1)^{n+1} J_{\nu}(nz) = \frac{1}{2} \text{ for } 0 < z < \pi \]

**Dirichlet series**

These are series of the form \( \sum_{n=1}^{\infty} \frac{a_n}{n^x} \). They converge for \( x > x_0 \), where \( x_0 \) is the **abscissa of convergence**. Assuming the limits exist:

- If \( \sum a_n \) diverges, then \( x_0 = \lim_{n \to \infty} \log |a_1 + \cdots + a_n| \log n \).
- If \( \sum a_n \) converges, then \( x_0 = \lim_{n \to \infty} \log |a_{n+1} + a_{n+2} + \cdots| \log n \).

For example:
- **Riemann zeta function**:
  \[ \zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}, \quad x_0 = 1 \]
- \( \sum_{n=1}^{\infty} \frac{\mu(n)}{n^x} = \frac{1}{\zeta(x)}, \quad x_0 = 1 \) (\( \mu(n) \) denotes the Möbius function)
- \( \sum_{n=1}^{\infty} \frac{d(n)}{n^x} = [\zeta(x)]^2, \quad x_0 = 1 \) (\( d(n) \) is the number of divisors of \( n \))

**Fourier series**

If \( f(x) \) satisfies certain properties, then (see page 46 for details)
\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \] (1.3.1)
If \( f(x) \) has the Laplace transform \( F(k) \), then

\[
\sum_{k=1}^{\infty} F(k) \cos(kt) = \frac{1}{2} \int_0^\infty \frac{\cos(t) - e^{-x}}{\cosh(x) - \cos(t)} f(x) \, dx,
\]

\[
\sum_{k=1}^{\infty} F(k) \sin(kt) = \frac{1}{2} \int_0^\infty \frac{f(x)}{\cosh(x) - \cos(t)} \, dx.
\]

(1.3.2)

Since the cosine transform of \( (\cosh(x) - \cos(t))^{-1} \) with respect to \( x \) is \( \pi \csc(t) \), we find that

\[
\sum_{k=1}^{\infty} k \sin(kt) = \frac{\pi}{2} \frac{\sinh(\pi - t)y}{\sinh(\pi y)}.
\]

\[
\sum_{k=1}^{\infty} \frac{\sin(2\pi x)}{k^{\pi+1}} = \frac{(-1)^{k-1}}{2} \frac{(2\pi)^{2k+1}}{2^{2k+1}} B_{2k+1}(x), \text{ for } 0 < x < \frac{1}{2}
\]

\[
\sum_{n=1}^{\infty} \frac{\cos(2\pi x)}{n^n} = \frac{(-1)^{k-1}}{2} \frac{(2\pi)^{2k}}{(2^{2k})} B_k(x) \text{ for } 0 < x < \frac{1}{2}
\]

\[
\sum_{n=1}^{\infty} \frac{\sin(2\pi x - \pi k/2)}{(2n+1)^{\pi+1}} = \frac{\pi^{k+1}}{4^{k+1}} E_k(x)
\]

\[
\sum_{n=1}^{\infty} a^n \sin(nx) = \frac{a \sin(x)}{1 - 2a \cos(x) + a^2} \text{ for } |a| < 1
\]

\[
\sum_{n=0}^{\infty} a^n \cos(nx) = \frac{1 - a \cos(x)}{1 - 2a \cos(x) + a^2} \text{ for } |a| < 1
\]

**Hypergeometric series**

The hypergeometric function is

\[
{\mbox{pFq}} \left( \begin{array}{c} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{array} \mid k \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n \ldots (a_p)_n}{(b_1)_n(b_2)_n \ldots (b_q)_n} \frac{x^n}{n!}
\]

where \((a)_n = \Gamma(a+n)/\Gamma(a)\) is the shifted factorial. Any infinite series \( \sum A_n \) with \( A_{n+1}/A_n \) a rational function of \( n \) is of this type. These include series of products and quotients of binomial coefficients. For example:

\[
\begin{align*}
2F_1 \left( \begin{array}{c} a, \ b \\ c \end{array} \mid 1 \right) & = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \text{ (Gauss)} \\
3F_2 \left( \begin{array}{c} -n, \ a, \ b \\ c, \ 1+a+b-c-n \end{array} \mid 1 \right) & = \frac{(c-a)_n(c-b)_n}{(c)(c-a-b)_n} \text{ (Saalschutz)} \\
4F_3 \left( \begin{array}{c} a, \ a/2, \ b, \ 1+a/2 \ 1+a/2, \ b, \ 1+2b-n \end{array} \mid 1 \right) & = \frac{(a-2b)_n(-b)_n}{(1+a-b)_n(-2b)_n} \text{ (Bailey)}
\end{align*}
\]

\[
\sum_{m=0}^{2n} (-1)^m \frac{(2m)_m(2n+m+1)}{(m+2m+2)} (3 + 2\sqrt{2})^m = \frac{(3/4)_n(5/4)_n}{(7/8)_n(9/8)_n}
\]

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Power series

1. The values of $x$ for which the power series $\sum_{n=0}^{\infty} a_n x^n$ converges, form an interval (interval of convergence) which may or may not include one or both endpoints.

2. A power series may be integrated and differentiated term-by-term within its interval of convergence.

3. Note that
$$\left[1 + \sum_{n=1}^{\infty} a_n x^n\right]^{-1} = 1 - \sum_{n=1}^{\infty} b_n x^n,$$
where $b_1 = a_1$ and $b_n = a_n + \sum_{k=1}^{n-1} b_{n-k} a_k$ for $n \geq 2$.

4. Inversion of power series: If
$$s = \sum_{n=1}^{\infty} a_n x^n,$$ then
$$x = \sum_{n=1}^{\infty} A_n s^n,$$ where
$$A_1 = \frac{1}{a_1}, A_2 = -a_2/a_1^3, A_3 = (2a_2^2 - a_1 a_3)/a_1^5, A_4 = (5a_1 a_2 a_3 - a_2^2 a_4 - 5a_3^2)/a_1^7, A_5 = (6a_1^2 a_2 a_4 + 3a_1^2 a_3^2 + 14a_1^4 - a_1^3 a_5 - 21a_1 a_2^2 a_3)/a_1^9.$$
\[ \sum_{n=1}^{\infty} \left( \frac{x^n}{(n+1)(n+3)} \right) = \frac{1}{2} \int_0^1 f'(u) \, du + \int_0^x 0 \, dt \sum_{n=0}^{\infty} t^n = \frac{1}{2x^3} \left[ x + \frac{1}{2} x^2 + (1 - x^2) \log(1 - x) \right] \text{ for } |x| < 1 \]

\[ \sum_{k=1}^{\infty} \frac{x^k}{k^2} = \operatorname{Li}_2(x) \text{ (polylogarithm)} \]

**Telescoping series**

If \( \lim_{n \to \infty} F(n) = 0 \), then \( \sum_{n=1}^{\infty} [F(n) - F(n + 1)] = F(1) \). For example,

\[ \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \sum_{n=1}^{\infty} \left[ \frac{1}{n+1} - \frac{1}{n+2} \right] = \frac{1}{2}. \]

The GWZ symbolic computer algorithm expresses a proposed identity in the form of a telescoping series \( \sum_{n=1}^{\infty} [F(n + 1, k) - F(n, k)] = 0 \), then searches for a \( G(n, k) \) that satisfies \( F(n + 1, k) - F(n, k) = G(n, k + 1) - G(n, k) \) and \( G(n, \pm \infty) = 0 \). The search assumes that \( G(n, k) = R(n, k)F(n, k - 1) \) where \( R(n, k) \) is a rational expression in \( n \) and \( k \). When \( R \) is found, the proposed identity is verified. For example, the Pfaff–Saalschutz identity has the following proof:

\[ \sum_{k=-\infty}^{\infty} \frac{(a + k)(b + k)(c - a - b + n - 1 - k)!}{(k + 1)!(n - k)!(c + k)!} = \frac{(c - a + n)(c - b + n)!}{(n + 1)!(c + n)!}, \]

\[ R(n, k) = -\frac{(b + k)(a + k)}{(c - b + n + 1)(c - a + n + 1)}. \]

**Other types of series**

1. Arithmetic series:

\[ \sum_{n=1}^{N} (a + nd) = N a + \frac{1}{2} N(N + 1)d. \]

2. Arithmetic power series:

\[ \sum_{n=1}^{N} (a \cdot nb)x^n = \frac{a - (a + bN)x^{N+1}}{1 - x} + \frac{bx(1 - x^N)}{(1 - x)^2}, \quad (x \neq 1). \]

3. Geometric series:

\[ 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}, \quad (|x| < 1). \]

4. Arithmetic–geometric series:

\[ a + (a + b)x + (a + 2b)x^2 + (a + 3b)x^3 + \cdots = \frac{a}{1-x} + \frac{bx}{(1-x)^2}, \quad (|x| < 1). \]
5. Combinatorial sums:
- $\sum_{k=0}^{n} \binom{t-k}{n-k} = \binom{t+1}{n}$
- $\sum_{k=-\infty}^{m} (-1)^k \binom{t}{k} = (-1)^m \binom{t-1}{m}$
- $\sum_{k=0}^{n} \binom{k+m}{k} = \binom{m+n+1}{n}$
- $\sum_{k=-\infty}^{m} (-1)^k \binom{x+m}{k} = \binom{x}{m}$
- $\sum_{k=0}^{\infty} \binom{x}{m+n} = \binom{x+y}{m+n}$
- $\sum_{k=-\infty}^{\infty} \binom{l+k}{n+k} = \binom{l+m}{l-m-n}$
- $\sum_{k=-\infty}^{\infty} (-1)^k \binom{l+k}{m+k+n} = (-1)^{l+m} \binom{m-1}{n-l}$
- $\sum_{k=0}^{l} \binom{l-k}{q+k} = \binom{l+q+1}{m+n+1}$ (if $m \geq q$)

6. Generating functions:
- Bessel functions: $\sum_{k=0}^{\infty} J_k(x) z^k = \exp \left( \frac{1}{2} x^2 - \frac{1}{2} \right)$
- Chebyshev polynomials: $\sum_{n=0}^{\infty} T_n(x) z^n = \frac{2(xz+1)}{2x^2-1}$
- Hermite polynomials: $\sum_{n=0}^{\infty} H_n(x) z^n = \exp(2xz-z^2)$
- Laguerre polynomials: $\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) z^n = (1-z)^{\alpha} \exp \left( \frac{1}{1-z} \right)$
- Legendre polynomials: $\sum_{n=0}^{\infty} P_n(z) z^n = \frac{1}{\sqrt{1-z+z^2}}$, for $|x| < 1$

7. Multiple series:
- $\sum_{n=0}^{\infty} \frac{(-1)^n}{(m+1)^n} = \sqrt{3}$ for $-\infty < l, m, n < \infty$ not all zero
- $\sum_{n=0}^{\infty} \frac{b_1}{(m+n+1)^2} = 4 \beta(z) \bar{\zeta}(z)$ for $-\infty < m, n < \infty$ not both zero
- $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(n+1/2)}{\Gamma(m+n+1/2)} \frac{\Gamma(n+1/2)}{\Gamma(m+n+1/2)} \frac{m+n}{z^{m+n}} = \sqrt{\pi} e^{-z} \text{erf}(\sqrt{z})$ for $z > 0$
- $\sum_{n=0}^{\infty} \frac{m^2-n^2}{(m+n+1)^2} = \frac{\pi}{4}$
- $\sum_{k_1, k_2, \ldots, k_n} \frac{1}{k_1 k_2 \ldots k_n} = \frac{n!}{(2n+1)!}$ for $1 \leq k_1 < \ldots < k_n < \infty$

1.3.5 SUMMATION FORMULAE

1. Euler–Maclaurin summation formula: As $n \to \infty$,

$$\sum_{k=0}^{n} f(k) \sim \frac{1}{2} f(n) + \int_{0}^{n} f(x) \, dx + C + \sum_{j=1}^{\infty} (-1)^{j+1} B_{j+1} \frac{f^{(j)}(n)}{(j+1)!}$$

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where

\[
C = \lim_{m \to \infty} \left[ \sum_{j=1}^{m} (-1)^j B_{j+1} \frac{f^{(j)}(0)}{(j+1)!} + \frac{1}{2} \int_{0}^{\infty} B_{m+1}(x-[x]) f^{(m+1)}(x) \, dx \right].
\]

2. **Poisson summation formula**: If \( f \) is continuous,

\[
\frac{1}{2} f(0) + \sum_{n=1}^{\infty} f(n) = \int_{0}^{\infty} f(x) \, dx + 2 \sum_{n=1}^{\infty} \left[ \int_{0}^{\infty} f(x) \cos(2\pi n x) \, dx \right].
\]

For example:

- \( \sum_{k=1}^{n} \frac{1}{k} \sim \log n + \gamma + \frac{1}{2n} - \ldots \) where \( \gamma \) is Euler’s constant.
- \( 1 + 2 \sum_{n=1}^{\infty} e^{-\pi^2/n^2} = \frac{\pi}{\sqrt{3}} \left[ 1 + 2 \sum_{n=1}^{\infty} e^{-\pi^2/n^2} \right] \) (Jacobi)

### 1.3.6 IMPROVING CONVERGENCE: SHANKS TRANSFORMATION

Let \( s_n \) be the \( n \)th partial sum. The sequences \( \{ S(s_n) \}, \{ S(S(s_n)) \}, \ldots \) often converge successively more rapidly to the same limit as \( \{ s_n \} \), where

\[
S(s_n) = \frac{s_{n+1}s_{n-1} - s_n^2}{s_{n+1} + s_{n-1} - 2s_n}.
\]

For example, for \( s_n = \sum_{k=0}^{n} (-1)^k z^k \), we find \( S(s_n) = \frac{1}{1+z} \) for all \( n \).

### 1.3.7 SUMMABILITY METHODS

Unique values can be assigned to divergent series in a variety of ways which preserve the values of convergent series.

1. **Abel summation**: \( \sum_{n=0}^{\infty} a_n = \lim_{r \to 1^-} \sum_{n=0}^{\infty} a_nr^n. \)
2. **Cesaro (C, 1)-summation**: \( \sum_{n=0}^{\infty} a_n = \lim_{N \to \infty} \frac{a_0 + \ldots + a_N}{N+1} \) where \( s_n = \sum_{m=0}^{n} a_m. \)

For example:

- \( 1 - 1 + 1 - 1 + \ldots = \frac{1}{2} \) (in the sense of Abel summation)
- \( 1 - 1 + 0 + 1 - 1 + 0 + 1 - \ldots = \frac{1}{3} \) (in the sense of Cesaro summation)

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1.3.8 OPERATIONS WITH SERIES

Let \( y = a_1 x + a_2 x^2 + a_3 x^3 + \ldots \), and let \( z = z(y) = b_1 x + b_2 x^2 + b_3 x^3 + \ldots \).

<table>
<thead>
<tr>
<th>( z )</th>
<th>( b_0 )</th>
<th>( b_1 )</th>
<th>( b_2 )</th>
<th>( b_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{1 - y} )</td>
<td>1</td>
<td>( a_1 )</td>
<td>( a_1^2 + a_2 )</td>
<td>( a_1^3 + 2a_1a_2 + a_3 )</td>
</tr>
<tr>
<td>( \sqrt{1 + y} )</td>
<td>1</td>
<td>( \frac{1}{2}a_1 )</td>
<td>( -\frac{1}{2}a_1^2 + \frac{1}{2}a_2 )</td>
<td>( \frac{1}{16}a_1^3 - \frac{1}{4}a_1a_2 + \frac{1}{4}a_3 )</td>
</tr>
<tr>
<td>( (1 + y)^{-1/2} )</td>
<td>1</td>
<td>( -\frac{1}{2}a_1 )</td>
<td>( \frac{3}{8}a_1^2 - \frac{1}{2}a_2 )</td>
<td>( -\frac{5}{16}a_1^3 + \frac{3}{4}a_1a_2 - \frac{1}{2}a_3 )</td>
</tr>
<tr>
<td>( e^y )</td>
<td>1</td>
<td>( a_1 )</td>
<td>( \frac{1}{2}a_1^2 + a_2 )</td>
<td>( \frac{1}{8}a_1^3 + a_1a_2 + a_3 )</td>
</tr>
<tr>
<td>( \log(1 + y) )</td>
<td>0</td>
<td>( a_1 )</td>
<td>( a_2 - \frac{1}{2}a_1^2 )</td>
<td>( a_3 - a_1a_2 + \frac{3}{2}a_1^3 )</td>
</tr>
<tr>
<td>( \sin y )</td>
<td>0</td>
<td>( a_1 )</td>
<td>( a_2 )</td>
<td>( -\frac{1}{8}a_1^3 + a_3 )</td>
</tr>
<tr>
<td>( \cos y )</td>
<td>1</td>
<td>0</td>
<td>( -\frac{1}{2}a_1^2 )</td>
<td>( -a_1a_2 )</td>
</tr>
<tr>
<td>( \tan y )</td>
<td>0</td>
<td>( a_1 )</td>
<td>( a_2 )</td>
<td>( \frac{1}{8}a_1^3 + a_3 )</td>
</tr>
</tbody>
</table>

1.3.9 MISCELLANEOUS SUMS AND SERIES

- \( \sum_{n=1}^{N} (-1)^{n+1} n^k = \frac{1}{2} (-1)^{N+1} [E_k(N+1) + (-1)^N E_k] \)
- \( \sum_{n=0}^{N} \binom{N}{k} \binom{m+n+k-1}{m} = \binom{N+n}{m} \binom{m+n+k-1}{m} \)
- \( \sum_{n=0}^{\infty} \frac{1}{n^{k+1}} = \frac{\pi^k b}{k!} (\cot(\pi a) - \cot(\pi b)) \)
- \( \sum_{n=0}^{\infty} \frac{1}{n^{k+1}} = \frac{1}{2\pi} \left( \frac{\sin(\pi \sqrt{k+1})}{\sin(\pi \sqrt{k})} + \frac{\sin(\pi \sqrt{k})}{\sin(\pi \sqrt{k+1})} \right) \)
- \( \sum_{n=1}^{\infty} \frac{1}{n^{k+1}} = \frac{1}{2\pi} \left( 1 - \frac{1}{8\pi} \right) \)
- The series \( \sum_{n=3}^{\infty} \frac{1}{n \log n (\log \log n)^2} \) converges to 38.43 \ldots so slowly that it requires \( 10^{3.14 \cdot 10^{10}} \) terms to give two-decimal accuracy
- The series \( \sum_{n=3}^{\infty} \frac{1}{n \log n (\log \log n)} \) diverges, but the partial sums exceed 10 only after a googolplex of terms have appeared
1.3.10 INFINITE SERIES

Algebraic functions

\[(x + y)^n = x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \ldots.\]

\[(1 \pm x)^n = 1 \pm \binom{n}{1} x + \binom{n}{2} x^2 \pm \binom{n}{3} x^3 + \ldots, \quad (x^2 < 1).\]

\[(1 \pm x)^{-n} = 1 \mp \binom{n}{1} x \pm \binom{n+1}{2} x^2 \mp \binom{n+2}{3} x^3 + \ldots, \quad (x^2 < 1).\]

\[\sqrt{1 + x} = 1 + \frac{1}{2} x - \frac{1}{8} x^2 + \frac{1}{16} x^3 - \frac{5}{128} x^4 + \ldots, \quad (x^2 < 1).\]

\[(1 + x)^{-1/2} = 1 - \frac{1}{2} x + \frac{3}{8} x^2 - \frac{5}{16} x^3 + \frac{35}{128} x^4 + \ldots, \quad (x^2 < 1).\]

\[(1 \pm x)^{-1} = 1 \mp x + x^2 \mp x^3 \mp x^4 + \ldots, \quad (x^2 < 1).\]

Exponential functions

\[e = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} + \ldots, \quad \text{(all real values of } x)\]

\[e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots,\]

\[= e^a \left[ 1 + (x - a) + \frac{(x - a)^2}{2!} + \ldots + \frac{(x - a)^n}{n!} + \ldots \right].\]

\[a^x = 1 + x \log_e a + \frac{(x \log_e a)^2}{2!} + \ldots + \frac{(x \log_e a)^n}{n!} + \ldots.\]
Logarithmic functions

\[
\log x = \frac{x - 1}{x} + \frac{1}{2} \left( \frac{x - 1}{x} \right)^2 + \cdots + \frac{1}{n} \left( \frac{x - 1}{x} \right)^n + \cdots, \quad (x > 1/2),
\]

\[
= (x - 1) - \frac{1}{2} (x - 1)^2 + \frac{1}{3} (x - 1)^3 - \cdots, \quad (2 \geq x > 0),
\]

\[
= 2 \left[ \frac{x - 1}{x + 1} + \frac{1}{3} \left( \frac{x - 1}{x + 1} \right)^3 + \frac{1}{5} \left( \frac{x - 1}{x + 1} \right)^5 + \cdots \right], \quad (x > 0).
\]

\[
= \log a + \frac{(x - a)}{a} - \frac{(x - a)^2}{2a^2} + \frac{(x - a)^3}{3a^2} - \cdots, \quad (0 < x \leq 2a).
\]

\[
\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad -1 < x \leq 1.
\]

\[
\log(n + 1) = \log(n - 1) + 2 \left[ \frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \cdots \right].
\]

\[
\log(a + x) = \log a + 2 \left[ \frac{x}{2a + x} + \frac{1}{3} \left( \frac{x}{2a + x} \right)^3 + \frac{1}{5} \left( \frac{x}{2a + x} \right)^5 + \cdots \right], \quad (a > 0, -a < x).
\]

\[
\log \frac{1 + x}{1 - x} = 2 \left[ x + \frac{x^3}{3} + \cdots + \frac{x^{2n-1}}{2n-1} + \cdots \right], \quad (-1 < x < 1).
\]

Trigonometric functions

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{(all real values of } x). \]

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{(all real values of } x). \]

\[
\tan x = x + \frac{2x^3}{3} + \frac{2x^5}{15} + \cdots + (-1)^{n-1} 2^{2n} (2^{2n} - 1) B_{2n} x^{2n-1} + \cdots \quad \frac{(2n)!}{B_{2n} (2n)!}, \quad (x^2 < \pi^2/4, B_n \text{ is the } n^{th} \text{ Bernoulli number}).
\]

\[
\cot x = \frac{1}{x} - \frac{x^3}{3} - \frac{2x^5}{945} - \frac{x^7}{4725} - \cdots + \frac{(-1)^{n+1} 2^{2n} B_{2n}}{(2n)!} x^{2n-1} + \cdots \quad (x^2 < \pi^2, B_n \text{ is the } n^{th} \text{ Bernoulli number}).
\]

\[
\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \frac{277x^8}{8064} + \cdots + \frac{(-1)^n E_{2n}}{(2n)!} x^{2n} + \cdots \quad (x^2 < \pi^2/4, E_n \text{ is the } n^{th} \text{ Euler number}).
\]

\[
\csc x = 1 + \frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15120} + \cdots + \frac{(-1)^{n+1} 2 (2^{2n-1} - 1) B_{2n}}{(2n)!} x^{2n-1} + \cdots
\]

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$\log \sin x = \log x - \frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} - \ldots \quad (x^2 < \pi^2).

\log \cos x = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \frac{17x^8}{2520} - \ldots \quad (x^2 < \pi^2/4).

\log \tan x = \log x + \frac{x^2}{3} + \frac{7x^4}{90} + \frac{62x^6}{2835} + \ldots \quad (x^2 < \pi^2/4).

\sin x = \sin a + (x - a) \cos a - \frac{(x - a)^2}{2!} \sin a + \frac{(x - a)^3}{3!} \cos a + \ldots \quad (|x| < \pi, B_n \text{ is the } n^{th} \text{ Bernoulli number}).

\cos ^{-1} x = \frac{\pi}{2} - \left( x + \frac{1}{2} \cdot x^3 + \frac{1 \cdot 3}{2 \cdot 4} x^5 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} x^7 + \ldots \right) \quad (x^2 < 1, 0 < \cos ^{-1} x < \frac{\pi}{2}).

\tan ^{-1} x = \frac{\pi}{2} - \left( \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \right) \quad (x^2 < 1),

\cot ^{-1} x = \frac{\pi}{2} - x + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \ldots \quad (x^2 < 1).
Hyperbolic functions

\[ \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots \]

\[ \sinh ax = \frac{2}{a} \sinh \pi a \left[ \frac{\sin x}{a^2 + 1^2} - \frac{2 \sin 2x}{a^2 + 2^2} + \frac{3 \sin 3x}{a^2 + 3^2} + \cdots \right] \quad (|x| < \pi). \]

\[ \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots \]

\[ \cosh ax = \frac{2a}{\pi} \sinh \pi a \left[ \frac{1}{2a^2} - \frac{\cos x}{a^2 + 1^2} + \frac{\cos 2x}{a^2 + 2^2} - \frac{\cos 3x}{a^2 + 3^2} + \cdots \right] \quad (|x| < \pi). \]

\[ \tanh x = x - \frac{1}{3} x^3 + \frac{2}{15} x^5 - \cdots + \frac{2^{2n} (2^{2n} - 1) B_{2n} x^{2n-1}}{(2n)!} + \cdots \quad (|x| < \pi/2), \]

\[ = 1 - 2 e^{-2x} + 2 e^{-4x} - 2 e^{-6x} + \cdots \quad (\text{Re } x > 0), \]

\[ = 2 x \left[ \frac{1}{(\frac{x}{2})^2 + x^2} + \frac{1}{(\frac{3x}{2})^2 + x^2} + \frac{1}{(\frac{5x}{2})^2 + x^2} + \cdots \right]. \]

\[ \coth x = 1 + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} + \cdots + \frac{2^{2n} B_{2n} x^{2n-1}}{(2n)!} + \cdots \quad (0 < |x| < \pi), \]

\[ = 1 + 2 e^{-2x} + 2 e^{-4x} + 2 e^{-6x} + \cdots \quad (\text{Re } x > 0), \]

\[ = \frac{1}{x} + 2 x \left[ + \frac{1}{\pi^2 + x^2} + \frac{1}{(2\pi)^2 + x^2} + \frac{1}{(3\pi)^2 + x^2} + \cdots \right] \quad (\text{Re } x > 0). \]

\[ \text{sech } x = 1 - \frac{1}{2!} x^2 + \frac{5}{4!} x^4 - \frac{61}{6!} x^6 + \cdots + \frac{E_{2n}}{(2n)!} x^{2n} + \cdots \quad (|x| < \pi/2, E_n \text{ is the } n^{\text{th}} \text{ Euler number}), \]

\[ = 2 \left( e^{-x} - e^{-3x} + e^{-5x} - e^{-7x} + \cdots \right) \quad (\text{Re } x > 0), \]

\[ = 4 \pi \left[ \frac{1}{\pi^2 + 4x^2} - \frac{3}{(3\pi)^2 + 4x^2} + \frac{5}{(5\pi)^2 + 4x^2} + \cdots \right]. \]

\[ \text{csch } x = \frac{1}{x} - \frac{7x^3}{360} + \cdots + \frac{2 (2^{2n-1} - 1) B_{2n} x^{2n-1}}{(2n)!} + \cdots \quad (0 < |x| < \pi), \]

\[ = 2 \left( e^{-x} + e^{-3x} + e^{-5x} + e^{-7x} + \cdots \right) \quad (\text{Re } x > 0), \]

\[ = \frac{1}{x} - \frac{2x}{\pi^2 + x^2} + \frac{2x}{(2\pi)^2 + x^2} - \frac{2x}{(3\pi)^2 + x^2} + \cdots \].

\[ \sinh nu = \sinh u \left[ (2 \cosh u)^{n-1} - \frac{(n - 2)}{1!} (2 \cosh u)^{n-3} \right. \]

\[ \left. + \frac{(n - 3)(n - 4)}{2!} (2 \cosh u)^{n-5} - \frac{(n - 4)(n - 5)(n - 6)}{3!} (2 \cosh u)^{n-7} + \cdots \right]. \]
For the sequence of complex numbers \(\{a_k\}\), the infinite product is defined as \(\prod_{k=1}^{\infty} (1 + a_k)\). A necessary condition for convergence is that \(\lim_{n \to \infty} a_n = 0\). A necessary and sufficient condition for convergence is that \(\sum_{k=1}^{\infty} \log(1 + a_k)\) converges. Examples:

\[
\begin{align*}
\text{z!} &= \prod_{k=1}^{\infty} \frac{(1 + \frac{1}{k})^z}{1 + \frac{1}{k}}, \\
\sin z &= z \prod_{k=1}^{\infty} \cos \frac{z}{2k}, \\
\sin \pi z &= \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right), \\
\cos \pi z &= \prod_{k=1}^{\infty} \left(1 - \frac{\frac{1}{2}z^2}{(k - \frac{1}{2})^2}\right).
\end{align*}
\]
\[ \sin(a + z) = (\sin a) \prod_{k=0, \pm 1, \pm 2, \ldots} \left( 1 + \frac{z}{a + k\pi} \right) \]
\[ \cos(a + z) = (\cos a) \prod_{k=\pm 1, \pm 3, \pm 5, \ldots} \left( 1 + \frac{2z}{2a + k\pi} \right) \]
\[ \sinh z = z \prod_{k=1}^{\infty} \left( 1 + \frac{z^2}{k^2\pi^2} \right) \]
\[ \cosh z = \prod_{k=0}^{\infty} \left( 1 + \frac{4z^2}{(2k + 1)^2\pi^2} \right) \]

**Weierstrass theorem**

Define \( E(w, m) = (1 - w) \exp \left( w + \frac{w^2}{2} + \cdots + \frac{w^m}{m} \right) \). For \( k = 1, 2, \ldots \) let \( \{b_k\} \) be a sequence of complex numbers such that \( |b_k| \to \infty \).

Then the infinite product \( P(z) = \prod_{k=1}^{\infty} E \left( \frac{z}{b_k}, k \right) \) is an entire function with zeros at \( b_k \) and at these points only. The multiplicity of the root at \( b_n \) is equal to the number of indices \( j \) such that \( b_j = b_n \).

### 1.3.12 Infinite Products and Infinite Series

1. The Rogers–Ramanujan identities (for \( a = 0 \) or \( a = 1 \)) are

\[ 1 + \sum_{k=1}^{\infty} \frac{q^{k^2 + ak}}{(1 - q)(1 - q^2) \cdots (1 - q^k)} = \prod_{j=0}^{\infty} \frac{1}{(1 - q^{5j + a + 1})(1 - q^{5j-a+4})}. \]

(1.3.4)

2. Jacobi’s triple product identity is

\[ \sum_{k=-\infty}^{\infty} q^{(\frac{1}{2})} x^k = \prod_{j=1}^{\infty} (1 - q^j)(1 + x^{-1}q^j)(1 + xq^{j-1}). \]

(1.3.5)

3. The quintuple product identity is

\[ \sum_{k=-\infty}^{\infty} (-1)^k q^{(3k^2 - k)/2} x^{3k} (1 + xq^k) \]

\[ = \prod_{j=1}^{\infty} (1 - q^j)(1 + x^{-1}q^j)(1 + xq^{j-1})(1 + x^{-2}q^{2j-1})(1 + x^2q^{2j-1}). \]

(1.3.6)
1.4 FOURIER SERIES

If \( f(x) \) is a bounded periodic function of period \( 2L \) (that is, \( f(x + 2L) = f(x) \)) and satisfies the Dirichlet conditions,

1. In any period, \( f(x) \) is continuous, except possibly for a finite number of jump discontinuities.
2. In any period \( f(x) \) has only a finite number of maxima and minima.

Then \( f(x) \) may be represented by the Fourier series,

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),
\]

(1.4.1)

where \( \{a_n\} \) and \( \{b_n\} \) are determined as follows:

\[
a_n = \frac{1}{L} \int_{\alpha}^{\alpha+2L} f(x) \cos \frac{n\pi x}{L} \, dx
text{ for } n = 0, 1, 2, \ldots,
\]

\[
= \frac{1}{L} \int_{0}^{2L} f(x) \cos \frac{n\pi x}{L} \, dx,
\]

\[
= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx;
\]

\[
b_n = \frac{1}{L} \int_{\alpha}^{\alpha+2L} f(x) \sin \frac{n\pi x}{L} \, dx\text{ for } n = 1, 2, 3, \ldots,
\]

\[
= \frac{1}{L} \int_{0}^{2L} f(x) \sin \frac{n\pi x}{L} \, dx,
\]

\[
= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx,
\]

(1.4.2) (1.4.3)

where \( \alpha \) is any real number (the second and third lines of each formula represent \( \alpha = 0 \) and \( \alpha = L \) respectively).

The series in Equation (1.4.1) will converge (in the Cesaro sense) to every point where \( f(x) \) is continuous, and to \( \frac{f(x^+) + f(x^-)}{2} \) (i.e., the average of the left hand and right hand limits) at every point where \( f(x) \) has a jump discontinuity.

1.4.1 SPECIAL CASES

1. If, in addition to the Dirichlet conditions, \( f(x) \) is an even function (i.e., \( f(x) = f(-x) \)), then the Fourier series becomes

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}.
\]

(1.4.4)

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That is, every $b_n = 0$. In this case, the $\{a_n\}$ may be determined from

$$a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L} \, dx \quad n = 0, 1, 2, \ldots$$  \hspace{1cm} (1.4.5)

If, in addition to the above requirements, $f(x) = f(L - x)$, then $a_n$ will be zero for all even values of $n$. In this case the expansion becomes

$$f(x) = \sum_{m=1}^{\infty} a_{2m-1} \cos \left( \frac{(2m - 1)\pi x}{L} \right).$$  \hspace{1cm} (1.4.6)

2. If, in addition to the Dirichlet conditions, $f(x)$ is an odd function (i.e., $f(x) = -f(-x)$), then the Fourier series becomes

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$  \hspace{1cm} (1.4.7)

That is, every $a_n = 0$. In this case, the $\{b_n\}$ may be determined from

$$b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} \, dx \quad n = 1, 2, 3, \ldots.$$  \hspace{1cm} (1.4.8)

If, in addition to the above requirements, $f(x) = f(L - x)$, then $b_n$ will be zero for all even values of $n$. In this case the expansion becomes

$$f(x) = \sum_{m=1}^{\infty} b_{2m-1} \sin \left( \frac{(2m - 1)\pi x}{L} \right).$$  \hspace{1cm} (1.4.9)

The series in Equation (1.4.6) and Equation (1.4.9) are known as odd harmonic series, since only the odd harmonics appear. Similar rules may be stated for even harmonic series, but when a series appears in even harmonic form, it means that $2L$ has not been taken to be the smallest period of $f(x)$. Since any integral multiple of a period is also a period, series obtained in this way will also work, but, in general, computation is simplified if $2L$ is taken as the least period.

Writing the trigonometric functions in terms of complex exponentials, we obtain the complex form of the Fourier series known as the complex Fourier series or as the exponential Fourier series. It is represented by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n x}$$  \hspace{1cm} (1.4.10)

where $\omega_n = \frac{n\pi}{L}$ for $n = 0, \pm 1, \pm 2, \ldots$ and the $\{c_n\}$ are determined from

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i\omega_n x} \, dx.$$  \hspace{1cm} (1.4.11)

The set of coefficients $\{c_n\}$ is often referred to as the Fourier spectrum.
1.4.2 ALTERNATE FORMS

The Fourier series in Equation (1.4.1) may be represented in the alternate forms:

1. When \( \phi_n = \tan^{-1}(-a_n/b_n) \), \( a_n = c_n \sin \phi_n \), \( b_n = -c_n \cos \phi_n \), and \( c_n = \sqrt{a_n^2 + b_n^2} \), then

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n \sin \left( \frac{n\pi x}{L} + \phi_n \right).
\] (1.4.12)

2. When \( \phi_n = \tan^{-1}(a_n/b_n) \), \( a_n = c_n \sin \phi_n \), \( b_n = c_n \cos \phi_n \), and \( c_n = \sqrt{a_n^2 + b_n^2} \), then

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n \cos \left( \frac{n\pi x}{L} + \phi_n \right).
\] (1.4.13)

1.4.3 USEFUL SERIES

1. \( 1 = \frac{4}{\pi} \left[ \frac{\pi x}{k} + \frac{1}{3} \sin \frac{3\pi x}{k} + \frac{1}{5} \sin \frac{5\pi x}{k} + \ldots \right] \quad [0 < x < k]. \)

2. \( x = \frac{2k}{\pi} \left[ \frac{\sin \frac{\pi x}{k}}{\frac{\pi}{2}} - \frac{1}{3} \sin \frac{2\pi x}{k} + \frac{1}{5} \sin \frac{3\pi x}{k} + \ldots \right] \quad [-k < x < k]. \)

3. \( x^2 = \frac{2k^2}{\pi^2} \left[ \left( \frac{\pi^2}{2} - \frac{1}{2} \right) \sin \frac{\pi x}{k} - \frac{\pi^2}{2} \sin \frac{2\pi x}{k} + \left( \frac{\pi^2}{3} - \frac{4}{3} \right) \sin \frac{3\pi x}{k} \right. \]

\( \left. - \frac{\pi^2}{4} \sin \frac{4\pi x}{k} + \left( \frac{\pi^2}{5} - \frac{4}{5} \right) \sin \frac{5\pi x}{k} + \ldots \right] \quad [0 < x < k]. \)

4. \( x^2 = \frac{k^2}{3} - \frac{4k^2}{\pi^2} \left[ \cos \frac{\pi x}{k} - \frac{1}{2} \cos \frac{2\pi x}{k} + \frac{1}{3} \cos \frac{3\pi x}{k} - \frac{1}{4} \cos \frac{4\pi x}{k} + \ldots \right] \quad [-k < x < k]. \)

\[
1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}.
\]

\[
1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6}.
\]

\[
1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}.
\]

\[
1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}.
\]

\[
1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \ldots = \frac{\pi^2}{24}.
\] (1.4.14)

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1.4.4 EXPANSIONS OF BASIC PERIODIC FUNCTIONS

- \( f(x) = \frac{4}{\pi} \sum_{n=1,3,5,\ldots} \frac{1}{n} \sin \frac{n\pi x}{L} \)

- \( f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left( \cos \frac{n\pi c}{L} - 1 \right) \sin \frac{n\pi x}{L} \)

- \( f(x) = \frac{c}{L} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \frac{n\pi c}{L} \cos \frac{n\pi x}{L} \)

- \( f(x) = \frac{2}{L} \sum_{n=1}^{\infty} \sin \frac{n\pi}{2} \sin(n\pi c/2L) \sin \frac{n\pi x}{L} \)

- \( f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \frac{n\pi x}{L} \)

- \( f(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n=1,3,5,\ldots} \frac{1}{n^2} \cos \frac{n\pi x}{L} \)

- \( f(x) = \frac{8}{\pi^2} \sum_{n=1,3,5,\ldots} (-1)^{n-1/2} \frac{1}{n^2} \sin \frac{n\pi x}{L} \)

- \( f(x) = \frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{L} \)

- \( f(x) = \frac{1}{2} + \frac{a}{2} + \frac{2}{\pi^2(1-a)} \sum_{n=1}^{\infty} \frac{(-1)^n \cos n\pi a - 1}{n^2} \cos \frac{n\pi x}{L} \) \( (a = \frac{c}{2L}) \)
\[ f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left[ 1 + \frac{\sin n\pi a}{n\pi(1-a)} \right] \sin \frac{n\pi x}{L} \quad (a = \frac{c}{2L}) \]

\[ f(x) = \frac{1}{2} - \frac{4}{\pi^2(1-2a)} \sum_{n=1,3,5, \ldots} \frac{1}{n^2} \cos n\pi a \cos \frac{n\pi x}{L} \quad (a = \frac{c}{2L}) \]

\[ f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left[ 1 + \frac{1 + (-1)^n}{n\pi(1-2a)} \sin n\pi a \right] \sin \frac{n\pi x}{L} \quad (a = \frac{c}{2L}) \]

\[ f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{4} \sin n\pi a \sin \frac{n\pi x}{L} \quad (a = \frac{c}{2L}) \]

\[ f(x) = \frac{9}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi a}{3} \sin \frac{n\pi x}{L} \quad (a = \frac{c}{2L}) \]

\[ f(x) = \frac{32}{3\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi a}{4} \sin \frac{n\pi x}{L} \quad (a = \frac{c}{2L}) \]
\[ f(x) = \frac{1}{\pi} + \frac{1}{2} \sin \omega t - \frac{2}{\pi} \sum_{n=3,4,6,\ldots} \frac{1}{n^2 - 1} \cos n\omega t \]

1.5 COMPLEX ANALYSIS

1.5.1 DEFINITIONS

A complex number \( z \) has the form \( z = x + iy \) where \( x \) and \( y \) are real numbers, and \( i = \sqrt{-1} \); the number \( i \) is sometimes called the imaginary unit. We write \( x = \text{Re} \ z \) and \( y = \text{Im} \ z \). The number \( x \) is called the real part of \( z \) and \( y \) is called the imaginary part of \( z \). This form is also called the Cartesian form of the complex number.

Complex numbers can also be written in polar form, \( z = re^{i\theta} \), where \( r \), called the modulus, is given by \( r = |z| = \sqrt{x^2 + y^2} \), and \( \theta \) is called the argument: \( \theta = \text{arg} \ z = \tan^{-1} \frac{y}{x} \). The geometric relationship between Cartesian and polar forms is shown below.

The complex conjugate of \( z \), denoted \( \bar{z} \), is defined as \( \bar{z} = x - iy = re^{-i\theta} \). Note that \( |z| = |\bar{z}| \), \( \text{arg} \bar{z} = -\text{arg} \ z \), and \( |z| = \sqrt{\bar{z}z} \). In addition, \( \bar{z} = z \), \( \bar{z_1} + \bar{z_2} = \bar{z_1} + \bar{z_2} \), and \( \bar{z_1}z_2 = \bar{z_1}\bar{z_2} \).

1.5.2 OPERATIONS ON COMPLEX NUMBERS

Addition and subtraction:

\[ z_1 \pm z_2 = (x_1 + iy_1) \pm (x_2 + iy_2) = (x_1 \pm x_2) + i(y_1 \pm y_2). \]

Multiplication:

\[ z_1z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) = r_1r_2e^{i(\theta_1 + \theta_2)}. \]
\[ |z_1 z_2| = |z_1| |z_2|, \quad \arg(z_1 z_2) = \arg z_1 + \arg z_2 = \theta_1 + \theta_2. \]

**Division:**

\[
\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{z_2 \bar{z}_2} = \frac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2} = \frac{r_1 e^{i(\theta_1 - \theta_2)}}{r_2}.
\]

\[ \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}, \quad \arg \left( \frac{z_1}{z_2} \right) = \arg z_1 - \arg z_2 = \theta_1 - \theta_2. \]

---

### 1.5.3 POWERS AND ROOTS OF COMPLEX NUMBERS

**Powers:**

\[ z^n = r^n e^{in\theta} = r^n (\cos n\theta + i \sin n\theta) \text{ DeMoivre's Theorem}. \]

**Roots:**

\[ z^{1/n} = r^{1/n} e^{i\theta/n} = r^{1/n} \left( \cos \frac{\theta + 2k\pi}{n} + i \sin \frac{\theta + 2k\pi}{n} \right), \quad k = 0, 1, 2, \ldots, n - 1. \]

The principal root has \(-\pi < \theta \leq \pi\) and \(k = 0\).

---

### 1.5.4 FUNCTIONS OF A COMPLEX VARIABLE

A complex function

\[ w = f(z) = u(x, y) + i v(x, y) = |w| e^{i\phi}, \]

where \(z = x + iy\), associates one or more values of the complex dependent variable \(w\) with each value of the complex independent variable \(z\) for those values of \(z\) in a given domain.

---

### 1.5.5 CAUCHY–RIEMANN EQUATIONS

A function \( w = f(z) \) is said to be analytic at a point \( z_0 \) if it is differentiable in a neighborhood (i.e., at each point of a circle centered on \( z_0 \) with an arbitrarily small radius) of \( z_0 \). That is, \( \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \) exists. A function is called analytic in a connected domain if it is analytic at every point in that domain.

A necessary and sufficient condition for \( f(z) = u(x, y) + i v(x, y) \) to be analytic is that it satisfy the Cauchy–Riemann equations,

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \quad (1.5.1)
\]

**Examples:**

1. \( f(z) = z^n \) is analytic everywhere when \( n \) is a nonnegative integer. If \( n \) is a negative integer, then \( f(z) \) is analytic except at the origin.
2. \( f(z) = \bar{z} \) is nowhere analytic.
3. \( f(z) = e^z \) is analytic everywhere.
1.5.6 CAUCHY INTEGRAL THEOREM

If \( f(z) \) is analytic at all points within and on a simple closed curve \( C \), then

\[
\oint_C f(z) \, dz = 0. \tag{1.5.2}
\]

1.5.7 CAUCHY INTEGRAL FORMULA

If \( f(z) \) is analytic inside and on a simple closed contour \( C \) and if \( z_0 \) is interior to \( C \), then

\[
f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} \, dz. \tag{1.5.3}
\]

Moreover, if the derivatives \( f'(z) \), \( f''(z) \), \ldots of all orders exist, then

\[
f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz. \tag{1.5.4}
\]

1.5.8 TAYLOR SERIES EXPANSIONS

If \( f(z) \) is analytic inside of and on a circle \( C \) of radius \( r \) centered at the point \( z_0 \), then a unique and uniformly convergent series expansion exists in powers of \( (z - z_0) \) of the form

\[
f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad |z - z_0| < r, \quad z_0 \neq \infty, \tag{1.5.5}
\]

where

\[
a_n = \frac{1}{n!} f^{(n)}(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz. \tag{1.5.6}
\]

If \( M(r) \) is an upper bound of \( |f(z)| \) on \( C \), then

\[
|a_n| = \frac{1}{n!} |f^{(n)}(z_0)| \leq \frac{M(r)}{r^n} \quad \text{(Cauchy’s inequality).} \tag{1.5.7}
\]

If the series is truncated with the term \( a_n(z - z_0)^n \), the remainder \( R_n(z) \) is given by

\[
R_n(z) = \frac{(z - z_0)^{n+1}}{2\pi i} \oint_C \frac{f(s)}{(s - z)(s - z_0)^{n+1}} \, ds, \tag{1.5.8}
\]

and

\[
|R_n(z)| \leq \left( \frac{|z - z_0|}{r} \right)^n \frac{r M(r)}{r - |z - z_0|}. \tag{1.5.9}
\]
1.5.9 LAURENT SERIES EXPANSIONS

If \( f(z) \) is analytic inside the annular domain between the concentric circles \( C_1 \) and \( C_2 \) centered at \( z_0 \) with radii \( r_1 \) and \( r_2 \) \((r_1 < r_2)\), respectively, then a unique series expansion exists in terms of positive and negative powers of \( z - z_0 \) of the following form:

\[
f(z) = \sum_{n=1}^{\infty} b_n (z - z_0)^{-n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n,
\]

\[
= \cdots + \frac{b_2}{(z - z_0)^2} + \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots
\]

(1.5.10)

where

\[
a_n = \frac{1}{2\pi i} \int_{C} \frac{f(s)}{(s - z_0)^{n+1}} \, ds, \quad n = 0, 1, 2, \ldots
\]

(1.5.11)

and

\[
b_n = \frac{1}{2\pi i} \int_{C} f(s)(s - z_0)^{n-1} \, ds, \quad n = 1, 2, 3, \ldots
\]

(1.5.12)

1.5.10 ZEROS AND SINGULARITIES

The points \( z \) for which \( f(z) = 0 \) are called zeros of \( f(z) \). A function \( f(z) \) which is analytic at \( z_0 \) has a zero of order \( m \) there, where \( m \) is a positive integer, if and only if the first \( m \) coefficients \( a_0, a_1, \ldots, a_{m-1} \) in the Taylor expansion about \( z_0 \) vanish.

A singular point or singularity of the function \( f(z) \) is any point at which \( f(z) \) is not analytic. An isolated singularity of \( f(z) \) at \( z_0 \) may be classified in one of three ways:

1. A removable singularity if and only if all coefficients \( b_n \) in the Laurent series expansion of \( f(z) \) about \( z_0 \) vanish.
2. A pole of order \( m \) if and only if \( (z - z_0)^m f(z) \), but not \( (z - z_0)^{m-1} f(z) \), is analytic at \( z_0 \) (i.e., if and only if \( b_m \neq 0 \) and \( 0 = b_{m+1} = b_{m+2} = \ldots \) in the Laurent series expansion of \( f(z) \) about \( z_0 \)). Equivalently, \( f(z) \) has a pole of order \( m \) if \( 1/f(z) \) is analytic at \( z_0 \) and has a zero of order \( m \) there.
3. An isolated essential singularity if and only if the Laurent series expansion of \( f(z) \) about \( z_0 \) has an infinite number of terms involving negative powers of \( z - z_0 \).
1.5.11 RESIDUES

Given a point $z_0$ where $f(z)$ is either analytic or has an isolated singularity, the residue of $f(z)$ is the coefficient of $(z - z_0)^{-1}$ in the Laurent series expansion of $f(z)$ about $z_0$, or

$$\text{Res}(z_0) = b_1 = \frac{1}{2\pi i} \int_C f(z) \, dz. \quad (1.5.13)$$

If $f(z)$ is either analytic or has a removable singularity at $z_0$, then $b_1 = 0$ there. If $z_0$ is a pole of order $m$, then

$$b_1 = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)] \bigg|_{z=z_0}. \quad (1.5.14)$$

For every simple closed contour $C$ enclosing at most a finite number of singularities $z_1, z_2, \ldots, z_n$ of an analytic function continuous on $C$,

$$\int_C f(z) \, dz = 2\pi i \sum_{k=1}^n \text{Res}(z_k), \quad (1.5.15)$$

where $\text{Res}(z_k)$ is the residue of $f(z)$ at $z_k$.

1.5.12 THE ARGUMENT PRINCIPLE

Let $f(z)$ be analytic on a simple closed curve $C$ with no zeros on $C$ and analytic everywhere inside $C$ except possibly at a finite number of poles. Let $\Delta_C \arg f(z)$ denote the change in the argument of $f(z)$ (final value − initial value) as $z$ transverses the curve once in the positive sense. Then

$$\frac{1}{2\pi} \Delta_C \arg f(z) = N - P, \quad (1.5.16)$$

where $N$ is number of zeros of $f(z)$ inside $C$, and $P$ is the number of poles inside $C$. The zeros and poles are counted according to their multiplicities.
1.5.13 TRANSFORMATIONS AND MAPPINGS

A function \( w = f(z) = u(z) + iv(z) \) maps points of the \( z \)-plane into corresponding points of the \( w \)-plane. At every point \( z \) such that \( f(z) \) is analytic and \( f'(z) \neq 0 \), the mapping is \textit{conformal}, i.e., the angle between two curves in the \( z \)-plane through such a point is reproduced in magnitude and sense by the angle between the corresponding curves in the \( w \)-plane. A table giving real and imaginary parts, zeros, and singularities for frequently used functions of a complex variable and a table illustrating a number of special transformations of interest are at the end of this section.

A function is said to be simple in a domain \( D \) if it is analytic in \( D \) and assumes no value more than once in \( D \). \textit{Riemann’s mapping theorem} states

If \( D \) is a simply connected domain in the complex \( z \) plane, which is neither the \( z \) plane nor the extended \( z \) plane, then there is a simple function \( f(z) \) such that \( w = f(z) \) maps \( D \) onto the disc \(|w| < 1\).

1.5.14 BILINEAR TRANSFORMATIONS

The \textit{bilinear} transformation is defined by \( w = \frac{az + b}{cz + d} \), where \( a, b, c, \) and \( d \) are complex numbers and \( ad \neq bc \). It is also known as the \textit{Möbius} or \textit{linear fractional} transformation. The bilinear transformation is defined for all \( z \neq -d/c \). The bilinear transformation is conformal and maps circles and lines into circles and lines.

The inverse transformation is given by \( z = \frac{-dw + b}{cw - a} \), which is also a bilinear transformation. Note that \( w \neq a/c \).

The \textit{cross ratio} of four distinct complex numbers \( z_k \) (for \( k = 1, 2, 3, 4 \)) is given by

\[
(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}.
\]

If any of the \( z_k \) is complex infinity, the cross ratio is redefined so that the quotient of the two terms on the right containing \( z_k \) is equal to 1. Under the bilinear transformation, the cross ratio of four points is invariant: \((w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)\).
1.5.15 TABLE OF TRANSFORMATIONS

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<tr>
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<td>$\frac{\sinh 2y}{\cos 2x + \cosh 2y}$</td>
<td>$z = k\pi, m = 1$</td>
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<td>$\frac{\sin 2y}{\cosh 2x + \cos 2y}$</td>
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</tr>
<tr>
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<td>$z = 1, m = 1$</td>
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</tbody>
</table>

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1.5.16 TABLE OF CONFORMAL MAPPINGS

In the following functions $z = x + iy$ and $w = u + iv = \rho e^{i\phi}$.

\[ w = z^2. \]

$w = z^2$; $A'$, $B'$ on the parabola $\rho = \frac{2k^2}{1 + \cos \phi}$.

$w = 1/z$. 

$w = e^z$. 

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$w = e^{z}$.

$w = e^{z}$.

$w = \sin z$.

$w = \sin z$.

$w = \sin z$; $BCD: y = k, B'C'D'$ is on the ellipse

\[
\left(\frac{u}{\cosh k}\right)^2 + \left(\frac{v}{\sinh k}\right)^2 = 1.
\]

$w = \frac{z - 1}{z + 1}$.

$w = \frac{i - z}{i + z}$.
\[ w = \frac{z - a}{a z - 1}; \quad a = \frac{1 + x_1 x_2 + \sqrt{(1 - x_1^2)(1 - x_2^2)}}{x_1 + x_2}; \]
\[ R_0 = \frac{1 - x_1 x_2 + \sqrt{(1 - x_1^2)(1 - x_2^2)}}{x_1 - x_2} \quad (a > 1 \text{ and } R_0 > 1 \text{ when } -1 < x_2 < x_1 < 1). \]

\[ w = \frac{z - a}{a z - 1}; \quad a = \frac{1 + x_1 x_2 + \sqrt{(x_1^2 - 1)(x_2^2 - 1)}}{x_1 + x_2}; \]
\[ R_0 = \frac{x_1 x_2 - 1 - \sqrt{(x_1^2 - 1)(x_2^2 - 1)}}{x_1 - x_2} \quad (x_2 < a < x_1 \text{ and } 0 < R_0 < 1 \text{ when } 1 < x_2 < x_1). \]

\[ w = z + 1/z. \]

\[ w = z + 1/z. \]

\[ \left( \frac{k u}{k^2 + 1} \right)^2 + \left( \frac{k v}{k^2 - 1} \right)^2 = 1. \]
\[ w = \log \frac{z - 1}{z + 1}. \]

\[ w = \log \frac{z - 1}{z + 1}; \ A B C \text{ is on the circle } x^2 + y^2 - 2y \cot k = 1. \]

\[ w = \log \frac{z + 1}{z - 1}; \text{ relationship between centers and radii: centers of circles at } z_n = \coth c_n, \text{ radii are } \text{csch } c_n, n = 1, 2. \]

\[ w = k \log \frac{k}{1-k} + \log 2(1-k) + i\pi - k \log(z+1) - (1-k) \log(z-1); \]

\[ x_1 = 2k - 1. \]

\[ w = \tan^2(z/2). \]

\[ w = \coth(z/2). \]

\[ w = \log \coth(z/2). \]
\[ w = \pi i + z - \log z. \]

\[ w = 2(z + 1)^{1/2} + \log \left(\frac{(z + 1)^{1/2} - 1}{(z + 1)^{1/2} + 1}\right). \]

\[ w = \frac{i}{k} \log \frac{1 + ikt}{1 - ikt} + \log \frac{1 + t}{1 - t}; \]

\[ t = \left(\frac{z - 1}{z + k^2}\right)^{1/2}. \]

\[ w = \frac{h}{\pi} \left[(z^2 - 1)^{1/2} + \cosh^{-1} z\right]. \]

\[ w = \cosh^{-1} \left(\frac{2z - k - 1}{k - 1}\right) - \frac{1}{k} \cosh^{-1} \left[\frac{(k + 1)z - 2k}{(k - 1)z}\right]. \]
1.6 REAL ANALYSIS

1.6.1 RELATIONS

For two sets $A$ and $B$, the product $A \times B$ is the set of all ordered pairs $(a, b)$ where $a$ is in $A$ and $b$ is in $B$. Any subset of the product $A \times B$ is called a relation. A relation $R$ on a product $A \times A$ is called an equivalence relation if the following three properties hold:

1. **Reflexive**: $(a, a)$ is in $R$ for every $a$ in $A$.
2. **Symmetric**: If $(a, b)$ is in $R$, then $(b, a)$ is in $R$.
3. **Transitive**: If $(a, b)$ and $(b, c)$ are in $R$, then $(a, c)$ is in $R$.

When $R$ is an equivalence relation then the equivalence class of an element $a$ in $A$ is the set of all $b$ in $A$ such that $(a, b)$ is in $R$.

Example: The set of rational numbers has an equivalence relation “=” defined by the requirement that an ordered pair $(\frac{a}{b}, \frac{c}{d})$ belongs in the relation if and only if $ad = bc$. The equivalence class of $\frac{1}{2}$ is the set {$\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \ldots$, $-\frac{1}{2}, -\frac{2}{4}, \ldots$}.

1.6.2 FUNCTIONS (MAPPINGS)

A relation $f$ on a set $X \times Y$ is a function (or mapping) from $X$ into $Y$ if $(x, y)$ and $(x, z)$ in the relation implies that $y = z$, and each $x \in X$ has a $y \in Y$ such that $(x, y)$ is in the relation. The last condition means that there is a unique pair in $f$ whose first element is $x$. We write $f(x) = y$ to mean that $(x, y)$ is in the relation $f$, and emphasize the idea of mapping by the notation $f : X \to Y$. The domain of a function $f$ is the set of all $x$ for which there is a pair $(x, y)$ in the relation. The range of a function $f$ is a set containing all the $y$ for which there is a pair $(x, y)$ in the relation. The image of a set $A$ in the domain of a function $f$ is the set of all $y$ in $Y$ such that $y = f(x)$ for some $x$ in $A$. The notation for the image of $A$ under $f$ is $f[A]$. The inverse image of a set $B$ in the range of a function $f$ is the set of all $x$ in $X$ such that $f(x) = y$ for some $y$ in $B$. The notation is $f^{-1}[B]$.

A function $f$ is one-to-one (or univalent, or injective) if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. A function $f : X \to Y$ is onto (or surjective) if for every $y$ in $Y$ there is some $x$ in $X$ such that $f(x) = y$. A function is bijective if it is both one-to-one and onto.

Examples:

- $f(x) = e^x$, as a mapping from $\mathbb{R}$ to $\mathbb{R}$, is one-to-one because $e^{x_1} = e^{x_2}$ implies $x_1 = x_2$ (by taking the natural logarithm). It is not onto because $-1$ is not the value of $e^x$ for any $x$ in $\mathbb{R}$.

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• $g(x) = x^3 - x$, as a mapping from $\mathbb{R}$ to $\mathbb{R}$, is onto because every real number is attained as a value of $g(x)$, for some $x$. It is not one-to-one because $g(-1) = g(0) = g(1)$.

• $h(x) = x^3$, as a mapping from $\mathbb{R}$ to $\mathbb{R}$, is bijective.

For an injective function $f$ mapping $X$ into $Y$, there is an inverse function $f^{-1}$ mapping the range of $f$ into $X$ which is defined by: $f^{-1}(y) = x$ if and only if $f(x) = y$. Example: The function $f(x) = e^x$ mapping $\mathbb{R}$ into $\mathbb{R}^+$ (the set of positive reals) is bijective. Its inverse is $f^{-1}(x) = \ln(x)$ which maps $\mathbb{R}^+$ into $\mathbb{R}$.

For functions $f : X \to Y$ and $g : Y \to Z$, with the range of $f$ contained in the domain of $g$, the composition $(g \circ f) : X \to Z$ is a function defined by $(g \circ f)(x) = g(f(x))$ for all $x$ in domain of $f$.

1. Note that $g \circ f$ may not be the same as $f \circ g$. For example, for $f(x) = x + 1$, and $g(x) = 2x$, we have $(g \circ f)(x) = g(f(x)) = 2f(x) = 2(x + 1) = 2x + 2$. However $(f \circ g)(x) = f(g(x)) = g(x) + 1 = 2x + 1$.

2. For every function $f$ and its inverse $f^{-1}$, we have $(f \circ f^{-1})(x) = x$, for all $x$ in the domain of $f^{-1}$, and $(f^{-1} \circ f)(x) = x$ for all $x$ in the domain of $f$. Note that the inverse function $f^{-1}$ is not the same as $\frac{1}{f}$ (unless $f(x) = x$).

### 1.6.3 SETS OF REAL NUMBERS

A sequence is the range of a function having the natural numbers as its domain. It can be denoted by $\{x_n \mid n$ is a natural number$\}$ or simply $\{x_n\}$. For a chosen natural number $N$, a finite sequence is the range of a function having natural numbers less than $N$ as its domain. Sets $A$ and $B$ are in a one-to-one correspondence if there is a bijective function from $A$ into $B$. Two sets $A$ and $B$ have the same cardinality if there is a one-to-one correspondence between them. A set which is equivalent to the set of natural numbers is denumerable (or countably infinite). A set which is empty or is equivalent to a finite sequence is finite (or finite countable).

Examples: The set of letters in the English alphabet is finite. The set of rational numbers is denumerable. The set of real numbers is uncountable.

**Axioms of order**

1. There is a subset $P$ (positive numbers) of $\mathbb{R}$ for which $x + y$ and $xy$ are in $P$ for every $x$ and $y$ in $P$.

2. Exactly one of the following conditions can be satisfied by a number $x$ in $\mathbb{R}$ (trichotomy): $x \in P$, $-x \in P$, or $x = 0$.

**Definitions**

A number $b$ is an upper (lower) bound of a subset $S$ in $\mathbb{R}$ if $x \leq b$ ($x \geq b$) for every $x$ in $S$. A number $c$ is a least upper bound (lub, supremum, or sup) of a subset $S$ in $\mathbb{R}$ if $c$ is an upper bound of $S$ and $b \geq c$ for every upper bound $b$ of $S$. A number $c$ is
a greatest lower bound (glb, infimum, or inf) if \( c \) is a lower bound of \( S \) and \( c \geq b \) for every lower bound \( b \) of \( S \).

**Completeness (or least upper bound) axiom**
If a nonempty set of real numbers has an upper bound, then it has a least upper bound.

**Characterization of the real numbers**
The real numbers are the smallest complete ordered field that contain the rationals. Alternatively, the properties of a field, the order properties, and the least upper bound axiom characterize the set of real numbers. The least upper bound axiom distinguishes the set of real numbers from other ordered fields.

**Archimedean property of \( \mathbb{R} \):** For every real number \( x \), there is an integer \( N \) such that \( x < N \). For every pair of real numbers \( x \) and \( y \) with \( x < y \), there is a rational number \( r \) such that \( x < r < y \). This is sometimes stated: The set of rational numbers is dense in \( \mathbb{R} \).

**Definitions**
The *extension* of \( \mathbb{R} \) by \( +\infty \) is accomplished by including the symbols \( \infty \) and \( -\infty \) with the following definitions (for all \( x \in \mathbb{R} \))

- \( -\infty < x < \infty \), for all \( x \) in \( \mathbb{R} \)
- \( x + \infty = \infty \), for all \( x \) in \( \mathbb{R} \)
- \( x - \infty = -\infty \), for all \( x \) in \( \mathbb{R} \)
- \( \frac{x}{\infty} = \frac{-x}{-\infty} = 0 \), for all \( x \) in \( \mathbb{R} \)
- \( x \cdot \infty = \infty \), if \( x > 0 \)
- \( x \cdot (-\infty) = -\infty \), if \( x > 0 \)
- \( \infty + \infty = \infty \)
- \( -\infty - \infty = -\infty \)
- \( \infty \cdot \infty = \infty \)
- \( -\infty \cdot (-\infty) = \infty \)

### 1.6.4 TOPOLOGY
A topology on a set \( X \) is a collection \( T \) of subsets of \( X \) (called open sets) having the following properties:

1. The empty set and \( X \) are in \( T \).
2. The union of elements in an arbitrary subcollection of \( T \) is in \( T \).
3. The intersection of elements in a finite subcollection of \( T \) is in \( T \).
The complement of an open set is a closed set. A set is compact if every open cover has a finite subcover.

Notes

1. A subset \( E \) of \( X \) is closed if and only if \( E \) contains all its limit points.
2. The union of finitely many closed sets is closed.
3. The intersection of an arbitrary collection of closed sets is closed.
4. The image of a compact set under a continuous function is compact.

1.6.5 METRIC SPACE

A norm on a vector space \( E \) with scalar field \( \mathbb{R} \) is a function \( || \cdot || \) from \( E \) into \( \mathbb{R} \) that satisfies the following conditions:

1. Positive definiteness: \( ||x|| \geq 0 \) for all \( x \) in \( E \), and \( ||x|| = 0 \) if and only if \( x = 0 \).
2. Scalar homogeneity: For every \( x \) in \( E \) and \( a \) in \( \mathbb{R} \), \( ||ax|| = |a|||x|| \).
3. Triangle inequality: \( ||x + y|| \leq ||x|| + ||y|| \) for all \( x, y \) in \( E \).

A metric (or distance function) on a set \( E \) is a function \( \rho : E \times E \to \mathbb{R} \) that satisfies the following conditions:

1. Positive definiteness: \( \rho(x, y) > 0 \) for all \( x, y \) in \( E \), and \( \rho(x, y) = 0 \) if and only if \( x = y \).
2. Symmetry: \( \rho(x, y) = \rho(y, x) \) for all \( x, y \) in \( E \).
3. Triangle inequality: \( \rho(x, y) \leq \rho(x, z) + \rho(z, y) \) for all \( x, y, z \) in \( E \).

Every norm \( || \cdot || \) gives rise to a metric \( \rho \) by defining: \( \rho(x, y) = ||x - y|| \). Examples:

1. \( \mathbb{R} \) with absolute value as norm has metric \( \rho(x, y) = |x - y| \).
2. \( \mathbb{R} \times \mathbb{R} \) (denoted \( \mathbb{R}^2 \)) with Euclidean norm \( ||(x, y)|| = \sqrt{x^2 + y^2} \), has metric \( \rho((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \).

A \( \delta \) neighborhood of a point \( x \) in a metric space \( E \) is the set of all \( y \) in \( E \) such that \( \rho(x, y) < \delta \). For example, a \( \delta \) neighborhood of \( x \) in \( \mathbb{R} \) is the interval centered at \( x \) with radius \( \delta \), \( (x - \delta, x + \delta) \). In a metric space the topology is generated by the \( \delta \) neighborhoods.

1. A subset \( G \) of \( \mathbb{R} \) is open if, for every \( x \) in \( G \), there is a \( \delta > 0 \) such that every \( y \) in \( \mathbb{R} \) with \( |x - y| < \delta \) is in \( G \) also. For example, intervals \((a, b), (a, \infty), (-\infty, b)\) are open in \( \mathbb{R} \).
2. A number \( x \) is a limit (or a point of closure, or an accumulation point) of a set \( F \) if, for every \( \delta > 0 \), there is a point \( y \) in \( F \) such that \( |x - y| < \delta \).
3. A subset $F$ of $\mathbb{R}$ is **closed** if it contains all of its limits. For example, intervals $[a,b]$, $(-\infty,b]$, and $[a,\infty)$ are closed in $\mathbb{R}$.

4. A subset $F$ is **dense** in $\mathbb{R}$ if every element of $\mathbb{R}$ is a limit point of $F$.

5. A metric space is **separable** if it contains a denumerable dense set. For example, $\mathbb{R}$ is separable because the subset of rationals is a denumerable dense set.

- *(Bolzano–Weierstrass theorem)* Any bounded infinite set of real numbers has a limit point in $\mathbb{R}$.
- *(Heine–Borel theorem)* A closed, bounded subset of $\mathbb{R}$ is compact.

### 1.6.6 CONVERGENCE IN $\mathbb{R}$

A number $L$ is a **limit point** of a sequence $\{x_n\}$ if, for every $\epsilon > 0$, there is a natural number $N$ such that $|x_n - L| < \epsilon$ for all $n > N$. If it exists, a limit point of a sequence is unique. A sequence is said to **converge** if it has a limit. A number $L$ is a **cluster point** of a sequence $\{x_n\}$ if, for every $\epsilon > 0$, there is an index $N$ such that $|x_n - L| < \epsilon$ for some $n > N$.

Example: The limit of a sequence is a cluster point, as in $\{\frac{1}{n}\}$, which converges to 0. However, cluster points are not necessarily limits, as in $\{(−1)^n\}$, which has cluster points $+1$ and $−1$ but no limit.

#### Limits

Let $\{x_n\}$ be a sequence. A number $L$ is the **limit superior** ($\limsup$) if, for every $\epsilon > 0$, there is a natural number $N$ such that $x_n > L - \epsilon$ for infinitely many $n \geq N$, and $x_n > L + \epsilon$ for only finitely many terms. An equivalent definition of the limit superior is given by

$$\limsup x_n = \inf_N \sup_{k \geq N} x_k.$$   

The **limit inferior** ($\liminf$) is defined in a similar way by

$$\liminf x_n = \sup_N \inf_{k \geq N} x_k.$$  

For example, the sequence $\{x_n\}$ with $x_n = 1 + (-1)^n + \frac{1}{n}$ has $\limsup x_n = 2$, and $\liminf x_n = 0$.

*(Theorem)* Every bounded sequence $\{x_n\}$ in $\mathbb{R}$ has a $\limsup$ and a $\liminf$. In addition, if $\lim x_n = \limsup x_n = \liminf x_n$, then the sequence converges to their common value.

A sequence $\{x_n\}$ is a **Cauchy sequence** if, for any $\epsilon > 0$, there exists a positive integer $N$ such that $|x_n - x_m| < \epsilon$ for every $n > N$ and $m > N$.

*(Theorem)* A sequence $\{x_n\}$ in $\mathbb{R}$ converges if and only if it is a Cauchy sequence. A metric space, in which every Cauchy sequence converges to a point in the space, is called **complete**. For example, $\mathbb{R}$ with the metric $\rho(x, y) = |x - y|$ is complete.

A number $L$ is a **limit** of a function $f$ as $x$ approaches a number $a$ if, for every $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all $x$ with $|x - a| < \delta$. This
is represented by the notation \( \lim_{x \to a} f(x) = L \). The symbol \( \infty \) is the limit of a function \( f \) as \( x \) approaches a number \( a \) if, for every positive number \( M \), there is a \( \delta > 0 \) such that \( f(x) > M \) for all \( x \) with \( |x-a| < \delta \). Notation is \( \lim_{x \to a} f(x) = \infty \). A number \( L \) is a limit of a function \( f \) as \( x \) approaches \( \infty \) if, for every \( \epsilon > 0 \), there is a positive number \( M \) such that \( |f(x) - L| < \epsilon \) for all \( x > M \); this is written \( \lim_{x \to \infty} f(x) = L \). The number \( L \) is said to be the limit at infinity.

For example, \( \lim_{x \to 2} 3x - 1 = 5 \), \( \lim_{x \to 0} \frac{1}{x^2} = \infty \), and \( \lim_{x \to \infty} \frac{1}{x} = 0 \).

### Pointwise and uniform convergence

A sequence of functions \( \{f_n(x)\} \) is said to converge pointwise to the function \( f(x) \) on a set \( E \) if for every \( \epsilon > 0 \) and \( x \in E \) there is a positive integer \( N \) such that \( |f(x) - f_n(x)| < \epsilon \) for every \( n \geq N \). A sequence of functions \( \{f_n(x)\} \) is said to converge uniformly to the function \( f \) on a set \( E \) if, for every \( \epsilon > 0 \), there exists a positive integer \( N \) such that \( |f(x) - f_n(x)| < \epsilon \) for all \( x \in E \) and \( n \geq N \).

Note that these formulations of convergence are not equivalent. For example, the functions \( f_n(x) = x^n \) on the interval \([0, 1]\) converge pointwise to the function \( f(x) = 0 \) for \( 0 \leq x < 1 \), \( f(1) = 1 \). They do not converge uniformly because, for \( \epsilon = 1/2 \), there is no \( N \) such that \( |f_n(x) - f(x)| < 1/2 \) for all \( x \) in \([0, 1]\) and every \( n \geq N \).

A function \( f \) is Lipschitz if there exists \( k > 0 \) in \( \mathbb{R} \) such that \( |f(x) - f(y)| \leq k|x - y| \) for all \( x \) and \( y \) in its domain. The function is a contraction if \( 0 < k < 1 \).

(Fixed point or contraction mapping theorem) If the function \( f : [a, b] \to [a, b] \) is a contraction, then there is a unique point \( x \) in \([a, b]\) such that \( f(x) = x \). The point \( x \) is called a fixed point of \( f \).

Example: Newton’s method for finding a zero of \( f(x) = (x + 1)^2 - 2 \) on the interval \([0, 1]\) produces \( x_{n+1} = g(x_n) \) with the contraction \( g(x) = \frac{x}{2} - \frac{1}{2} + \frac{1}{x+1} \). This has the unique fixed point \( \sqrt{2} - 1 \) in \([0, 1]\).

### 1.6.7 CONTINUITY IN \( \mathbb{R} \)

A function \( f : \mathbb{R} \to \mathbb{R} \) is continuous at a point \( a \) if \( f \) is defined at \( a \) and \( \lim_{x \to a} f(x) = f(a) \). The function \( f \) is continuous on a set \( E \) if it is continuous at every point of \( E \). A function \( f \) is uniformly continuous on a set \( E \) if, for every \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that \( |f(x) - f(y)| < \epsilon \) for every \( x \) and \( y \) in its domain with \( |x - y| < \delta \).

A sequence \( \{f_n(x)\} \) of continuous functions on the interval \([a, b]\) is equicontinuous if, for every \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that \( |f_n(x) - f_n(y)| < \epsilon \) for every \( n \) and for all \( x \) and \( y \) in \([a, b]\) with \( |x - y| < \delta \). Examples:

1. A function can be continuous without being uniformly continuous. The function \( g(x) = \frac{1}{x} \) is continuous but not uniformly continuous on the open interval \((0, 1)\).

2. A collection of continuous functions can be bounded on a closed interval without having a uniformly convergent sub-sequence. The continuous functions \( f_n(x) = \frac{x^n}{x^n + 1} \) are each bounded by 1 in the closed interval \([0, 1]\) and for every \( x \) there is the limit: \( \lim_{n \to \infty} f_n(x) = 0 \). However, \( f_n(\frac{1}{n}) = 1 \) for every \( n \), so
that no sub-sequence can converge uniformly to 0 everywhere on [0, 1]. This sequence is not equicontinuous.

*Theorem* Let \( \{ f_n(x) \} \) be a sequence of functions mapping \( \mathbb{R} \) into \( \mathbb{R} \) which converges uniformly to a function \( f \). If each \( f_n(x) \) is continuous at a point \( a \), then \( f(x) \) is also continuous at \( a \).

*Theorem* If a function \( f \) is continuous on a closed bounded set \( E \), then it is uniformly continuous on \( E \).

*Ascoli–Arzelà theorem* Let \( K \) be a compact set in \( \mathbb{R} \). If \( \{ f_n(x) \} \) is uniformly bounded and equicontinuous on \( K \), then \( \{ f_n(x) \} \) contains a uniformly convergent sub-sequence on \( K \).

### 1.6.8 CONVERGENCE IN \( L_p \)

In the context of elementary measure theory, two measurable functions \( f \) and \( g \) are *equivalent* if they are equal except on a set of measure zero. They are said to be equal *almost everywhere*. This is denoted by \( f = g \ a.e. \)

The (vector) space of measurable functions \( f \) on \( [a, b] \), for which \( \int_a^b |f(x)|^p \, dx < \infty \) with \( 0 < p < \infty \), is denoted by \( L_p[a, b] \) or simply \( L_p \). The space of bounded measurable functions on \( [a, b] \) is denoted by \( L_\infty \).

The \( L_p \) norm for \( 0 < p < \infty \) is defined by

\[
\|f\|_p = \left( \int_a^b |f(x)|^p \, dx \right)^{1/p}.
\]

The \( L_\infty \) norm is defined by

\[
\|f\|_\infty = \operatorname{ess\ sup}_{a \leq x \leq b} |f(x)|,
\]

where

\[
\operatorname{ess\ sup}_{a \leq x \leq b} |f(x)| = \inf \{ M \mid m \{ t : f(t) > M \} = 0 \}.
\]

Let \( \{ f_n(x) \} \) be a sequence of functions in \( L_p \) \((1 \leq p < \infty)\) and \( f \) be some function in \( L_p \). We say that \( \{ f_n \} \) converges in the mean of order \( p \) to \( f \) if \( \lim_{n \to \infty} \| f_n - f \|_p = 0 \).

*Riesz–Fischer theorem* The \( L_p \) spaces are complete.

### Inequalities

1. **Minkowski inequality**: If \( f \) and \( g \) are in \( L_p \) with \( 1 \leq p \leq \infty \), then \( \| f + g \|_p \leq \| f \|_p + \| g \|_p \). That is,

\[
\left( \int_a^b |f + g|^p \right)^{1/p} \leq \left( \int_a^b |f|^p \right)^{1/p} + \left( \int_a^b |g|^p \right)^{1/p} \quad \text{for } (1 \leq p < \infty),
\]

\[
\operatorname{ess\ sup} |f + g| \leq \operatorname{ess\ sup} |f| + \operatorname{ess\ sup} |g|.
\]
2. Hölder inequality: If $p$ and $q$ are nonnegative extended real numbers such that $1/p + 1/q = 1$ and $f \in L_p$ and $g \in L_q$, then $\|fg\|_1 \leq \|f\|_p \|g\|_q$. That is
\[
\int_a^b |fg| \leq \left(\int_a^b |f|^p \right)^{1/p} \left(\int_a^b |g|^q \right)^{1/q} \quad \text{for } (1 \leq p < \infty),
\]
(1.6.4)
\[
\int_a^b |fg| \leq (\text{ess sup } |f|) \int_a^b |g|.
\]
(1.6.5)

3. Schwartz (or Cauchy–Schwarz) inequality: If $f$ and $g$ are in $L_2$, then $\|fg\|_1 \leq \|f\|_2 \|g\|_2$. This is the special case of Hölder’s inequality with $p = q = 2$.

1.6.9 CONVERGENCE IN $L_2$

Two functions $f$ and $g$ in $L_2[a, b]$ are orthogonal if $\int_a^b fg = 0$. A set of $L_2$ functions $\{\phi_n\}$ are orthogonal if $\int_a^b \phi_m \phi_n = 0$ for $m \neq n$. The set is orthonormal if, in addition, each member has norm 1. That is, $\|\phi_n\|_2 = 1$. For example, the functions $\{\sin nx\}$ are mutually orthogonal on $(-\pi, \pi)$. The functions $\{\frac{\sin nx}{\sqrt{\pi}}\}$ form an orthonormal set on $(-\pi, \pi)$.

Let $\phi_n$ be an orthonormal set in $L_2$ and $f$ be in $L_2$. The numbers $c_n = \int_a^b f \phi_n dx$ are the generalized Fourier coefficients of $f$ with respect to $\{\phi_n\}$, and the series $\sum_{n=1}^{\infty} c_n \phi_n(x)$ is called the generalized Fourier series of $f$ with respect to $\{\phi_n\}$.

For a function $f$ in $L_2$, the mean square error of approximating $f$ by the sum $\sum_{n=1}^{N} a_n \phi_n$ is $\frac{1}{N} \int_a^b |f(x) - \sum_{n=1}^{N} a_n \phi_n(x)|^2 dx$. An orthonormal set $\{\phi_n\}$ is complete if the only measurable function $f$ that is orthogonal to every $\phi_n$ is zero. That is, $\int_a^b f \phi_n dx = 0$ a.e.

(Theorem) The generalized Fourier series of $f$ in $L_2$ converges in the mean (of order 2) to $f$.

(Theorem) Parseval’s identity holds:
\[
\int_a^b |f(x)|^2 dx = \sum_{n=1}^{\infty} c_n^2.
\]

Bessel’s inequality: For a function $f$ in $L_2$ having generalized Fourier coefficients $\{c_n\}$, $\sum_{n=1}^{\infty} c_n^2 \leq \int_a^b |f(x)|^2 dx$.

(Theorem) The mean square error of approximating $f$ by the series $\sum_{n=1}^{\infty} a_n \phi_n$ is minimum when all coefficients $a_n$ are the Fourier coefficients of $f$ with respect to $\{\phi_n\}$.

(Riesz–Fischer theorem) Let $\{\phi_n\}$ be an orthonormal set in $L_2$ and let $\{c_n\}$ be constants such that $\sum_{n=1}^{\infty} c_n^2$ converges. Then a unique function $f$ in $L_2$ exists such that the $c_n$ are the Fourier coefficients of $f$ with respect to $\{\phi_n\}$ and $\sum_{n=1}^{\infty} c_n \phi_n$ converges in the mean (or order 2) to $f$.

Example: suppose that the series $\sum_{n=1}^{\infty} (a_n x^2 + b_n^2)$ converges. Then the trigonometric series $\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ is the Fourier series of some function in $L_2$. 

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1.6.10 ASYMPTOTIC RELATIONSHIPS

Asymptotic relationships are indicated by the symbols $O$, $\Omega$, $\Theta$, $o$, and $\sim$.

1. The symbol $O$ (pronounced “big-oh”): $f(x) \in O(g(x))$ as $x \to x_0$ if a positive constant $C$ exists such that $|f(x)| \leq C |g(x)|$ for all $x$ sufficiently close to $x_0$. Note that $O(g(x))$ is a class of functions. Sometimes the statement $f(x) \in O(g(x))$ is written (imprecisely) as $f = O(g)$.

2. The symbol $\Omega$ (pronounced “omega”): $f(x) \in \Omega(g(x))$ as $x \to x_0$ if a positive constant $C$ exists such that $g(x) \leq C f(x)$ for all $x$ sufficiently close to $x_0$.

3. The symbol $\Theta$ (pronounced “theta”): $f(x) \in \Theta(g(x))$ as $x \to x_0$ if positive constants $c_1$ and $c_2$ exist such that $c_1 g(x) \leq f(x) \leq c_2 g(x)$ for all $x$ sufficiently close to $x_0$. This is equivalent to: $f(x) = O(g(x))$ and $g(x) = O(f(x))$. The symbol $\approx$ is often use for $\Theta$.

4. The symbol $o$ (pronounced “little-oh”): $f(x) \in o(g(x))$ as $x \to x_0$ if, given any $\mu > 0$, we have $|f(x)| < \mu |g(x)|$ for all $x$ sufficiently close to $x_0$.

5. The symbol $\sim$ (pronounced “asymptotic to”): $f(x) \sim g(x)$ as $x \to x_0$ if $f(x)/g(x) \sim 1$ as $x \to x_0$.

6. Two functions, $f(x)$ and $g(x)$, are asymptotically equivalent as $x \to x_0$ if $f(x)/g(x) \sim 1$ as $x \to x_0$.

7. A sequence of functions, $\{g_k(x)\}$, forms an asymptotic series at $x_0$ if $g_{k+1}(x) = o(g_k(x))$ as $x \to x_0$.

8. Given a function $f(x)$ and an asymptotic series $\{g_k(x)\}$ at $x_0$, the formal series $\sum_{k=0}^{\infty} a_k g_k(x)$ is an asymptotic expansion of $f(x)$ if $f(x) - \sum_{k=0}^{n} a_k g_k(x) = o(g_n(x))$ as $x \to x_0$ for every $n$; this is expressed as $f(x) \sim \sum_{k=0}^{\infty} a_k g_k(x)$. Partial sums of this formal series are called asymptotic approximations to $f(x)$. This formal series need not converge.

Think of $O$ being an upper bound on a function, $\Omega$ being a lower bound, and $\Theta$ being both an upper and lower bound. For example: $\sin x \in O(x)$ as $x \to 0$, $\log n \in o(n)$ as $n \to \infty$, and $n^9 \in \Omega(n^9 + n^2)$ as $n \to \infty$.

1.7 GENERALIZED FUNCTIONS

Dirac’s delta function is defined by $\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$, and is normalized so that $\int_{-\infty}^{\infty} \delta(x) \, dx = 1$. Properties include (assuming that $f(x)$ is continuous):

1. $\int_{-\infty}^{\infty} f(x) \delta(x - a) \, dx = f(a)$.
2. $\int_{-\infty}^{\infty} f(x) \frac{d^n \delta(x)}{dx^n} \, dx = (-1)^m \frac{d^n f(0)}{dx^n}$. 

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3. $x \delta(x)$, as a distribution, equals zero.
4. $\delta(ax) = \frac{1}{|a|} \delta(x)$ when $a \neq 0$.
5. $\delta(x^2 - a^2) = \frac{1}{2|a|} [\delta(x + a) + \delta(x - a)]$.
6. $\delta(x) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \cos \frac{n\pi x}{L}$ (Fourier series).
7. $\delta(x) = \frac{1}{L} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \sin \frac{n\pi x}{L}$ for $0 < \xi < L$ (Fourier sine series).
8. $\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \, dk$ (Fourier transform).
9. $\delta(\rho - \rho') = \rho \int_{0}^{\infty} k J_m(k\rho) J_m(k\rho') \, dk$.

Sequences of functions $\{\phi_n\}$ that approximate the delta function as $n \to \infty$ are known as delta sequences. For example:

- $\phi_n(x) = \frac{n}{\pi} \frac{1}{n^2 + x^2}$
- $\phi_n(x) = \frac{n}{\sqrt{n}} e^{-n^2 x^2}$
- $\phi_n(x) = \frac{1}{n\pi} \sin \frac{n\pi x}{x}$
- $\phi_n(x) = \begin{cases} 0 & |x| \geq 1/n \\ n/2 & |x| < 1/n \end{cases}$

The Heaviside function, or step function, is defined as

$$H(x) = \int_{-\infty}^{x} \delta(t) \, dt = \begin{cases} 0 & x < 0 \\ 1 & x > 1 \end{cases}.$$ Sometimes $H(0)$ is stated to be 1/2. This function has the representations:

1. $H(x) = \frac{1}{2} + \frac{1}{\pi} \sum_{n=odd}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{L}$
2. $H(x) = \frac{1}{\pi} \int_{-\infty}^{x} \frac{e^{ikx}}{k} \, dk$

The related signum function gives the sign of its argument:

$$\text{sgn}(x) = 2H(x) - 1 = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$ (1.7.1)

The delta function $\delta(x - x') = \delta(x_1 - x'_1) \delta(x_2 - x'_2) \delta(x_3 - x'_3)$ in terms of the coordinates $(\xi_1, \xi_2, \xi_3)$, related to $(x_1, x_2, x_3)$, via the Jacobian $J(x_i, \xi_j)$, is written

$$\delta(x - x') = \frac{1}{|J(x_i, \xi_j)|} \delta(\xi_1 - \xi'_1) \delta(\xi_2 - \xi'_2) \delta(\xi_3 - \xi'_3).$$ (1.7.2)

For example, in spherical polar coordinates

$$\delta(x - x') = \frac{1}{r^2} \delta(r - r') \delta(\phi - \phi') \delta(\cos \theta - \cos \theta').$$ (1.7.3)

The solutions to differential equations involving delta functions are called Green’s functions (see Sections 5.6.5 and 5.7.4).
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2.1 ELEMENTARY ALGEBRA

2.1.1 BASIC ALGEBRA

Algebraic equations

A polynomial equation in one variable is an equation of the form

\[ f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0. \]

The degree of the equation is \( n \) (where \( a_n \neq 0 \)).

A complex number \( z \) is a root of the polynomial \( f(x) \) if \( f(z) = 0 \). A complex number \( z \) is a root of multiplicity \( k \) if \( f(z) = f'(z) = f''(z) = \cdots = f^{(k-1)}(z) = 0 \), but \( f^{(k)}(z) \neq 0 \). A root of multiplicity 1 is called a simple root. A root of multiplicity 2 is called a double root, and a root of multiplicity 3 is called a triple root.

Fundamental theorem of algebra

A polynomial equation of degree \( n \) has exactly \( n \) complex roots, where a double root is counted twice, a triple root three times, and so on. If the \( n \) roots of the polynomial \( f(x) \) are \( z_1, z_2, \ldots, z_n \) (where a double root is listed twice, a triple root three times, and so on), then

\[ f(x) = a_n(x - z_1)(x - z_2) \cdots (x - z_n). \quad (2.1.1) \]

If the coefficients \( a_0, a_1, \ldots, a_n \) are real numbers, then the polynomial will always have an even number of complex roots occurring in pairs. That is, if \( z \) is a complex root, then so is \( \overline{z} \). If the polynomial has an odd degree and the coefficients are real, then it must have at least one real root.

The coefficients of the polynomial may be expressed as symmetric functions of the roots. For example, the elementary symmetric functions are

\[ s_1 = z_1 + z_2 + \cdots + z_n = \frac{a_{n-1}}{a_n}, \]

\[ s_2 = z_1z_2 + z_1z_3 + z_2z_3 + \cdots = \sum_{i>j} z_iz_j = \frac{a_{n-2}}{a_n}, \]

\[ \vdots \]

\[ s_n = z_1z_2z_3 \cdots z_n = (-1)^na_0 \quad (\text{2.1.2}) \]

where \( s_k \) is the sum of \( \binom{n}{k} \) products, each product combining \( k \) factors without repetition.

The discriminant of the polynomial is defined by \( \prod_{i>j}(z_i - z_j)^2 \), where the ordering of the roots is irrelevant. The discriminant can always be written as a polynomial combination of \( a_0, a_1, \ldots, a_n \), divided by \( a_n \).
Resultants

Let \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \) and \( g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0 \), where \( a_n \neq 0 \) and \( b_m \neq 0 \). The resultant of \( f \) and \( g \) is the determinant of the \((m + n) \times (m + n)\) matrix

\[
\begin{vmatrix}
  a_n & a_{n-1} & \cdots & a_0 & 0 & \cdots & 0 \\
  0 & a_n & a_{n-1} & \cdots & a_1 & a_0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  b_m & b_{m-1} & \cdots & b_0 & 0 & \cdots & 0 \\
  0 & b_m & b_{m-1} & \cdots & b_0 & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  0 & \cdots & 0 & b_m & b_1 & b_0
\end{vmatrix}
\quad (2.1.3)
\]

The resultant of \( f(x) \) and \( g(x) \) is 0 if and only if \( f(x) \) and \( g(x) \) have a common root. The resultant of \( f(x) \) and \( f'(x) \) is zero if and only if \( f(x) \) has a multiple root. For example:

1. If \( f(x) = x^2 + 2x + 3 \) and \( g(x; \alpha) = 4x^3 + 5x^2 + 6x + (7 + \alpha) \), then the resultant of \( f(x) \) and \( g(x; \alpha) \) is

\[
\det \begin{bmatrix}
  1 & 2 & 3 & 0 & 0 \\
  0 & 1 & 2 & 3 & 0 \\
  0 & 0 & 1 & 2 & 3 \\
  4 & 5 & 6 & 7 + \alpha & 0 \\
  0 & 4 & 5 & 6 & 7 + \alpha
\end{bmatrix} = (16 + \alpha)^2
\]

Note that \( g(x, -16) = (4x - 3)(x^2 + 2x + 3) = (4x - 3)f(x) \).

2. The resultant of \( ax + b \) and \( cx + d \) is \( da - bc \).

3. The resultant of \( (x + a)^5 \) and \( (x + b)^5 \) is \( (b - a)^{25} \).
Algebraic identities

\[(a \pm b)^2 = a^2 \pm 2ab + b^2.\]
\[(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3.\]
\[(a \pm b)^4 = a^4 \pm 4a^3b + 6a^2b^2 \pm 4ab^3 + b^4.\]
\[(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}\] where \(\binom{n}{k} = \frac{n!}{k!(n-k)!} \).
\[a^2 + b^2 = (a + bi)(a - bi).\]
\[a^4 + b^4 = (a^2 + \sqrt{2}ab + b^2)(a^2 - \sqrt{2}ab + b^2).\]
\[a^2 - b^2 = (a - b)(a + b).\]
\[a^3 + b^3 = (a - b)(a^2 + ab + b^2).\]
\[a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}).\]
\[(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.\]
\[(a + b + c)^3 = a^3 + b^3 + c^3 + 3(a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2) + 6abc.\]

Laws of exponents

Assuming all quantities are real, then

\[a^x a^y = a^{x+y}, \quad \frac{a^x}{a^y} = a^{x-y}, \quad (ab)^x = a^x b^x,\]
\[a^0 = 1 \text{ if } a \neq 0, \quad a^{-x} = \frac{1}{a^x}, \quad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x},\]
\[(a^x)^y = a^{xy}, \quad a^\frac{x}{y} = \sqrt[y]{a}, \quad \sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b},\]
\[\sqrt[n]{\sqrt[y]{a}} = \sqrt[n y]{a}, \quad a^z = \sqrt[n]{a^x} = (\sqrt[n]{a})^x, \quad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.\]

Proportion

If \(\frac{a}{b} = \frac{c}{d}\), then \(\frac{a}{c} = \frac{b}{d}\), \(ad = bc\), \(\frac{a+b}{b} = \frac{c+d}{d}\), \(\frac{a-b}{b} = \frac{c-d}{d}\), and \(\frac{a-b}{a+b} = \frac{c-d}{c+d}\).

If \(\frac{a}{b} = \frac{c}{d}\), where \(a, b, c,\) and \(d\) are all positive numbers and \(a\) is the largest of the four numbers, then \(a + d > b + c\).

2.1.2 PROGRESSIONS

Arithmetic progression

An arithmetic progression is a sequence of numbers such that the difference of any two consecutive numbers is constant. If the sequence is \(a_1, a_2, \ldots, a_n\), where \(a_{i+1} - a_i = d\), then \(a_k = a_1 + (k-1)d\) and

\[a_1 + a_2 + \cdots + a_n = \frac{n}{2}(2a_1 + (n-1)d).\]
In particular, the sequence 1, 2, ..., n is an arithmetic progression with sum \(n(n+1)/2\).

**Geometric progression**

A geometric progression is a sequence of numbers such that the ratio of any two consecutive numbers is constant. If the sequence is \(a_1, a_2, \ldots, a_n\), where \(a_{i+1}/a_i = r\), then \(a_k = a_1 r^{k-1}\).

\[
a_1 + a_2 + \cdots + a_n = \begin{cases} a_1 \frac{1-r^n}{1-r} & r \neq 1 \\ na_1 & r = 1. \end{cases} \tag{2.1.4}
\]

If \(|r| < 1\), then the infinite geometric series \(a_1(1 + r + r^2 + r^3 + \cdots)\) converges to \(\frac{a_1}{1-r}\). For example, \(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2\).

**Means**

The arithmetic mean of \(a\) and \(b\) is given by \(\frac{a+b}{2}\). More generally, the arithmetic mean of \(a_1, a_2, \ldots, a_n\) is given by \((a_1 + a_2 + \cdots + a_n)/n\).

The geometric mean of \(a\) and \(b\) is given by \(\sqrt{ab}\). More generally, the geometric mean of \(a_1, a_2, \ldots, a_n\) is given by \(\sqrt[n]{a_1 a_2 \cdots a_n}\). The geometric mean of \(n\) numbers is less than the arithmetic mean, unless all of the numbers are equal.

The harmonic mean of \(a\) and \(b\) is given by \(\frac{2ab}{a+b}\). If \(A\), \(G\), and \(H\) represent the arithmetic, geometric, and harmonic means of \(a\) and \(b\), then \(AH = G^2\).

**2.1.3 DEMOIVRE’S THEOREM**

A complex number \(a + bi\) can be written in the form \(re^{i\theta}\), where \(r^2 = a^2 + b^2\) and \(\tan \theta = b/a\). Because \(e^{i\theta} = \cos \theta + i \sin \theta\),

\[(a + bi)^n = r^n (\cos n\theta + i \sin n\theta),\]

and \(\sqrt[n]{1} = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}, \quad k = 0, 1, \ldots, n - 1\).

**2.1.4 PARTIAL FRACTIONS**

The technique of partial fractions allows a quotient of two polynomials to be written as a sum of simpler terms.

Given the fraction \(\frac{f(x)}{g(x)}\), where both \(f(x)\) and \(g(x)\) are polynomials, begin by dividing \(f(x)\) by \(g(x)\) to produce a quotient \(q(x)\) and a remainder \(r(x)\), where the degree of \(r(x)\) is less than the degree of \(g(x)\), so that \(\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}\). Therefore, assume that the rational function has the form \(\frac{r(x)}{g(x)}\), where the degree of the numerator is less than the degree of the denominator.

The techniques used depend on the factorization of \(g(x)\).
Single linear factor
Suppose that \( g(x) = (x - a)h(x) \), where \( h(a) \neq 0 \). Then
\[
\frac{r(x)}{g(x)} = \frac{A}{x-a} + \frac{s(x)}{h(x)},
\]
where the number \( A \) is given by \( r(a)/h(a) \). For example,
\[
\frac{2x}{x^2 - 1} = \frac{1}{x - 1} + \frac{1}{x + 1}.
\]

Repeated linear factor
Suppose that \( g(x) = (x - a)^k h(x) \), where \( h(a) \neq 0 \). Then
\[
\frac{r(x)}{g(x)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_k}{(x-a)^k} + \frac{s(x)}{h(x)},
\]
where
\[
A_k = \frac{r(a)}{h(a)},
\]
\[
A_{k-1} = \frac{d}{dx} \left( \frac{r(x)}{g(x)} \right) \bigg|_{x=a},
\]
\[
A_{k-2} = \frac{1}{2!} \frac{d^2}{dx^2} \left( \frac{r(x)}{g(x)} \right) \bigg|_{x=a},
\]
\[
A_{k-j} = \frac{1}{j!} \frac{d^j}{dx^j} \left( \frac{r(x)}{g(x)} \right) \bigg|_{x=a}.
\]

Single quadratic factor
Suppose that \( g(x) = (x^2 + bx + c)h(x) \), where \( b^2 - 4c < 0 \) (so that \( x^2 + bx + c \) does not factor into real linear factors) and \( h(x) \) is relatively prime to \( x^2 + bx + c \). Then
\[
\frac{r(x)}{g(x)} = \frac{Ax + B}{x^2 + bx + c} + \frac{s(x)}{h(x)}.
\]
In order to determine \( A \) and \( B \), multiply the equation by \( g(x) \) so that there are no denominators remaining, and substitute any two values for \( x \), yielding two equations for \( A \) and \( B \).

Repeated quadratic factor
Suppose that \( g(x) = (x^2 + bx + c)^k h(x) \), where \( b^2 - 4c < 0 \) (so that \( x^2 + bx + c \) does not factor into real linear factors) and \( h(x) \) is relatively prime to \( x^2 + bx + c \). Then
\[
\frac{r(x)}{g(x)} = \frac{A_1x + B_1}{x^2 + bx + c} + \frac{A_2x + B_2}{(x^2 + bx + c)^2} + \frac{A_3x + B_3}{(x^2 + bx + c)^3} + \cdots + \frac{A_kx + B_k}{(x^2 + bx + c)^k} + \frac{s(x)}{h(x)}.
\]
In order to determine $A_i$ and $B_i$, multiply the equation by $g(x)$ so that there are no denominators remaining, and substitute any $2k$ values for $x$, yielding $2k$ equations for $A_i$ and $B_i$.

## 2.2 POLYNOMIALS

### 2.2.1 QUADRATIC POLYNOMIALS

The solution of the equation $ax^2 + bx + c = 0$, where $a \neq 0$, is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$  \hspace{1cm} (2.2.1)

The discriminant of this equation is $b^2 - 4ac$. Suppose that $a$, $b$, and $c$ are all real. If the discriminant is negative, then the two roots are complex numbers which are conjugate. If the discriminant is positive, then the two roots are unequal real numbers. If the discriminant is 0, then the two roots are equal.

### 2.2.2 CUBIC POLYNOMIALS

To solve the equation $ax^3 + bx^2 + cx + d = 0$, where $a \neq 0$, begin by making the substitution $y = x + \frac{b}{3a}$. That gives the equation $y^3 + 3py + q = 0$, where $p = \frac{3ac - b^2}{9a^2}$ and $q = \frac{2b^3 - 9abc + 27a^2d}{27a^3}$. The discriminant of this polynomial is $4p^3 + q^2$.

The solutions are given by $\sqrt[3]{\alpha} - \sqrt[3]{\beta}$, $\sqrt[3]{\alpha} e^{\frac{2\pi i}{3}} - \sqrt[3]{\beta}$, and $\sqrt[3]{\alpha} e^{\frac{4\pi i}{3}} - \sqrt[3]{\beta}$, where

$$\alpha = \frac{-q + \sqrt{q^2 + 4p^3}}{2} \quad \text{and} \quad \beta = \frac{-q - \sqrt{q^2 + 4p^3}}{2}.$$  \hspace{1cm} (2.2.2.2)

Suppose that $p$ and $q$ are real numbers. If the discriminant is positive, then one root is real, and two are complex conjugates. If the discriminant is 0, then there are three real roots, of which at least two are equal. If the discriminant is negative, then there are three unequal real roots.

### Trigonometric solution of cubic polynomials

In the event that the roots of the polynomial $y^3 + 3py + q = 0$ are all real, meaning that $q^2 + 4p^3 \leq 0$, then the expressions above involve complex numbers. In that case one can also express the solution in terms of trigonometric functions. Define $r$ and $\theta$ by

$$r = \sqrt[3]{-p^3} \quad \text{and} \quad \theta = \cos^{-1} \frac{-q}{2r}.$$  \hspace{1cm} (2.2.2.3)
Then the three roots are given by

\[ 2\sqrt{r} \cos \frac{\theta}{3}, \quad 2\sqrt{r} \cos \frac{\theta + 2\pi}{3}, \quad \text{and} \quad 2\sqrt{r} \cos \frac{\theta + 4\pi}{3} \]

### 2.2.3 QUARTIC POLYNOMIALS

To solve the equation \( ax^4 + bx^3 + cx^2 + dx + e = 0 \), where \( a \neq 0 \), start with the substitution \( y = x + \frac{b}{4a} \). That gives \( y^4 + py^2 + qy + r = 0 \), where \( p = \frac{8ac - 3b^2}{8a^2} \), \( q = \frac{b^2 - 4ac + 3b^2}{8a^4} \), and \( r = \frac{16abc - 3b^3 - 125ac^2}{256a^4} \).

The cubic resolvent of this polynomial is defined as \( t^3 - pt^2 - 4rt + (4pr - q^2) = 0 \). If \( u \) is a root of the cubic resolvent, then the solutions of the original quartic are given by the solutions of

\[ y^2 \pm \sqrt{u - p} \left( y - \frac{q}{2(u - p)} \right) + \frac{u}{2} = 0. \]

### 2.2.4 QUINTIC POLYNOMIALS

Some quintic equations are solvable by radicals. If the function \( f(x) = x^5 + ax + b \) (with \( a \) and \( b \) rational) is irreducible, then \( f(x) = 0 \) is solvable by radicals if, and only if, numbers \( \epsilon, c, \) and \( e \) exist (with \( \epsilon = \pm 1, c \geq 0, \) and \( e \neq 0 \)) such that

\[ a = \frac{5e^4(3 - 4\epsilon c)}{c^2 + 1} \quad \text{and} \quad b = \frac{-4e^5(11\epsilon + 2c)}{c^2 + 1}. \]

In this case, the roots are given by \( x = e (\omega^j u_1 + \omega^{j+1} u_2 + \omega^{j+2} u_3 + \omega^{j+3} u_4) \) for \( j = 0, 1, 2, 3, 4 \), where \( \omega \) is a fifth root of unity (\( \omega = \exp(2\pi i/5) \)) and

\[
\begin{align*}
    u_1 &= \left( \frac{v_1^2 v_3}{D^2} \right), \\
    u_2 &= \left( \frac{v_2^2 v_4}{D^2} \right), \\
    u_3 &= \left( \frac{v_3^2 v_1}{D^2} \right), \\
    u_4 &= \left( \frac{v_4^2 v_2}{D^2} \right),
\end{align*}
\]

\[
\begin{align*}
    v_1 &= \sqrt{D} + \sqrt{D - \epsilon \sqrt{D}}, \\
    v_2 &= -\sqrt{D} - \sqrt{D + \epsilon \sqrt{D}}, \\
    v_3 &= -\sqrt{D} + \sqrt{D - \epsilon \sqrt{D}}, \\
    v_4 &= \sqrt{D} - \sqrt{D - \epsilon \sqrt{D}}, \quad \text{and} \quad D = c^2 + 1.
\end{align*}
\]

**Example 2.2.1**

The quintic \( f(x) = x^5 + 15x + 12 \) has the values \( \epsilon = -1, \ c = 4/3, \) and \( e = 1. \) Hence the unique real root is given by

\[
\begin{align*}
    x &= \left( \frac{-75 + 21\sqrt{10}}{125} \right)^{1/5} + \left( \frac{-75 - 21\sqrt{10}}{125} \right)^{1/5} \\
    &\quad + \left( \frac{225 + 72\sqrt{10}}{125} \right)^{1/5} + \left( \frac{225 - 72\sqrt{10}}{125} \right)^{1/5}.
\end{align*}
\]

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2.2.5 TSCHIRNHAUS' TRANSFORMATION

The \( n \)th degree polynomial equation

\[
a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0
\]

can be transformed to one with up to three fewer terms,

\[
z^n + b_{n-4} z^{n-4} + \cdots + b_1 z + b_0 = 0
\]

by making a transformation of the form

\[
z_j = \gamma_4 x^4_j + \gamma_3 x^3_j + \gamma_2 x^2_j + \gamma_1 x_j + \gamma_0
\]

for \( j = 1, \ldots, n \) where the \( \{\gamma_i\} \) can be computed, in terms of radicals, from the \( \{a_i\} \). Hence, the general quintic polynomial can be transformed to the form \( z^5 + az + b = 0 \).

2.2.6 POLYNOMIAL NORMS

The polynomial \( P(x) = \sum_{j=0}^{n} a_j x^j \) has the norms:

\[
\|P\|_1 = \int_0^{2\pi} |P(e^{i\theta})| \frac{d\theta}{2\pi} \quad |P|_1 = \sum_{j=0}^{n} |a_j| . \tag{2.2.2}
\]

\[
\|P\|_2 = \left( \int_0^{2\pi} |P(e^{i\theta})|^2 \frac{d\theta}{2\pi} \right)^{1/2} \quad |P|_2 = \left( \sum_{j=1}^{n} |a_j|^2 \right)^{1/2} . \tag{2.2.3}
\]

\[
\|P\|_{\infty} = \max_{|z|=1} |P(z)| \quad |P|_{\infty} = \max_{j} |a_j| . \tag{2.2.4}
\]

For the double bar norms, \( P \) is considered as a function on the unit circle; for the single bar norms, \( P \) is identified with its coefficients. These norms are comparable:

\[
|P|_{\infty} \leq \|P\|_1 \leq \|P\|_2 = \|P\|_2 \leq \|P\|_{\infty} \leq |P|_1 \leq n|P|_{\infty} . \tag{2.2.5}
\]

2.2.7 GALOIS GROUP OF A POLYNOMIAL

Consider the polynomial \( P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = a_n (x - z_1)(x - z_2) \cdots (x - z_n) \).

There are certain relations among the roots that do not depend on how the roots are numbered. For example, this is true of the elementary symmetric functions; see Equation (2.1.2). The set of permutations that leave all polynomials with rational coefficients, \( H(z_1, \ldots, z_n) \), invariant is the Galois group of the equation.
For example, consider the polynomial \( f(x) = (x^2 - 2)(x^2 - 3) \), with roots \( \{z_1 = -\sqrt{2}, z_2 = \sqrt{2}, z_3 = -\sqrt{3}, z_4 = \sqrt{3}\} \). There are four permutations that leave all allowable relationships between the roots invariant:

\[
\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad \quad p_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} \\
p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad \quad p_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}
\]

That is, all permutations that switch \( z_1 \) and \( z_2 \), or switch \( z_3 \) and \( z_4 \), or both, will leave allowable relations invariant. To demonstrate that neither \( z_1 \) nor \( z_2 \) can be switched with \( z_3 \) or \( z_4 \), consider the expressions \( H_1(z_i) = z_1z_2 + 2 \) and \( H_2(z_i) = z_3z_4 + 3 \).

### 2.2.8 OTHER POLYNOMIAL PROPERTIES

1. For polynomial \( P(x) = \sum_{j=0}^{n} a_jx^j \), with \( a_0 \neq 0 \), Jensen’s inequality is

\[
\int_0^{2\pi} \log \left| P(e^{i\theta}) \right| \frac{d\theta}{2\pi} \geq \log |a_0|
\]

2. The polynomial \( P(x_1, \ldots, x_n) = \sum_{|\alpha|=m} a_{\alpha} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \), where \( \alpha = (\alpha_1, \ldots, \alpha_N) \)

   can be written in the symmetric form

   \[
P(x_1, \ldots, x_n) = \sum_{i_1, \ldots, i_m=1}^{N} c_{i_1, \ldots, i_m} x_{i_1} x_{i_2} \cdots x_{i_m}
\]

   with \( c_{i_1, \ldots, i_m} = \frac{a_{\alpha}}{\alpha_1! \cdots \alpha_m!} \). This means that the \( x_1x_2 \) term is written as \( \frac{1}{2}(x_1x_2 + x_2x_1) \), the term \( x_1^2 x_2^2 \) becomes \( \frac{1}{4}(x_1x_2x_1^2 + x_2x_1x_2 + x_2x_2x_1) \).

3. A valuation of the polynomial \( P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = a_n(x - z_1)(x - z_2) \cdots (x - z_m) \) is given by \( M(P) = a_n \prod_{i=1}^{m} \max(1, |z_i|) \). This valuation satisfies the properties:

   - \( M(P) M(Q) = M(PQ) \)
   - \( M(P(x^k)) = M(P(x^k)) \) for \( k \geq 1 \)
   - \( M(x^n P(x^{-1})) = M(P(x)) \)
2.3 NUMBER THEORY

2.3.1 CONGRUENCES

Definitions

1. If the integers \(a\) and \(b\) leave the same remainder when divided by the number \(n\), then \(a\) and \(b\) are congruent modulo \(n\). This is written \(a \equiv b \pmod{n}\).

2. If the congruence \(x^2 \equiv a \pmod{p}\) has a solution, then \(a\) is a quadratic residue of \(p\). Otherwise, \(a\) is a quadratic nonresidue of \(p\).

- Let \(p\) be a prime. Legendre’s symbol \(\left(\frac{a}{p}\right)\) has the value +1 if \(a\) is a quadratic residue of \(p\), and the value −1 if \(a\) is a quadratic nonresidue of \(p\).

- The Jacobi symbol generalizes the Legendre symbol to non-prime moduli. If \(n = \prod_{i=1}^{k} p_i^{b_i}\) then the Jacobi symbol can be written in terms of the Legendre symbol

\[
\left(\frac{a}{n}\right) = \prod_{i=1}^{k} \left(\frac{a}{p_i}\right)^{b_i}
\]  

(2.3.1)

3. Carmichael numbers are composite numbers \(\{n\}\) that satisfy \(a^{n-1} \equiv 1 \pmod{n}\) for every \(a\) \((1 < a < n)\) relatively prime to \(n\).

Properties

1. If \(a \equiv b \pmod{n}\), then \(b \equiv a \pmod{n}\).

2. If \(a \equiv b \pmod{n}\), and \(b \equiv c \pmod{n}\), then \(a \equiv c \pmod{n}\).

3. If \(a \equiv a' \pmod{n}\), and \(b \equiv b' \pmod{n}\), then \(a \pm b \equiv a' \pm b' \pmod{n}\).

4. If \(a \equiv a' \pmod{n}\), then \(a^2 \equiv (a')^2 \pmod{n}\), \(a^3 \equiv (a')^3 \pmod{n}\), etc.

5. If \((k, m) = d\), then the congruence \(kx \equiv n \pmod{m}\) is solvable if and only if \(d\) divides \(n\). It then has \(d\) solutions.

6. If \(p\) is a prime, then \(a^p \equiv a \pmod{p}\).

7. If \(p\) is a prime, and \(p\) does not divide \(a\), then \(a^{p-1} \equiv 1 \pmod{p}\).

8. If \((a, m) = 1\) then \(a^{\phi(m)} \equiv 1 \pmod{m}\). (See Section 2.3.16 for \(\phi(m)\).)

9. If \(p\) is an odd prime and \(a\) is not a multiple of \(p\), then \((p - 1)! \equiv -\left(\frac{a}{p}\right) a^{(p-1)/2} \pmod{p}\).

10. If \(p\) and \(q\) are odd primes, then the law of quadratic reciprocity states that

\[
\left(\frac{p}{q}\right) \left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4}.
\]
11. The number $-1$ is a quadratic residue of primes of the form $4k + 1$ and a nonresidue of primes of the form $4k + 3$.

12. The number 2 is a quadratic residue of primes of the form $8k \pm 1$ and a nonresidue of primes of the form $8k \pm 3$.

13. The number $-3$ is a quadratic residue of primes of the form $6k + 1$ and a nonresidue of primes of the form $6k + 5$.

14. The number 3 is a quadratic residue of primes of the form $12k \pm 1$ and a nonresidue of primes of the form $12k \pm 5$.

Values
There are infinitely many Carmichael numbers. There are 32 Carmichael numbers less than one million; they are 561, 1105, 1729, 2465, 2821, 6601, 8911, 10585, 15841, 29341, 41041, 46657, 63973, 75361, 101101, 115921, 126217, 162401, 172081, 188461, 252601, 278545, 294409, 314821, 334153, 340561, 399001, 410041, 449065, and 488881.

2.3.2 CHINESE REMAINDER THEOREM

Let $m_1, m_2, \ldots, m_r$ be pairwise relatively prime integers. Then the system of congruences

\begin{align*}
&x \equiv a_1 \pmod{m_1} \\
&x \equiv a_2 \pmod{m_2} \\
&\quad \vdots \\
&x \equiv a_r \pmod{m_r}
\end{align*}

has a unique solution modulo $M = m_1 m_2 \cdots m_r$. This unique solution can be written as

$$x = a_1 M_1 y_1 + a_2 M_2 y_2 + \cdots + a_r M_r y_r$$

(2.3.2)

where $M_k = M/m_k$, and $y_k$ is the inverse of $M_k$ (modulo $m_k$).

Example: For the system of congruences

\begin{align*}
&x \equiv 1 \pmod{3} \\
&x \equiv 2 \pmod{5} \\
&x \equiv 3 \pmod{7}
\end{align*}

we have $M = 3 \cdot 5 \cdot 7 = 105$. Hence $M_1 = 35$, $M_2 = 21$, and $M_3 = 15$. The equation for $y_1$ is $M_1 y_1 = 35 y_1 \equiv 1 \pmod{3}$ with solution $y_1 \equiv 2 \pmod{3}$. Likewise, $y_2 \equiv 1 \pmod{5}$ and $y_3 \equiv 1 \pmod{7}$. This results in $x = 1 \cdot 35 \cdot 2 + 2 \cdot 21 \cdot 1 + 3 \cdot 15 \cdot 1 \equiv 52 \pmod{105}$.

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2.3.3 CONTINUED FRACTIONS

The symbol \([a_0, a_1, \ldots, a_N]\) (with \(N\) finite or infinite and the \(\{a_i\}\) being positive integers) represents the simple continued fraction,

\[
[a_0, a_1, \ldots, a_N] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots + \frac{1}{a_N}}}}
\]  

(2.3.3)

The \(n^{th}\) convergent (with \(0 < n < N\)) of \([a_0, a_1, \ldots, a_N]\) is defined to be \([a_0, a_1, \ldots, a_n]\).

If \(\{p_n\}\) and \(\{q_n\}\) are defined by

\[
p_0 = a_0, \quad p_1 = a_1a_0 + 1, \quad p_n = a_n p_{n-1} + p_{n-2} \quad (2 \leq n \leq N)
\]

\[
q_0 = 1, \quad q_1 = a_1, \quad q_n = a_n q_{n-1} + q_{n-2} \quad (2 \leq n \leq N)
\]

then \([a_0, a_1, \ldots, a_n] = p_n/q_n\). The continued fraction is convergent if and only if the infinite series \(\sum a_i\) is divergent.

If the positive rational number \(x\) can be represented by a simple continued fraction with an odd (even) number of terms, then it is also representable by one with an even (odd) number of terms. (Specifically, if \(a_n = 1\) then \([a_0, a_1, \ldots, a_{n-1}, 1] = [a_0, a_1, \ldots, a_{n-1} + 1]\), and if \(a_n \geq 2\), then \([a_0, a_1, \ldots, a_n] = [a_0, a_1, \ldots, a_n - 1, 1]\).) Aside from this indeterminacy, the simple continued fraction of \(x\) is unique. The error in approximating by a convergent is bounded by

\[
\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}.
\]  

(2.3.4)

The algorithm for finding a continued fraction expansion of a number is to remove the integer part of the number (this becomes \(a_i\)), take the reciprocal, and repeat. For the number \(\pi\):

\[
\beta_0 = \pi \approx 3.14159 \quad a_0 = \lfloor \beta_0 \rfloor = 3
\]

\[
\beta_1 = 1/(\beta_0 - a_0) \approx 7.062 \quad a_1 = \lfloor \beta_1 \rfloor = 7
\]

\[
\beta_2 = 1/(\beta_1 - a_1) \approx 15.997 \quad a_2 = \lfloor \beta_2 \rfloor = 15
\]

\[
\beta_3 = 1/(\beta_2 - a_2) \approx 1.0034 \quad a_3 = \lfloor \beta_3 \rfloor = 1
\]

\[
\beta_4 = 1/(\beta_3 - a_3) \approx 292.6 \quad a_4 = \lfloor \beta_4 \rfloor = 292
\]

Approximations to \(\pi\) and \(e\) may be found from \(\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, \ldots]\) and \(e = [2, 1, 2, 1, 1, 4, 1, 1, 6, \ldots, 1, 1, 2n, \ldots]\). The convergents for \(\pi\) are \(\frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \ldots\). The convergents for \(e\) are \(\frac{8}{3}, \frac{11}{4}, \frac{19}{7}, \frac{87}{32}, \ldots\).
A periodic continued fraction is an infinite continued fraction in which \(a_l = a_{l+k}\) for all \(l \geq L\). The set of partial quotients \(a_L, a_{L+1}, \ldots, a_{L+k-1}\) is the period. A periodic continued fraction may be written as

\[
[a_0, a_1, \ldots, a_{L-1}, a_L, a_{L+1}, \ldots, a_{L+k-1}].
\]

(2.3.5)

For example, \(\sqrt{2} = [1, \dot{2}]\), \(\sqrt{3} = [1, 1, \dot{2}]\), \(\sqrt{5} = [2, 4]\), and \(\sqrt{7} = [2, 1, 1, 4]\). If \(x = [b, \dot{a}]\) then \(x = \frac{1}{2}(b + \sqrt{b^2 + 4a})\). For example, \([1] = (1+\sqrt{5})/2\), \([2] = 1+\sqrt{2}\), and \([2, 1] = 1+\sqrt{3}\).

Functions can be represented as continued fractions. Using the notation

\[
b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \frac{a_4}{b_4 + \ldots}}}} = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \frac{a_4}{b_4 + \ldots}}}}
\]

(2.3.6)

then (allowable values of \(z\) may be restricted in the following)

- \(\ln(1+z) = \frac{z}{1} + \frac{\frac{2z}{3}}{1 + \frac{\frac{2z}{3}}{1 + \frac{\frac{2z}{3}}{1 + \frac{\frac{2z}{3}}{1 + \ldots}}}}\)
- \(e^z = \frac{z}{1} + \frac{\frac{2z}{3}}{1 + \frac{\frac{2z}{3}}{1 + \frac{\frac{2z}{3}}{1 + \frac{\frac{2z}{3}}{1 + \ldots}}}}\)
- \(\tan z = \frac{z}{1} + \frac{\frac{2z}{3}}{1 + \frac{\frac{2z}{3}}{1 + \frac{\frac{2z}{3}}{1 + \frac{\frac{2z}{3}}{1 + \ldots}}}}\)
- \(\tanh z = \frac{z}{1} + \frac{\frac{2z}{3}}{1 + \frac{\frac{2z}{3}}{1 + \frac{\frac{2z}{3}}{1 + \frac{\frac{2z}{3}}{1 + \ldots}}}}\)

### 2.3.4 DIOPHANTINE EQUATIONS

A diophantine equation is one which requires the solutions to come from the set of integers.

Apart from the trivial solutions (with \(x = y = 0\) or \(x = u\)), the general solution to the equation \(x^3 + y^3 = u^3 + v^3\) is given by

\[
\begin{align*}
x &= \lambda \left[1 - (a - 3b)(a^2 + 3b^2)\right] \\
y &= \lambda \left[(a + 3b)(a^2 + 3b^2) - 1\right] \\
u &= \lambda \left[(a + 3b) - (a^2 + 3b^2)^2\right] \\
v &= \lambda \left[(a^2 + 3b^2)^2 - (a - 3b)\right]
\end{align*}
\]

(2.3.7)

where \(\{\lambda, a, b\}\) are any rational numbers except that \(\lambda \neq 0\).

A parametric solution to \(x^4 + y^4 = u^4 + v^4\) is given by

\[
\begin{align*}
x &= a^4b - 3a^2b^3 - 2a^4b^4 + 3a^2b^5 + ab^6 \\
y &= a^2b - 3a^2b^2 - 2a^4b^3 + a^2b^5 + b^7 \\
u &= a^2b - 3a^2b^2 - 2a^4b^3 + 3a^2b^5 + ab^6 \\
v &= a^2b + 3a^4b^2 - 2a^4b^3 + 2a^2b^5 + b^7
\end{align*}
\]

(2.3.8)

Fermat’s last theorem states that there are no integer solutions to \(x^n + y^n = z^n\), when \(n > 2\). This was proven by Andrew Wiles in 1995.
Pell’s equation

Pell’s equation is \(x^2 - My^2 = 1\). The solutions, integral values of \((x, y)\), arise from continued fraction convergents of \(\sqrt{M}\) (see page 88).

The number \(\sqrt{2}\) has the continued fraction expansion \([1, 2, 2, 2, 2, \ldots]\). Hence, the first few convergents are \(\frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \ldots\). In this case, every second convergent represents a solution:

\[
3^2 - 2 \cdot 2^2 = 1,
\]
\[
17^2 - 2 \cdot 12^2 = 1, \text{ and}
\]
\[
99^2 - 2 \cdot 70^2 = 1.
\]

Pythagorean triples

If the positive integers \(A, B,\) and \(C\) satisfy the relationship \(A^2 + B^2 = C^2\), then the triplet \((A, B, C)\) is a Pythagorean triple. It is possible to construct a right triangle with sides of length \(A\) and \(B\) and a hypotenuse of \(C\).

There are infinitely many Pythagorean triples. The most general solution to \(A^2 + B^2 = C^2\), with \((A, B) = 1\) and \(A\) even, is given by

\[
A = 2xy \quad B = x^2 - y^2 \quad C = x^2 + y^2,
\]

(2.3.9)

where \(x\) and \(y\) are relatively prime integers of opposite parity with \(x > y > 0\). The following shows some Pythagorean triples with the associated \((x, y)\) values.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
<th>(A)</th>
<th>(B)</th>
<th>(C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>4</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>8</td>
<td>15</td>
<td>17</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>12</td>
<td>35</td>
<td>37</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>16</td>
<td>63</td>
<td>65</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>20</td>
<td>99</td>
<td>101</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>12</td>
<td>5</td>
<td>13</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>20</td>
<td>21</td>
<td>29</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td>28</td>
<td>45</td>
<td>51</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>24</td>
<td>7</td>
<td>25</td>
</tr>
</tbody>
</table>

Waring’s problem

If each positive integer can be expressed as a sum of \(n\) \(k\)th powers, then there is a least value of \(n\) for which this is true: this is the number \(g(k)\). For all sufficiently large numbers, however, a smaller value of \(n\) may suffice: this is the number \(G(k)\). Lagrange’s theorem states: “Every positive integer is the sum of four squares;” this is equivalent to the statement \(g(2) = 4\). The following identity shows how a product can be written as the sum of four squares:

\[
(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) =
(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)^2 + (x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3)^2
+ (x_1y_3 - x_3y_1 + x_4y_2 - x_2y_4)^2 + (x_1y_4 - x_4y_1 + x_2y_3 - x_3y_2)^2
\]

(2.3.10)
Consider \( k = 3 \); all numbers can be written as the sum of not more than 9 cubes, so that \( g(3) = 9 \). However, only the two numbers

\[
23 = 2^3 + 2^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3,
\]
\[
239 = 4^3 + 4^3 + 3^3 + 3^3 + 3^3 + 1^3 + 1^3 + 1^3,
\]

require the use of 9 cubes; so \( G(3) \leq 8 \).

The current known values of \( g(k) \) include: \( g(3) = 9 \), \( g(4) \geq 19 \), \( g(5) = 37 \) and

\[
g(k) = \left\lfloor \left( \frac{3}{2} \right)^k \right\rfloor + 2^k - 2 \text{ for } 6 \leq k \leq 471600000.
\]

The value of \( G(k) \) is only known for two values of \( k \), \( G(2) = 4 \) and \( G(4) = 16 \). It is known that: \( 4 \leq G(3) \leq 7 \), \( G(5) \leq 23 \), \( G(6) \leq 36 \), \( G(7) \leq 137 \), and \( G(8) \leq 163 \). It is also known that

\[
G(k) \leq 6k \log k + \left( 4 + 3 \log \left( 3 + \frac{2}{k} \right) \right) k + 3. \tag{2.3.11}
\]

### 2.3.5 GREATEST COMMON DIVISOR

The greatest common divisor (GCD) of the integers \( n \) and \( m \) is the largest integer that evenly divides both \( n \) and \( m \). The Euclidean algorithm is frequently used for computing the GCD of two numbers; it utilizes the fact that \( a = \left\lfloor \frac{a}{b} \right\rfloor b + c \) where \( 0 \leq c < b \). For example, consider 78 and 21. Since \( 78 = 3 \cdot 21 + 15 \), the largest integer that evenly divides both 78 and 21 is also the largest integer that evenly divides both 21 and 15. Iterating results in

\[
78 = 3 \cdot 21 + 15 \]
\[
21 = 1 \cdot 15 + 6 \]
\[
15 = 2 \cdot 6 + 3 \]
\[
6 = 2 \cdot 3 + 0
\]

Hence \( \text{GCD}(78,21) = \text{GCD}(6,3) = 3 \). A common notation for \( \text{GCD}(n, m) \) is \( (n, m) \).

Two numbers, \( a \) and \( b \), are said to be relatively prime if they have no divisors in common; i.e., if \( \text{GCD}(a, b) = 1 \). The probability that two integers chosen randomly are relatively prime is \( \pi/6 \).

### 2.3.6 LEAST COMMON MULTIPLE

The least common multiple of the integers \( a \) and \( b \) (denoted LCM(\( a, b \))) is the smallest integer \( r \) that is divisible by both \( a \) and \( b \). The simplest way to find the LCM of \( a \) and \( b \) is via the formula \( \text{LCM}(a, b) = ab/\text{GCD}(a, b) \). For example, \( \text{LCM}(10, 4) = \frac{10 \cdot 4}{\text{GCD}(10, 4)} = \frac{40}{2} = 20 \).
2.3.7 FAREY SEQUENCES

The Farey series of order \( n \), \( \mathcal{F}_n \), is the ascending series of irreducible fractions between zero and one whose denominators do not exceed \( n \). The fraction \( \frac{k}{m} \) belongs to \( \mathcal{F}_n \) if \( 0 \leq k \leq m \leq n \) and \( k \) and \( m \) are relatively prime. If \( \frac{k}{m} \) and \( \frac{k'}{m'} \) are two successive terms of \( \mathcal{F}_n \) then \( mk' - km' = 1 \). \( \mathcal{F}_{n-1} \) can be obtained from \( \mathcal{F}_n \) by deleting terms with denominators of \( n \).

\[
\mathcal{F}_3 = \left( \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{1}{1} \right), \\
\mathcal{F}_5 = \left( \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1} \right).
\]

2.3.8 MÖBIUS FUNCTION

The Möbius function is defined by

- \( \mu(1) = 1 \)
- \( \mu(n) = 0 \) if \( n \) has a squared factor
- \( \mu(p_1p_2\ldots p_k) = (-1)^k \) if all the primes \( \{p_1, \ldots, p_k\} \) are distinct

The Möbius inversion formula states that, if \( g(n) = \sum_{d|n} f(d) \), then

\[
f(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) g(d) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right).
\]

For example, \( \phi(n) = n \sum_{d|n} \frac{\mu(d)}{d} \).

The generalized Möbius formula in one dimension is that: if \( g(x) = \sum_{n=1}^{\infty} f(n^\alpha x) \), then \( f(x) = \sum_{n=1}^{\infty} \mu(n) g(n^\alpha x) \), for any real \( \alpha \) except 0.

The table below can be derived from the table in Section 2.3.13 (For example, \( \mu(2) = -1, \mu(4) = 0, \) and \( \mu(6) = 1 \)).

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2.3.9 PRIME NUMBERS

- A prime number is a positive integer greater than 1 with no positive, integral divisors other than 1 and itself. There are infinitely many prime numbers, 2, 3, 5, 7, . . .

- Twin primes are prime numbers that differ by two: (3, 5), (5, 7), (11, 13), (17, 19), . . . It is not known whether there are infinitely many twin primes.

- For every integer \( n \geq 2 \), the numbers \( \{n!+2, n!+3, \ldots, n!+n\} \) are a sequence of \( n-1 \) consecutive composite numbers.

The function \( \pi(x) \) represents the number of primes less than \( x \). The prime number theorem states that \( \pi(x) \sim x/\log x \) as \( x \to \infty \).

![Table of \( \pi(x) \) values](image)

<table>
<thead>
<tr>
<th>( x )</th>
<th>100</th>
<th>1000</th>
<th>10000</th>
<th>10(^5)</th>
<th>10(^6)</th>
<th>10(^7)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi(x) )</td>
<td>25</td>
<td>168</td>
<td>1229</td>
<td>9592</td>
<td>78498</td>
<td>664579</td>
</tr>
</tbody>
</table>

**Prime formula**

The set of prime numbers is identical with the set of positive values taken on by the polynomial of degree 25 in the 26 variables \( \{a, b, \ldots, z\} \):
(k + 2)(1 - [wz + h + j - q]^2 - [(gk + 2g + k + 1)(h + j) + h - z]^2 - [2n + p + q + z - e]^2
- [16(k + 1)^3(k + 2)(a + 1)^2 + 1 - f]^2 - [a^2 + 2a + 1 - o]^2 - [(a^2 - 1)^2 + 1 - x^2]^2
- [16r^2y^2[(a - 1) + 1 - u]^2 - [((a + u^2)(u^2 - a))^2 - 1 - (n + 4dy)]^2 + 1 - (x + ca)^2]^2
- [(n + l + v - y]^2
- [(a - 1)^2 + 1 - m]^2 - [a_i + k + 1 - l - i]^2 - [p + l(a_n - 1) + b(2an + 2a - n^2 - 2a - 2) - m]^2
- [q + y(a - p - 1) + s(2ap + 2a - p^2 + 2p - 2) - x]^2 - [z + pl(a - p) + t(2ap - p^2 - 1) - pm]^2].

(2.3.12)

Although this polynomial appears to factor, the factors are improper, \( P = P \cdot 1. \) Note that this formula will also take on negative values, such as \(-76). 

There also exists a prime representing polynomial which contains 12 variables.

**Dirichlet’s theorem on primes in arithmetic progressions:** Let \( a \) and \( b \) be relatively prime positive integers. Then the arithmetic progression \( an + b \) (for \( n = 1, 2, \ldots \)) contains infinitely many primes.

**Proofs of primality**

**Lucas–Lehmer primality test:** Define the sequence, \( r_1 = 3, r_{m+1} = r_m^2 - 2. \) If \( p \) is a prime of the form \( 4n + 3 \) and \( M_p = 2^p - 1, \) then \( M_p \) will be prime (called a Mersenne prime) if, and only if, \( M_p \) divides \( r_{p-1}. \)

This simple test is the reason that the largest known prime numbers are Mersenne primes. For example, consider \( p = 7 \) and \( M_7 = 127. \) The \{\( r_n \)\} sequence is \{3, 7, 47, 2207 \equiv 48, 2302 \equiv 16, 254 \equiv 0\}; hence \( M_7 \) is prime.

It is also possible to give, for a general number \( p, \) a “certificate” that \( p \) is prime. It is easy to use the certificate to verify that a given number is prime (easier than it was to determine that it was prime in the first place). There are several types of certificates that can be given. Pratt’s certificate consists of a number \( a \) and the factorization of the number \( p - 1. \) This method is feasible if \( p - 1 \) is easy to factor.

The number \( p \) will be prime if there exists a primitive root \( a \) in the field \( \text{GF}[p]. \) This primitive root must satisfy the conditions \( a^{p-1} \equiv 1 \pmod{p} \) and \( a^{(p-1)/q} \not\equiv 1 \pmod{p} \) for any prime \( q \) that divides \( p. \) For example, the number \( p = 31 \) has \( p - 1 = 30 = 2 \cdot 3 \cdot 5, \) and a primitive root is given by \( a = 3. \) Hence, to verify that \( p = 31 \) is prime, we compute

\[
3^{(31-1)/2} = 3^{15} \equiv 14348907 \equiv -1 \neq 1 \quad \text{(mod 31)},
\]
\[
3^{(31-1)/3} = 3^{10} \equiv 59049 \equiv 25 \neq 1 \quad \text{(mod 31)},
\]
\[
3^{(31-1)/5} = 3^6 \equiv 719 \equiv 16 \neq 1 \quad \text{(mod 31)},
\]
\[
3^{(31-1)} = (3^{(31-1)/2})^2 \equiv (-1)^2 = 1 \quad \text{(mod 31)}.
\]

**Probabilistic primality test**

Let \( n \) be a number whose primality is to be determined. Probabilistic primality tests can return one of two results: either a proof that the number \( n \) is composite or a
statement of the form, “The probability that the number \( n \) is not prime is less than \( \epsilon \), where \( \epsilon \) can be specified by the user. Typically, we take \( \epsilon = 2^{-200} < 10^{-60} \).

From Fermat’s theorem, if \( b \neq 0 \), then \( b^{n-1} = 1 \pmod{n} \) whenever \( n \) is prime. If this holds, then \( n \) is a probable prime to the base \( b \). Given a value of \( n \), if a value of \( b \) can be found such that this does not hold, then \( n \) cannot be prime. It can happen, however, that a probable prime is not prime.

Let \( P(x) \) be the probability that \( n \) is composite under the hypotheses:

1. \( n \) is an odd integer chosen randomly from the range \([2, x]\);
2. \( b \) is an integer chosen randomly from the range \([2, n-2]\);
3. \( n \) is a probable prime to the base \( b \).

Then \( P(x) \leq (\log x)^{-197} \) for \( x \geq 10^{10000} \).

A different test can be obtained from the following theorem. Given the number \( n \), find \( s \) and \( t \) with \( n-1 = 2^st \), with \( t \) odd. Then choose a random integer \( b \) from the range \([2, n-2]\). If either

\[
b^t = 1 \pmod{n} \quad \text{or} \quad b^{2^i} = -1 \pmod{n},
\]

for some \( i < s \), then \( n \) is a strong probable prime to the base \( b \). Every odd prime must pass this test. If \( n > 1 \) is an odd composite, then the probability that it is a strong probable prime to the base \( b \), when \( b \) is chosen randomly, is less than \( 1/4 \).

A stronger test can be obtained by choosing \( k \) independent values for \( b \) in the range \([2, n-2]\) and checking the above relation for each value of \( b \). Let \( P_k(x) \) be the probability that \( n \) is found to be a strong probable prime to each base \( b \). Then \( P_k(x) \leq 4^{-(k-1)} P(x)/(1-P(x)) \).

### 2.3.10 PRIME NUMBERS LESS THAN 10,000

The prime number \( p_{10000} \) is found by looking at the row beginning with \( n_+ \) and at the column beginning with \( k \).
2.3.11 PRIME NUMBERS OF SPECIAL FORMS

1. As of September 1996, the largest known prime numbers, in descending order, are

<table>
<thead>
<tr>
<th>Number</th>
<th>Number of digits</th>
</tr>
</thead>
<tbody>
<tr>
<td>2^{1257787} - 1</td>
<td>$a$378,632</td>
</tr>
<tr>
<td>2^{859433} - 1</td>
<td>258,716</td>
</tr>
<tr>
<td>2^{756839} - 1</td>
<td>227,832</td>
</tr>
<tr>
<td>391581 · 2^{16193} - 1</td>
<td>$b$65,087</td>
</tr>
<tr>
<td>2^{216091} - 1</td>
<td>65,050</td>
</tr>
<tr>
<td>3 · 2^{157169} + 1</td>
<td>47,314</td>
</tr>
<tr>
<td>9 · 2^{149143} + 1</td>
<td>44,898</td>
</tr>
<tr>
<td>9 · 2^{147073} + 1</td>
<td>44,275</td>
</tr>
<tr>
<td>9 · 2^{145247} + 1</td>
<td>43,725</td>
</tr>
<tr>
<td>2^{132049} - 1</td>
<td>39,751</td>
</tr>
<tr>
<td>9 · 2^{127003} + 1</td>
<td>38,233</td>
</tr>
<tr>
<td>5 · 2^{125413} + 1</td>
<td>37,754</td>
</tr>
<tr>
<td>9 · 2^{114854} + 1</td>
<td>34,576</td>
</tr>
<tr>
<td>13 · 2^{114296} + 1</td>
<td>34,408</td>
</tr>
<tr>
<td>2110503 - 1</td>
<td>33,265</td>
</tr>
</tbody>
</table>

*a*Largest prime of the form $2^n - 1$, a Mersenne prime.

*b*Largest non-Mersenne prime.

Other large primes include:

- 134088 · 10^{15030} + 1 (largest prime of the form $k · 10^n + 1$, has 15,036 digits)
- $10^{1810} + 1465641 · 10^{5002} + 1$ (largest palindromic prime, has 11,811 digits)
- 3610! − 1 (largest prime factorial minus 1, has 11,277 digits)
- 2 · 3^{13782} + 1 (largest prime of the form $k · 3^n + 1$, has 6,576 digits)
- 3476!! − 1 (largest double factorial minus one, has 5,402 digits)
2. All of the known Mersenne primes of the form $2^n - 1$ are given by the values (note that $n$ must be prime itself), $n = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, 110503, 132049, 216091, 756839, and 859433.

3. The largest known twin primes are: $242206083 \cdot 2^{38880} \pm 1$ (with 11,713 digits), $570918348 \cdot 10^{5120} \pm 1$ (with 5,129 digits), and $697053813 \cdot 2^{16352} \pm 1$ (with 4,932 digits).

4. The $n^{th}$ repunit is $R_n = (10^n - 1)/9 = 11 \ldots 11$. The only known prime repunits correspond to $n = 2, 19, 23, 317, 1031$.

5. Prime numbers of the forms $2^n \pm a$ and $10^n \pm b$: Large prime numbers of a specified size are sometimes needed. In the following table, for a given $n$, $a_{\pm}$ and $b_{\pm}$ are the least values such that $2^n + a_{\pm}$ and $10^n + b_{\pm}$ are probably primes (a probabilistic primality test was used). For example, for $n = 3$, the numbers $2^3 - 1 = 7$, $2^3 + 3 = 11$, $10^3 - 3 = 997$, and $10^3 + 9 = 1009$ are all prime.
<table>
<thead>
<tr>
<th>$n$</th>
<th>$2^n + a$</th>
<th>$10^n + b$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$a_-$</td>
<td>$a_+$</td>
</tr>
<tr>
<td></td>
<td>$b_-$</td>
<td>$b_+$</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>-3</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
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6. Define \( p# \) to be the product of the prime numbers less than or equal to \( p \).

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### 2.3.12 PRIME PERIOD LENGTHS

When an integer is divided by a prime \( p \), and the result is represented as a decimal, the number of digits in the period is called \( \lambda(p) \). For example, \( 1/7 = 0.142857 \ldots \) so that \( \lambda(7) = 6 \), and \( 1/37 = 0.027027 \ldots \) so that \( \lambda(37) = 3 \). The period \( \lambda(p) \) always divides \( p - 1 \).

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### 2.3.13 FACTORIZATION TABLE

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©1996 CRC Press LLC
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| 41 | (2 \cdot 5 \cdot 41) | 3 \cdot 137 | 2 \cdot 103 \cdot 7 \cdot 59 | 2 \cdot 3^2 \cdot 23 \cdot 5 \cdot 83 | 5 \cdot 83 | 2 \cdot 14 \cdot 3 \cdot 19 | 5 \cdot 7 \cdot 3 \cdot 19 | 419 |
| 42 | (2^2 \cdot 3 \cdot 5 \cdot 7) | 421 | 2 \cdot 211 | 3 \cdot 47 | 2 \cdot 3 \cdot 53 | 5^2 \cdot 17 \cdot 2 \cdot 3 \cdot 17 | 2 \cdot 7 \cdot 1 \cdot 7 | 2 \cdot 3 \cdot 53 | 2 \cdot 107 | 3 \cdot 11 \cdot 13 |
| 43 | (2 \cdot 5 \cdot 43) | 431 | 2^4 \cdot 3^3 | 433 | 2 \cdot 7 \cdot 31 | 3 \cdot 5 \cdot 29 | 2 \cdot 109 | 19 \cdot 23 | 2 \cdot 3 \cdot 73 | 439 |
| 44 | (2^3 \cdot 5 \cdot 11) | 3^2 \cdot 7^2 | 2 \cdot 13 \cdot 17 | 443 | 2^2 \cdot 3 \cdot 37 | 5 \cdot 89 | 2 \cdot 223 | 3 \cdot 149 | 2 \cdot 7 | 449 |
| 45 | (2 \cdot 3^2 \cdot 5^2) | 11 \cdot 41 | 2 \cdot 113 | 3 \cdot 151 | 2 \cdot 227 | 5 \cdot 7 \cdot 13 | 2 \cdot 3 \cdot 19 | 457 | 2 \cdot 229 | 3 \cdot 17 |
| 46 | (2 \cdot 5 \cdot 5 \cdot 23) | 461 | 2 \cdot 3 \cdot 7 \cdot 11 | 463 | 2 \cdot 29 | 3 \cdot 5 \cdot 31 | 2 \cdot 3 \cdot 19 | 467 | 2 \cdot 3 \cdot 13 | 7 \cdot 67 |
| 47 | (2 \cdot 5 \cdot 8 \cdot 47) | 3 \cdot 157 | 2 \cdot 119 | 11 \cdot 43 | 2 \cdot 3 \cdot 79 | 5^2 \cdot 19 | 2 \cdot 2 \cdot 7 \cdot 17 | 3 \cdot 2 \cdot 23 | 2 \cdot 4 \cdot 3 \cdot 7 | 479 |
| 48 | (2^3 \cdot 3 \cdot 5 \cdot 13) | 13 \cdot 37 | 2 \cdot 241 | 3 \cdot 7 \cdot 23 | 2^2 \cdot 11^2 | 5 \cdot 97 | 2 \cdot 3 \cdot 5 | 487 | 2 \cdot 61 | 3 \cdot 163 |
| 49 | (2 \cdot 5 \cdot 7^2) | 491 | 2 \cdot 3 \cdot 41 | 17 \cdot 29 | 2 \cdot 13 \cdot 19 | 3 \cdot 2 \cdot 5 \cdot 11 | 2 \cdot 3 \cdot 11 | 4 \cdot 3 \cdot 7 | 7 \cdot 71 | 2 \cdot 3 \cdot 83 | 499 |</p>
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### 2.3.14 FACTORIZATION OF $2^m - 1$

| $2^3 - 1 = 7$ | $2^{19} - 1 = 524287$ |
| $2^4 - 1 = 3 \times 5$ | $2^{20} - 1 = 3 \times 5^2 \times 11 \times 31 \times 41$ |
| $2^5 - 1 = 31$ | $2^{21} - 1 = 7^2 \times 127 \times 337$ |
| $2^6 - 1 = 3^2 \times 7$ | $2^{22} - 1 = 3 \times 23 \times 89 \times 683$ |
| $2^7 - 1 = 127$ | $2^{23} - 1 = 47 \times 178481$ |
| $2^8 - 1 = 3 \times 5 \times 17$ | $2^{24} - 1 = 3^2 \times 5 \times 7 \times 13 \times 17 \times 241$ |
| $2^9 - 1 = 7 \times 73$ | $2^{25} - 1 = 31 \times 601 \times 1801$ |
| $2^{10} - 1 = 3 \times 11 \times 31$ | $2^{26} - 1 = 3 \times 2731 \times 8191$ |
| $2^{11} - 1 = 23 \times 89$ | $2^{27} - 1 = 7 \times 73 \times 262657$ |
| $2^{12} - 1 = 3^2 \times 5 \times 7 \times 13$ | $2^{28} - 1 = 3 \times 5 \times 29 \times 43 \times 113 \times 127$ |
| $2^{13} - 1 = 8191$ | $2^{29} - 1 = 233 \times 1103 \times 2089$ |
| $2^{14} - 1 = 3 \times 43 \times 127$ | $2^{30} - 1 = 3^2 \times 7 \times 11 \times 31 \times 151 \times 331$ |
| $2^{15} - 1 = 7 \times 31 \times 151$ | $2^{31} - 1 = 2147483647$ |
| $2^{16} - 1 = 3 \times 5 \times 17 \times 257$ | $2^{32} - 1 = 3 \times 5 \times 17 \times 257 \times 65537$ |
| $2^{17} - 1 = 131071$ | $2^{33} - 1 = 7 \times 23 \times 89 \times 599479$ |
| $2^{18} - 1 = 3^3 \times 7 \times 19 \times 73$ | $2^{34} - 1 = 3 \times 43691 \times 131071$ |

### 2.3.15 MAGIC SQUARES

A magic square is a square array of integers with the property that the sum of the integers in each row or column is the same. If $(c, n) = (d, n) = (e, n) = (f, n) = (cf - en, n) = 1$, then the array $A = (a_{ij})$ will be magic (and use the $n^2$ numbers $0, 1, \ldots, n^2 - 1$) if $a_{ij} = k$ with

$$ i \equiv ck + e \left\lfloor \frac{k}{n} \right\rfloor \pmod{n} \quad \text{and} \quad j \equiv dk + f \left\lfloor \frac{k}{n} \right\rfloor \pmod{n} $$

For example, with $c = 1, d = e = f = 2, \text{ and } n = 3$, a magic square is

| 6 | 1 | 5 |
| 2 | 3 | 7 |
| 4 | 8 | 0 |

### 2.3.16 TOTIENT FUNCTION

**Definitions**

- $\phi(n)$ is the totient function; is the number of integers not exceeding and relatively prime to $n$.
- $\sigma(n)$ is the sum of the divisors of $n$.
- $\tau(n)$ is the number of divisors of $n$.  

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Define $\sigma_k(n)$ to be the $k^{th}$ divisor function, the sum of the $k^{th}$ powers of the divisors of $n$. Then $\phi(n) = \sigma_0(n)$ and $\sigma(n) = \sigma_1(n)$.

A perfect number $n$ satisfies $\sigma(n) = 2n$.

**Example**

The numbers less than 6 and relatively prime to 6 are $\{1, 5\}$. Hence $\phi(6) = 2$. The divisors of 6 are $\{1, 2, 3, 6\}$. There are $\tau(6) = 4$ divisors. The sum of these numbers is $\sigma(6) = 1 + 2 + 3 + 6 = 12$.

**Properties**

1. $\phi$ is a multiplicative function: if $(n, m) = 1$, then $\phi(nm) = \phi(n)\phi(m)$.
2. Gauss’s theorem states: $n = \sum_{d|n} \phi(d)$.
3. When $n = \prod_i p_i^{\alpha_i}$,

$$\sigma_k(n) = \sum_{d|n} d^k = \prod_i p_i^{k(\alpha_i+1)} - 1 \over p_i - 1$$

4. Generating functions include

$$\sum_{n=0}^{\infty} \frac{\sigma_k(n)}{n^s} = \zeta(s)\zeta(s-k)$$

$$\sum_{n=0}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}$$

5. The positive integer $n$ is an even perfect number if, and only if, $n = 2^{m-1}(2^m - 1)$, where $m$ is a positive integer such that $M_m = 2^m - 1$ is prime ($M_m$ is called a Mersenne prime). The sequence of perfect numbers is $\{6, 28, 496, \ldots\}$ (corresponding to $m = 2, 3, 5, \ldots$).
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2.4 VECTOR ALGEBRA

2.4.1 NOTATION FOR VECTORS AND SCALARS

A vector is an ordered \( n \)-tuple of values. A vector is usually represented by a lower case, bold faced letter, such as \( \mathbf{v} \). The individual components of a vector \( \mathbf{v} \) are typically denoted by a lower case letter along with a subscript identifying the relative position of the component in the vector, such as \( \mathbf{v} = [v_1, v_2, \ldots, v_n] \). In this case, the vector is said to be \( n \)-dimensional. If the \( n \) individual components of the vector are real numbers, then \( \mathbf{v} \in \mathbb{R}^n \). Likewise, if the \( n \) components of \( \mathbf{v} \) are complex, then \( \mathbf{v} \in \mathbb{C}^n \).

Two vectors, \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \), are said to be equal if they have exactly the same components. A negative vector, written as \( -\mathbf{v} \), is one that acts in direction opposite to \( \mathbf{v} \), but is of equal magnitude.

2.4.2 PHYSICAL VECTORS

Any quantity that is completely determined by its magnitude is called a scalar. For example, mass, density, and temperature are scalars. Any quantity that is completely determined by its magnitude and direction is called, in physics, a vector. We often use a three-dimensional vector to represent a physical vector. Examples of physical vectors include velocity, acceleration, and force. A physical vector is represented by a directed line segment, the length of which represents the magnitude of the vector. Two vectors are said to be parallel if they have exactly the same direction, i.e., the angle between the two vectors equals zero.

2.4.3 FUNDAMENTAL DEFINITIONS

1. A row vector is a vector whose components are aligned horizontally. A column vector has its components aligned vertically. The transpose operator, denoted by the superscript \( ^T \), switches the orientation of a vector between horizontal and vertical.

**EXAMPLE 2.4.1**

\[
\mathbf{v} = [1, 2, 3, 4], \quad \mathbf{v}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad (\mathbf{v}^T)^T = [1, 2, 3, 4].
\]  

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Vectors are traditionally written with either rounded braces or with square brackets.

2. Two vectors, \( \mathbf{v}_i \) and \( \mathbf{v}_j \), are said to be orthogonal if \( \mathbf{v}_i^T \cdot \mathbf{v}_j = 0 \), where the operation \( \cdot \) denotes an inner product.

3. A set of vectors \( \{ \mathbf{v}_1, \ldots, \mathbf{v}_n \} \) is said to be orthogonal if \( \mathbf{v}_i^T \cdot \mathbf{v}_j = 0 \) for all \( i \neq j \).

4. A set of orthogonal vectors \( \{ \mathbf{v}_1, \ldots, \mathbf{v}_m \} \) is said to be orthonormal if, in addition to possessing the property of orthogonality, the set possesses the property that \( \mathbf{v}_i^T \cdot \mathbf{v}_i = 1 \) for all \( 1 \leq i \leq m \).

### 2.4.4 LAWS OF VECTOR ALGEBRA

1. The vector sum of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \), represented by \( \mathbf{v}_1 + \mathbf{v}_2 \), results in another vector of the same dimension, and is calculated by simply adding corresponding vector components, e.g., if \( \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n \), then \( \mathbf{v}_1 + \mathbf{v}_2 = [v_{11} + v_{21}, \ldots, v_{1n} + v_{2n}] \).

2. The vector subtraction of \( \mathbf{v}_2 \) from \( \mathbf{v}_1 \), represented by \( \mathbf{v}_1 - \mathbf{v}_2 \), is equivalent to the addition of \( \mathbf{v}_1 \) and \( -\mathbf{v}_2 \).

3. If \( r \) is a scalar, then \( r \mathbf{v} \) or \( \mathbf{v} r \) represents a scaling of the vector \( \mathbf{v} \) in the same direction as \( \mathbf{v} \), since the multiplicative scalar is distributed to each component of \( \mathbf{v} \).

4. If \( 0 \leq r < 1 \), then the scalar multiplication \( r \mathbf{v} \) shrinks the length of \( \mathbf{v} \), multiplication by \( r = 1 \) leaves \( \mathbf{v} \) unchanged, and, if \( r > 1 \), then \( r \mathbf{v} \) stretches the length of \( \mathbf{v} \). When \( r < 0 \), multiplication of \( \mathbf{v} \) by \( r \) has the same effects on the magnitude, or length of \( \mathbf{v} \), but results in a vector oriented in the direction opposite to \( \mathbf{v} \).

**EXAMPLE 2.4.2**

\[
4 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 12 \end{bmatrix}, \quad -4 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ -12 \end{bmatrix}.
\]

5. If \( r \) and \( s \) are scalars, and \( \mathbf{v}_1, \mathbf{v}_2, \) and \( \mathbf{v}_3 \) are vectors, the following rules of algebra are valid:

\[
\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1, \\
(r + s)\mathbf{v}_1 = r\mathbf{v}_1 + s\mathbf{v}_1 = \mathbf{v}_1 r + \mathbf{v}_1 s = \mathbf{v}_1 (r + s), \quad \text{and} \quad r(\mathbf{v}_1 + \mathbf{v}_2) = r\mathbf{v}_1 + r\mathbf{v}_2, \\
\mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3) = (\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3.
\]
2.4.5 VECTOR NORMS

1. A norm is the vector analog of absolute value for real scalars. Norms provide a distance measure for a vector space.
2. A vector norm applied to a vector \( \mathbf{v} \) is denoted by a double bar notation \( \| \mathbf{v} \| \).
3. An \( n \)-dimensional real number space \( \mathbb{R}^n \) together with a norm defines a metric space.
4. The properties of a vector norm are
   - For any vector \( \mathbf{x} \neq \mathbf{0} \), \( \| \mathbf{x} \| > 0 \),
   - \( \| \gamma \mathbf{x} \| = |\gamma| \| \mathbf{x} \| \), and
   - \( \| \mathbf{x} + \mathbf{y} \| \leq \| \mathbf{x} \| + \| \mathbf{y} \| \) (triangle inequality).
5. The three most commonly used vector norms are
   - The \( L_1 \) norm is defined as \( \| \mathbf{v} \|_1 = |v_1| + \cdots + |v_n| = \sum_{i=1}^{n} |v_i| \).
   - The \( L_2 \) norm (Euclidean norm) is defined as
     \[
     \| \mathbf{v} \|_2 = (|v_1|^2 + |v_2|^2 + \cdots + |v_n|^2)^{1/2} = \left( \sum_{i=1}^{n} v_i^2 \right)^{1/2}.
     \] (2.4.2)
   - The \( L_{\infty} \) norm is defined as \( \| \mathbf{v} \|_{\infty} = \max_{1 \leq i \leq n} |v_i| \).
6. In the absence of any subscript, the norm \( \| \cdot \| \) is usually assumed to be the \( L_2 \) norm.
7. A unit vector with respect to a particular norm \( \| \cdot \| \) is a vector that satisfies the property that \( \| \mathbf{v} \| = 1 \), and is sometimes denoted by \( \hat{\mathbf{v}} \).

2.4.6 DOT, SCALAR, OR INNER PRODUCT

1. The dot, scalar, or inner product of two vectors of the same dimension, represented by \( \mathbf{v}_1 \cdot \mathbf{v}_2 \) or \( \mathbf{x}^T \mathbf{y} \), has two common definitions, depending upon the context in which this product is encountered.
   - (a) In vector calculus and physics, the dot or scalar product is defined by
     \[
     \mathbf{v}_1 \cdot \mathbf{v}_2 = \| \mathbf{v}_1 \|_2 \| \mathbf{v}_2 \|_2 \cos \theta,
     \] (2.4.3)
     where \( \theta \) represents the angle determined by \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) (see Figure 2.4.1).
   - (b) In optimization, linear algebra, and computer science, the inner product of two vectors, \( \mathbf{x} \) and \( \mathbf{y} \), is equivalently defined as
     \[
     \mathbf{x}^T \mathbf{y} = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + \cdots + x_n y_n.
     \] (2.4.4)
From the first definition, it is apparent that the inner product of two perpendicular, or orthogonal, vectors is zero, since the cosine of 90° is zero.

2. The inner product of two parallel vectors (with \( \mathbf{v}_2 = r \mathbf{v}_1 \)) is given by \( \mathbf{v}_1 \cdot \mathbf{v}_2 = r \| \mathbf{v}_1 \|^2 \). For example, when \( r > 0 \),

\[
\mathbf{v}_1 \cdot \mathbf{v}_2 = \| \mathbf{v}_1 \|_2 \| \mathbf{v}_2 \|_2 \cos 0 = \| \mathbf{v}_1 \|_2 \| \mathbf{v}_2 \|_2 = r \| \mathbf{v}_1 \|_2^2.
\]

(2.4.5)

3. The dot, scalar, or inner product is distributive, e.g.,

\[
(\mathbf{v}_1 + \mathbf{v}_2) \cdot \mathbf{v}_3 = \mathbf{v}_1 \cdot \mathbf{v}_3 + \mathbf{v}_2 \cdot \mathbf{v}_3.
\]

(2.4.6)

4. For \( \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n \) with \( n > 1 \),

\[
\mathbf{a}^\top \mathbf{b} = \mathbf{a}^\top \mathbf{c} \not\Rightarrow \mathbf{b} = \mathbf{c}.
\]

(2.4.7)

However, it is valid to conclude that

\[
\mathbf{a}^\top \mathbf{b} = \mathbf{a}^\top \mathbf{c} \quad \Rightarrow \quad \mathbf{a}^\top (\mathbf{b} - \mathbf{c}) = 0,
\]

(2.4.8)

and that the vector \( \mathbf{a} \) is orthogonal to the vector \( (\mathbf{b} - \mathbf{c}) \).

**FIGURE 2.4.1**

*Depiction of right-hand rule.*

---

### 2.4.7 VECTOR OR CROSS PRODUCT

1. The vector or cross product of two nonzero three-dimensional vectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) is defined as

\[
\mathbf{v}_1 \times \mathbf{v}_2 = \hat{\mathbf{n}} \| \mathbf{v}_1 \|_2 \| \mathbf{v}_2 \|_2 \sin \theta,
\]

(2.4.9)

where \( \hat{\mathbf{n}} \) is the unit *normal* vector perpendicular to both \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) in the direction adhering to the *right-hand rule* (see Figure 2.4.1) and \( \theta \) is the angle between \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \).
2. If \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are parallel, then \( \mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0} \).

3. The quantity \( \| \mathbf{v}_1 \|_2 \| \mathbf{v}_2 \|_2 | \sin \theta | \) represents the area of the parallelogram determined by \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \).

4. The following rules apply for vector products:

\[
(\gamma \mathbf{v}_1) \times (\alpha \mathbf{v}_2) = (\gamma \alpha) \mathbf{v}_1 \times \mathbf{v}_2,
\]

\[
\mathbf{v}_1 \times \mathbf{v}_2 = -\mathbf{v}_2 \times \mathbf{v}_1,
\]

\[
\mathbf{v}_1 \times (\mathbf{v}_2 + \mathbf{v}_3) = \mathbf{v}_1 \times \mathbf{v}_2 + \mathbf{v}_1 \times \mathbf{v}_3,
\]

\[
(\mathbf{v}_1 + \mathbf{v}_2) \times \mathbf{v}_3 = \mathbf{v}_1 \times \mathbf{v}_3 + \mathbf{v}_2 \times \mathbf{v}_3,
\]

\[
\mathbf{v}_1 \times (\mathbf{v}_2 \times \mathbf{v}_3) = \mathbf{v}_2 (\mathbf{v}_3 \cdot \mathbf{v}_1) - \mathbf{v}_3 (\mathbf{v}_2 \cdot \mathbf{v}_1),
\]

\[
(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}),
\]

\[
(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{d})]\mathbf{c} - [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})]\mathbf{d}
\]

\[
= [\mathbf{a} \cdot (\mathbf{c} \times \mathbf{d})]\mathbf{b} - [\mathbf{b} \cdot (\mathbf{c} \times \mathbf{d})]\mathbf{a}.
\]

5. The pairwise cross products of the unit vectors \( \mathbf{i}, \mathbf{j}, \) and \( \mathbf{k} \), corresponding to the directions of \( \mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k} \), are given by

\[
\mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i} = \mathbf{k},
\]

\[
\mathbf{j} \times \mathbf{k} = -\mathbf{k} \times \mathbf{j} = \mathbf{i},
\]

\[
\mathbf{k} \times \mathbf{i} = -\mathbf{i} \times \mathbf{k} = \mathbf{j}, \quad \text{and}
\]

\[
\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}.
\]

6. If \( \mathbf{v}_1 = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} \) and \( \mathbf{v}_2 = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k} \), then

\[
\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
 a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 
\end{vmatrix},
\]

\[
= (a_2 b_3 - b_2 a_3)\mathbf{i} + (a_3 b_1 - b_3 a_1)\mathbf{j} + (a_1 b_2 - b_1 a_2)\mathbf{k}.
\]

### 2.4.8 SCALAR AND VECTOR TRIPLE PRODUCTS

1. The **scalar triple product** involving three three-dimensional vectors \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \), sometimes denoted by \( [\mathbf{a} \mathbf{b} \mathbf{c}] \) (not to be confused with a matrix containing three columns \( [\mathbf{a} \ \mathbf{b} \ \mathbf{c}] \)), can be computed using the determinant

\[
[\mathbf{a} \mathbf{b} \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot \begin{vmatrix}
b_2 & b_3 & \mathbf{i} \\
c_2 & c_3 & \mathbf{j} \\
 b_1 & b_3 & \mathbf{k} 
\end{vmatrix}
\]

\[
= a_1 \begin{vmatrix}
b_2 & b_3 \\
c_2 & c_3 
\end{vmatrix} - a_2 \begin{vmatrix}
b_1 & b_3 \\
c_1 & c_3 
\end{vmatrix} + a_3 \begin{vmatrix}
b_1 & b_2 \\
c_1 & c_2 
\end{vmatrix}
\]

\[
= a_1 a_2 a_3
\]

\[
= \| \mathbf{a} \| \| \mathbf{b} \| \| \mathbf{c} \| \cos \phi \sin \theta.
\]
where $\theta$ is the angle between $\mathbf{b}$ and $\mathbf{c}$, and $\phi$ is the angle between $\mathbf{a}$ and the normal to the plane defined by $\mathbf{b}$ and $\mathbf{c}$.

2. The absolute value of the triple scalar product calculates the volume of the parallelepiped determined by the three vectors. The answer you get is therefore independent of the order in which the triple product is taken.

3. Given three noncoplanar reference vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, the reciprocal system is given by $\mathbf{v}_1^*$, $\mathbf{v}_2^*$, and $\mathbf{v}_3^*$, where

$$
\mathbf{v}_1^* = \frac{\mathbf{v}_2 \times \mathbf{v}_3}{[\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3]}, \quad \mathbf{v}_2^* = \frac{\mathbf{v}_3 \times \mathbf{v}_1}{[\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3]}, \quad \mathbf{v}_3^* = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{[\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3]}.
$$

(2.4.14)

Note that

$$
1 = \|\mathbf{v}_1\| \|\mathbf{v}_1^*\| = \|\mathbf{v}_2\| \|\mathbf{v}_2^*\| = \|\mathbf{v}_3\| \|\mathbf{v}_3^*\|,
$$

and

$$
0 = \|\mathbf{v}_1\| \|\mathbf{v}_2^*\| = \|\mathbf{v}_1\| \|\mathbf{v}_3^*\| = \|\mathbf{v}_2\| \|\mathbf{v}_1^*\|,
$$

etc. (2.4.15)

The system $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is its own reciprocal.

4. The vector triple product involving three three-dimensional vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, given by $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, results in a vector, perpendicular to $\mathbf{a}$, lying in the plane of $\mathbf{b}$ and $\mathbf{c}$, and is defined as

$$
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c},
$$

$$
= \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
b_1 & a_2 & a_3 \\
c_2 & c_3 & c_1
\end{vmatrix}.
$$

(2.4.16)

5. $$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{acd}]\mathbf{b} - [\mathbf{bcd}]\mathbf{a} = [\mathbf{abd}]\mathbf{c} - [\mathbf{abc}]\mathbf{d}$$

### 2.5 LINEAR AND MATRIX ALGEBRA

#### 2.5.1 DEFINITIONS

1. An $m \times n$ matrix is a two-dimensional array of numbers consisting of $m$ rows and $n$ columns. By convention, a matrix is denoted by a capital letter emphasized with italics, as in $A$, $B$, $D$, or boldface, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{D}$, along with a subscript denoting the dimensions of the matrix, e.g., $A_{2 \times 3}$. If $\mathbf{A}$ is a real $n \times m$ matrix, then we write $\mathbf{A} \in \mathbb{R}^{n \times m}$.

2. $A_{m \times n}$ is called rectangular if $m \neq n$.

3. $A_{m \times n}$ is called square if $m = n$. 

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4. A particular element or component of a matrix is denoted by the lower case letter of that which names the matrix, along with two subscripts corresponding to the row $i$ and column $j$ location of the component in the array, e.g.,

$$A_{m \times n} \text{ has components } a_{ij};$$
$$B_{m \times n} \text{ has components } b_{ij}. $$

For example, $a_{23}$ is in the second row and third column of matrix $A$.

5. Any component $a_{ij}$ with $i = j$ is called a diagonal element.

6. Any component $a_{ij}$ with $i \neq j$ is called an off-diagonal element.

7. Two matrices $A$ and $B$ are said to be equal if they have the same number of rows ($m$) and columns ($n$), and $a_{ij} = b_{ij}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$.

8. An $m \times 1$ dimensional matrix is called a column vector. Similarly, a $1 \times n$ dimensional matrix is called a row vector.

9. A column (row) vector with all components equal to zero is called a null vector and is usually denoted by $0$.

10. A column vector with all components equal to one is sometimes denoted by $e$. The analogous row vector is denoted by $e^T$.

11. The unit vectors of order $n$ are usually $\{e_1, e_2, \ldots, e_n\}$ where $e_i$ is a $n \times 1$ vector of all zeros, except for the $i$th component, which is one.

12. The scalar $x^T x = \sum_{i=1}^{n} x_i^2$ is the sum of squares of all components of the vector $x$.

13. The weighted sum of squares is defined by $x^T D_w x = \sum_{i=1}^{n} w_i x_i^2$, when $x$ has $n$ components and the diagonal matrix $D_w$ is of dimension $(n \times n)$.

14. If $Q$ is a square matrix, then $x^T Q x$ is called a quadratic form.

15. An $n \times n$ matrix $A$ is called non-singular, or regular, if there exists an $n \times n$ matrix $B$ such that $AB = BA = I$. The unique matrix $B$ is called the inverse of $A$, and is denoted by $A^{-1}$.

16. The scalar $x^T y = \sum_{i=1}^{n} x_i y_i$ is the sum of products of the components of $x$ by those of $y$.

17. The weighted sum of products is $x^T D_w y = \sum_{i=1}^{n} w_i x_i y_i$, when $x$ and $y$ have $n$ components, and the diagonal matrix $D_w$ is $(n \times n)$.

18. The scalar $x^T Q y$ is called a bilinear form, where $Q$ is a matrix of any appropriate dimension.

19. The transpose of an $m \times n$ matrix $A$, denoted by $A^T$, is an $n \times m$ matrix with rows and columns interchanged, so that the $(i, j)$ component of $A$ is the $(j, i)$ component of $A^T$, and $(A^T)_{ji} = A_{ij}$.

20. The Hermitian conjugate of a matrix $A$, denoted by $A^H$, is obtained by transposing $A$ and replacing each element by its complex conjugate. Hence, if $a_{kl} = u_{kl} + i v_{kl}$, then $(A^H)_{kl} = u_{lk} - i v_{lk}$, with $i = \sqrt{-1}$.

21. If $Q$ is a square matrix, then $x^H Q x$ is called a Hermitian form.
2.5.2 TYPES OF MATRICES

1. The diagonal alignment of components in a matrix extending from the upper left to the lower right is called the principal or main diagonal.

2. A square matrix with all components off the principal diagonal equal to zero is called a diagonal matrix, typically denoted by the letter $D$ with a subscript indicating the typical element in the principal diagonal.

**EXAMPLE 2.5.1**

$$D_a = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}, \quad D_\lambda = \begin{bmatrix} \lambda_{11} & 0 & 0 \\ 0 & \lambda_{22} & 0 \\ 0 & 0 & \lambda_{33} \end{bmatrix}.$$  

3. The identity matrix, denoted by $I$, is the diagonal matrix with $a_{ii} = 1$ for all $i = j$, and $a_{ij} = 0$ for $i \neq j$. The $n \times n$ identity matrix is denoted $I_n$.

4. A matrix that has all main diagonal components equal to one and zeros everywhere else except a single nonzero component in location $(i, j)$ is called an elementary matrix, denoted by $E_{ij}$.

**EXAMPLE 2.5.2**

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$  

An elementary matrix can also be expressed as $E = I - \alpha u v^T$, where $I$ is the identity matrix, $\alpha$ is a scalar, and $u$ and $v$ are vectors of the same dimension. In this context, the elementary matrix is referred to as a rank one modification of an identity matrix.

5. Other elementary matrices are those of the form $E_{ij} = e_i e_j^T$. Note that, using these matrices, $A = \sum_i \sum_j a_{ij} E_{ij}$.

6. A matrix with all components above the principal diagonal equal to zero is called a lower triangular matrix.
EXAMPLE 2.5.3

\[ L = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \] is lower triangular.

7. The transpose of a lower triangular matrix is called an upper triangular matrix.

8. A matrix whose components are arranged in \( m \) rows and a single column is called a column matrix, or column vector, and is typically denoted using bold face, lower case letters, e.g., \( \mathbf{a} \) and \( \mathbf{b} \).

9. A matrix whose components are arranged in \( n \) columns and a single row is called a row matrix, or row vector, and is typically denoted as a transposed column vector, e.g., \( \mathbf{a}^T \) and \( \mathbf{b}^T \).

10. A square matrix is called symmetric if \( A = A^T \).

11. A square matrix is called skew symmetric if \( A^T = -A \).

12. A square matrix \( A \) is called Hermitian if it equals its conjugate transpose, or \( A = A^H \). All real symmetric matrices are Hermitian.

13. A square matrix \( Q \) with orthonormal columns is said to be orthogonal. It follows directly that the rows of \( Q \) must also be orthonormal, so that \( QQ^T = \mathbf{I} \) or \( \mathbf{Q}^T = \mathbf{Q}^{-1} \).

14. A principal submatrix of a symmetric matrix \( A \) is formed by deleting rows and columns of \( A \) simultaneously, e.g., row 1 and column 1; row 9 and column 9, etc.

15. A zero, or null, matrix is one whose elements are all zero.

16. An \( m \times n \) matrix \( A \) with orthonormal columns satisfies the property \( A^T A = \mathbf{I} \), where \( \mathbf{I} \) is an \( n \times n \) identity matrix.

17. A square matrix, whose elements are constant along each diagonal, is called a Toeplitz matrix.

EXAMPLE 2.5.4

\[ \mathbf{A} = \begin{bmatrix} a & d & e \\ b & a & d \\ c & b & a \end{bmatrix} \quad \text{and} \quad \mathbf{M} = \begin{bmatrix} 4 & 0 & 1 \\ -11 & 4 & 0 \\ 3 & -11 & 4 \end{bmatrix} \]

are Toeplitz matrices. Notice that Toeplitz matrices are symmetric about a diagonal extending from the upper right-hand corner element to the lower left-hand corner element. This type of symmetry is called persymmetry.

\(^1\)Note the inconsistency in terminology that has persisted.
18. A Vandermonde matrix is a square matrix \( V \in \mathbb{R}^{(n+1) \times (n+1)} \) in which each column contains unit increasing powers of a single matrix value:

\[
V = \begin{bmatrix}
v_1 & v_2 & \cdots & v_{(n+1)} \\
v_1^2 & v_2^2 & \cdots & v_{(n+1)}^2 \\
\vdots & \vdots & \ddots & \vdots \\
v_1^n & v_2^n & \cdots & v_{(n+1)}^n
\end{bmatrix}.
\] (2.5.1)

19. A square matrix \( U \) is said to be in upper Hessenberg form if \( u_{ij} = 0 \) whenever \( i > j + 1 \). An upper Hessenberg matrix is essentially an upper triangular matrix with an extra nonzero element immediately below the main diagonal entry in each column of \( U \). For example,

\[
U = \begin{bmatrix}
u_{11} & u_{12} & u_{13} & u_{14} \\
b_{21} & u_{22} & u_{23} & u_{24} \\
0 & b_{32} & u_{33} & u_{34} \\
0 & 0 & b_{43} & u_{44}
\end{bmatrix}
\] is upper Hessenberg.

20. If the sum of the components of each column of a matrix \( A \in \mathbb{R}^{n \times n} \) equals one, then \( A \) is called a Markov matrix.

21. A circulant matrix is an \( n \times n \) matrix of the form

\[
C = \begin{bmatrix}
c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\
c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\
c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0
\end{bmatrix},
\] (2.5.2)

where the components \( c_{ij} \) are such that \( (j - i) = k \mod n \) have the same value \( c_k \). These components comprise the \( k^{th} \) stripe of \( C \).

22. A matrix \( A \) is called graded across its rows if \( a_{ij} \leq a_{i,j+1} \), for all \( i, j \); graded up its columns if \( a_{ij} \leq a_{i+1,j} \); and doubly graded if both conditions apply.

### 2.5.3 Conformability for Addition and Multiplication

1. Two matrices \( A \) and \( B \) can be added (subtracted) if they are of the same dimension. The result is a matrix of the same dimension.

**Example 2.5.5**

\[
A_{2 \times 3} + B_{2 \times 3} = \begin{bmatrix}3 & 2 & -1 \\ 4 & 0 & 9\end{bmatrix} + \begin{bmatrix}11 & -2 & 3 \\ 0 & 1 & 1\end{bmatrix} = \begin{bmatrix}14 & 0 & 2 \\ 4 & 1 & 10\end{bmatrix}.
\]
2. Multiplication of a matrix or a vector by a scalar is achieved by multiplying each component by that scalar. If $B = \alpha A$, then $b_{ij} = \alpha a_{ij}$ for all components.

3. The matrix multiplication $AB$ is valid if the number of columns of $A$ is equal to the number of rows of $B$.

4. The multiplication of two matrices $A_{m \times n}$ and $B_{n \times q}$ results in a matrix $C_{m \times q}$ whose components are defined as

$$c_{mq} = \sum_{k=1}^{n} a_{mk} b_{kq}.$$  \hspace{1cm} (2.5.3)

Each $c_{mq}$ is the result of the inner (dot) product of the $m^{th}$ row of $A$ with the $q^{th}$ column of $B$. This rule applies similarly for matrix multiplication involving more than two matrices, so that if $ABCD = E$, then

$$e_{ij} = \sum_{k} \sum_{l} \sum_{m} a_{ik} b_{kl} c_{lm} d_{mj}.$$ \hspace{1cm} (2.5.4)

The second subscript for each matrix component must coincide with the first subscript of the next one.

**EXAMPLE 2.5.6**

$$\begin{bmatrix} 2 & -1 & 3 \\ -4 & 1 & 4 \end{bmatrix} \begin{bmatrix} 5 & -3 & -3 \\ 2 & 2 & -1 \\ -7 & 1 & 5 \end{bmatrix} = \begin{bmatrix} -13 & -5 & 10 \\ -46 & 18 & 31 \end{bmatrix}.$$  

5. In general, matrix multiplication is not commutative: $AB \neq BA$.


7. The distributive law of multiplication and addition holds: $C(A+B) = CA + CB$ and $(A+B)C = AC + BC$.

8. Both the transpose operator and the Hermitian operator reverse the order of matrix multiplication: $(ABC)^T = C^T B^T A^T$ and $(ABC)^H = C^H B^H A^H$.

9. Strassen algorithm: The matrix product

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

can be computed in the following way:

\[
\begin{align*}
m_1 &= (a_{12} - a_{22})(b_{21} + b_{22}), \\
m_2 &= (a_{11} + a_{22})(b_{11} + b_{22}), \\
m_3 &= (a_{11} - a_{21})(b_{11} + b_{12}), \\
m_4 &= (a_{11} + a_{12})b_{22}, \\
m_5 &= a_{11}(b_{12} - b_{22}), \\
m_6 &= a_{22}(b_{21} - b_{11}), \\
m_7 &= (a_{21} + a_{22})b_{11},
\end{align*}
\]

\[
\begin{align*}
c_{11} &= m_1 + m_2 - m_4 + m_6, \\
c_{12} &= m_2, \\
c_{21} &= m_6 + m_7, \quad \text{and} \quad (2.5.5) \\
c_{22} &= m_2 - m_3 + m_5 - m_7.
\end{align*}
\]
This computation uses 7 multiplications and 18 additions and subtractions. Using this formula recursively allows multiplication of two \( n \times n \) matrices using \( O(n^{\log_2 7}) = O(n^{2.807...}) \) scalar multiplications.

10. The order in which matrices are grouped together for multiplication can change the number of scalar multiplications required. The straightforward number of scalar multiplications required to multiply matrix \( X_{a \times b} \) by matrix \( Y_{b \times c} \) is \( abc \), without using clever algorithms such as Strassen’s. For example, consider the matrix product \( P = A_{10 \times 100}B_{100 \times 5}C_{5 \times 50} \). The parenthesization \( P = ((AB)C) \) requires \((10 \times 100 \times 5) + (10 \times 5 \times 50) = 7,500 \) scalar multiplications. The parenthesization \( P = (A(BC)) \) requires \((10 \times 100 \times 50) + (100 \times 5 \times 50) = 75,000 \) scalar multiplications.

### 2.5.4 DETERMINANTS AND PERMANENTS

1. The determinant of a square matrix \( A \), denoted by \( |A| \) or \( \det(A) \), is a scalar function of \( A \) defined as
   \[
   \det(A) = \sum_\sigma \text{sgn}(\sigma)a_{1,\sigma(1)}a_{2,\sigma(2)}\cdots a_{n,\sigma(n)} \tag{2.5.6}
   \]
   where the sum is taken over all permutations \( \sigma \) of \( \{1, 2, \ldots, n\} \). The signum function \( \text{sgn}(\sigma) \) is the number of successive transpositions required to change the permutation \( \sigma \) to the identity permutation. Note these properties of determinants:
   \[|A| |B| = |AB| \text{ and } |A| = |A^T|.\]

2. For a \( 2 \times 2 \) matrix,
   \[
   \begin{vmatrix}
   a_{11} & a_{12} \\
   a_{21} & a_{22}
   \end{vmatrix}
   = a_{11}a_{22} - a_{12}a_{21}.
   \]
   For a \( 3 \times 3 \) matrix,
   \[
   \begin{vmatrix}
   a_{11} & a_{12} & a_{13} \\
   a_{21} & a_{22} & a_{23} \\
   a_{31} & a_{32} & a_{33}
   \end{vmatrix}
   = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}
   - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}.
   \tag{2.5.7}
   \]
   In general, for \( m = n \),
   \[
   \begin{vmatrix}
   a_{11} & a_{12} & \cdots & a_{1n} \\
   a_{21} & a_{22} & \cdots & a_{2n} \\
   \vdots & \vdots & \ddots & \vdots \\
   a_{n1} & a_{n2} & \cdots & a_{nn}
   \end{vmatrix}
   = \sum_{\sigma} (-1)^\delta a_{i_{\sigma(1)}i_{\sigma(1)}} a_{i_{\sigma(2)}i_{\sigma(2)}} \cdots a_{i_{\sigma(n)}i_{\sigma(n)}} \tag{2.5.8}
   \]
   where the sum is over all permutations \( i_1 \neq i_2 \neq \cdots \neq i_n \), and \( \delta \) denotes the number of transpositions necessary to bring the sequence \( (i_1, i_2, \ldots, i_n) \) back into the natural order \( (1, 2, \ldots, n) \).

3. Interchanging two rows (or columns) of a matrix changes the sign of its determinant.

4. A determinant does not change its value if a linear combination of other rows (or columns) is added to or subtracted from any given row (or column).
5. Multiplying an entire row (or column) of \( A \) by a scalar \( \gamma \) causes the determinant to be multiplied by the same scalar \( \gamma \).

6. For an \( n \times n \) matrix \( A \), \( |\gamma A| = \gamma^n |A| \).

7. If \( \det (A) = 0 \), then \( A \) is singular; if \( \det (A) \neq 0 \), then \( A \) is nonsingular or invertible. The determinant of the identity matrix is one.

8. \( \det (A^{-1}) = 1/\det (A) \).

9. The size of the determinant of a square matrix \( A \) is not related to the condition number of \( A \).

10. When the edges of a parallelepiped \( P \) are defined by the rows (or columns) of \( A \), the absolute value of the determinant of \( A \) measures the volume of \( P \). Thus, if any row (or column) of \( A \) is dependent upon another row (or column) of \( A \), the determinant of \( A \) equals zero.

11. The cofactor of a square matrix \( A \), \( \text{cof}_{ij}(A) \), is the determinant of a submatrix obtained by striking the \( i \)th row and the \( j \)th column of \( A \) and choosing a positive (negative) sign if \( i + j \) is even (odd).

**Example 2.5.7**

\[
\text{cof}_{23} \begin{bmatrix} 2 & 4 & 3 \\ 6 & 1 & 5 \\ -2 & 1 & 3 \end{bmatrix} = (-1)^{2+3} \begin{vmatrix} 2 & 4 \\ -2 & 1 \end{vmatrix} = -(2 + 8) = -10.
\]

12. Let \( a_{ij} \) denote the components of \( A \) and \( a^{ij} \) those of \( A^{-1} \). Then,

\[
a^{ij} = \frac{\text{cof}_{ji}(A)}{|A|}. \tag{2.5.9}
\]

13. Partitioning of determinants: Let \( A = \begin{bmatrix} B & C \\ D & E \end{bmatrix} \). Assuming all inverses exist, then

\[
|A| = |E| \cdot |B - CE^{-1}D| = |B| \cdot |(E - DB^{-1}C)|. \tag{2.5.10}
\]

14. Laplace development: The determinant of \( A \) is a combination of row \( i \) (column \( j \)) and the cofactors of row \( i \) (column \( j \)), i.e.,

\[
|A| = a_{i1} \text{cof}_{11}(A) + a_{i2} \text{cof}_{12}(A) + \cdots + a_{in} \text{cof}_{in}(A), \tag{2.5.11}
\]

\[
= a_{1j} \text{cof}_{1j}(A) + a_{2j} \text{cof}_{2j}(A) + \cdots + a_{nj} \text{cof}_{nj}(A),
\]

for any row \( i \) or any column \( j \).

15. Omitting the signum function in Equation (2.5.6) yields the definition of permanent of \( A \), given by \( \text{per} A = \sum_{\sigma} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} \). Properties of the permanent include:

(a) If \( A \) is an \( m \times n \) matrix and \( B \) is an \( n \times m \) matrix, then

\[
|\text{per}(AB)|^2 \leq \text{per}(A^{H}A) \text{per}(B^{H}B). \tag{2.5.12}
\]
(b) If \( P \) and \( Q \) are permutation matrices, then \( \text{per} PAQ = \text{per} A \).
(c) If \( D \) and \( G \) are diagonal matrices, then \( \text{per} DAG = \text{per} D \text{ per } A \text{ per } G \).

### 2.5.5 MATRIX NORMS

1. The mapping \( g : \mathbb{R}^{m \times n} \Rightarrow \mathbb{R} \) is a matrix norm if \( g \) satisfies the same three properties as a vector norm:
   - \( g(A) \geq 0 \) for \( A \neq 0 \) and \( g(A) = 0 \) if and only if \( A \equiv 0 \), so that (in norm notation) \( \|A\| > 0 \) for all nonzero \( A \).
   - For two matrices \( A, B \in \mathbb{R}^{m \times n} \), \( g(A + B) \leq g(A) + g(B) \), so that \( \|A + B\| \leq \|A\| + \|B\| \).
   - \( g(rA) = |r| g(A) \), where \( r \in \mathbb{R} \), so that \( \|\gamma A\| = |\gamma| \|A\| \).

2. The most common matrix norms are the \( L_p \) matrix norm and the Frobenius norm. The \( L_p \) norm of a matrix \( A \) is the number defined by
   \[
   \|A\|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}
   \]  
   (2.5.13)

   where \( \|\cdot\|_p \) represents one of the \( L_p \) (vector) norms with \( p = 1, 2, \text{ or } \infty \).

3. The matrix 1-norm of \( A_{m \times n} \) is defined as \( \|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}| \).

4. The matrix 2-norm of \( A \) is the square root of the greatest eigenvalue of \( A^T A \), i.e., \( \|A\|_2^2 = \lambda_{\text{max}}(A^T A) \), which is the same as the largest singular value of \( A \), \( \|A\|_2 = \sigma_1(A) \). When \( A \) is symmetric, then \( \|A\| = \max |\lambda_j| \), where \( \lambda_j \) is the largest eigenvalue of \( A \).

5. The \( L_\infty \) norm is defined as \( \|A\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}| \).

6. The following properties hold:
   \[
   \frac{1}{\sqrt{m}} \|A\|_1 \leq \|A\|_2 \leq \sqrt{n} \|A\|_1 ,
   \]
   \[
   \max_{i,j} |a_{ij}| \leq \|A\|_2 \leq \sqrt{mn} \max_{i,j} |a_{ij}| , \quad \text{and} \quad (2.5.14)
   \]
   \[
   \frac{1}{\sqrt{n}} \|A\|_\infty \leq \|A\|_2 \leq \sqrt{m} \|A\|_\infty .
   \]

7. The matrix \( p \) norms satisfy the additional property of consistency, defined as \( \|AB\|_p \leq \|A\|_p \|B\|_p \).

8. The Frobenius or Hilbert-Schmidt norm of a matrix \( A \) is a number defined by
   \[
   \|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}
   \]  
   (2.5.15)
which satisfies $\|A\|_F^2 = \text{trace}(A^T A)$. Since $\mathbb{R}^{m \times n}$ is isomorphic to $\mathbb{R}^{mn}$, the Frobenius norm can be interpreted as the $L_2$ norm of an $nm \times 1$ column vector in which each column of $A$ is appended to the next in succession. See Section 2.5.20.

9. The Frobenius norm is compatible with the vector 2 norm, i.e., $\|Ax\|_F \leq \|A\|_F \|x\|_2$. Additionally, the Frobenius norm satisfies the condition $\|A\|_2 \leq \|A\|_F \leq \sqrt{n} \|A\|_2$.

### 2.5.6 SINGULARITY, RANK, AND INVERSES

1. An $n \times n$ matrix $A$ is called singular if there exists a vector $x \neq 0$ such that $Ax = 0$ or $A^T x = 0$. (Note that $x = 0$ means that all components of $x$ are zero).
   If a matrix is not singular, it is called nonsingular.
2. $(AB)^{-1} = B^{-1} A^{-1}$, provided all inverses exist.
3. $(A^{-1})^T = (A^T)^{-1}$.
4. $(\gamma A)^{-1} = (1/\gamma)A^{-1}$.
5. If $D_w$ is a diagonal matrix, then $D_w^{-1} = D_{1/w}$.
6. Partitioning: Let $A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$. Assuming that all inverses exist, then $A^{-1} = \begin{bmatrix} X & Y \\ Z & U \end{bmatrix}$, where
   
   $X = (B - CE^{-1})^{-1}, \quad U = (E - DB^{-1}C)^{-1}, \quad (2.5.16)$
   $Y = -B^{-1}CU, \quad Z = -E^{-1}DX. \quad (2.5.17)$
7. If $A$ and $B$ are both invertible, then
   
   $$(A + B)^{-1} = B^{-1} (A^{-1} + B^{-1})^{-1} A^{-1} = A^{-1} (A^{-1} + B^{-1})^{-1} B^{-1}. \quad (2.5.18)$$
8. The row rank of a matrix $A$ is defined as the number of linearly independent rows of $A$. Likewise, the column rank equals the number of linearly independent columns of $A$.
9. For any matrix, the row rank equals the column rank.
10. If $A \in \mathbb{R}^{n \times n}$ has rank of $n$, then $A$ is said to have full rank.
11. A square matrix is invertible if, and only if, it has full rank.
12. $\text{Rank}(AB) \leq \min \{ \text{rank}(A), \text{rank}(B) \}$.
13. $\text{Rank}(A^T A) = \text{rank}(AA^T) = \text{rank}(A)$.

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2.5.7 SYSTEMS OF LINEAR EQUATIONS

1. Suppose that \( A \) is a matrix. Then \( Ax = b \) is a system of linear equations. If \( A \)
   is square and nonsingular, there exists a unique solution \( x = A^{-1}b \).

2. For the linear system of equations involving \( n \) variables and \( m \) equations, written as \( Ax = c \) or

   \[
   a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = c_1, \\
   a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = c_2, \\
   \vdots \\
   a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = c_m,
   \]

   (2.5.19)

   three possible outcomes exist for the simultaneous determination of a solution:

   - No such solution exists and the system is called **inconsistent**.
   - A unique solution exists, and the system is called **consistent**.
   - Multiple solutions exist, the system has an infinite number of solutions, and the system is called **undetermined**.

3. For a system of linear equations \( Ax = b \) (for which \( A \) is nonsingular), the
   sensitivity of the solution \( x \) to perturbations in \( A \) and \( b \) is given in terms of the
   **condition number** of \( A \) defined by

   \[
   \text{cond}(A) = \| A^{-1} \| \| A \|. 
   \]
   (2.5.20)

   where \( \| \cdot \| \) is any of the \( p \) norms. In all cases, \( \text{cond}(A) \geq 1 \). When \( \text{cond}(A) \)
   is equal to one, \( A \) is said to be **perfectly conditioned**. Matrices with small
   condition numbers are called **well-conditioned**. If \( \text{cond}(A) \) is large, then \( A \) is
   called **ill-conditioned**.

4. The size of the determinant of a square matrix \( A \) is **not** related to the condition
   number of \( A \).

5. When \( A \) is singular, the definition of condition number is modified slightly, incor-
   porating the generalized, or pseudo inverse, of \( A \), and is defined by \( \text{cond}(A) = \| A^+ \| \| A \| \), where \( A^+ \) represents the pseudo inverse of \( A \) (see **Section 2.5.11**).

6. Let \( A = (a_{ij}) \) be an \( n \times n \) matrix. Using the \( L_2 \) condition number, \( \text{cond} A = \max_{j} |\lambda_j(A)| / \min_{j} |\lambda_j(A)| $:
Matrix $A_{n \times n} = (a_{ij})$

<table>
<thead>
<tr>
<th>Condition number</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{ij} = \sqrt{2/(n+1)} \sin((ij\pi)/(n+1))$</td>
</tr>
<tr>
<td>$a_{ij} = n \delta_{ij} + 1$</td>
</tr>
<tr>
<td>$a_{ij} = (i + j)/p$, $n = p - 1$, $p$ a prime</td>
</tr>
<tr>
<td>The circulant whose first row is $(1, 2, \ldots, n)$</td>
</tr>
<tr>
<td>$a_{ij} = \frac{i}{j}$ if $i \leq j$</td>
</tr>
<tr>
<td>$j/i$ if $i &gt; j$</td>
</tr>
<tr>
<td>$-2$ if $i = j$</td>
</tr>
<tr>
<td>$0$ if $</td>
</tr>
<tr>
<td>$1$ if $</td>
</tr>
<tr>
<td>$a_{ij} = 2 \text{min}(i, j) - 1$</td>
</tr>
<tr>
<td>$a_{ij} = (i + j - 1)^{-1}$ (Hilbert matrix)</td>
</tr>
</tbody>
</table>

### 2.5.8 OTHER MATRIX TRANSFORMATIONS

1. A Householder transformation, or Householder reflection, is an $n \times n$ matrix $H$ of the form $H = I - (2uu^T)/(u^Tu)$, where the Householder vector $u \in \mathbb{R}^n$ is nonzero.

2. A Givens rotation is defined as a rank two correction to the identity matrix given by

$$
\mathbf{G}(i, k, \theta) = \begin{bmatrix}
1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & c & \cdots & s & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & -s & \cdots & c & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 1
\end{bmatrix}
$$

where $c = \cos \theta$ and $s = \sin \theta$ for some angle $\theta$. Premultiplication by $\mathbf{G}(i, k, \theta)^T$ induces a counterclockwise rotation of $\theta$ radians in the $(i, k)$ plane. For $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} = \mathbf{G}(i, k, \theta)^T \mathbf{x}$, the components of $\mathbf{y}$ are given by

$$
y_j = \begin{cases}
c x_i - s x_k, & \text{for } j = i \\
s x_i + c x_k, & \text{for } j = k \\
x_j, & \text{for } j \neq i, k.
\end{cases}
$$
2.5.9 LINEAR SPACES AND LINEAR MAPPINGS

1. The projection matrix, $P$, onto a subspace $S$ of the nonzero $m \times n$ matrix $A$, denoted by $P_S$, is the unique $m \times m$ matrix possessing the three properties:
   
   (a) $P_S = P_S^T$;
   
   (b) $P_S^2 = P_S$ (the projection matrix is idempotent);
   
   (c) The vector $b_S$ lies in the subspace $S$ if, and only if, $b_S = P_Sv$ for some $m$-dimensional vector $v$. In other words, $b_S$ can be written as a linear combination of the columns of $P_S$.

2. Let $R(A)$ and $N(A)$ denote, respectively, the range space and null space of an $m \times n$ matrix $A$. They are defined by

   $$ R(A) = \{ y \mid y = Ax; x \in \mathbb{R}^n \} $$

   and

   $$ N(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \}. $$

3. When the $m \times n$ matrix $A$ has rank $n$, the projection of $A$ onto the four subspaces of $A$ is given by

   $$ P_{R(A)} = A(A^TA)^{-1}A^T, $$
   
   $$ P_{R(A^T)} = I, $$
   
   $$ P_{N(A)} = I, $$
   
   $$ P_{N(A^T)} = I - A(A^TA)^{-1}A^T. $$

   When $A$ is of rank $m$, the projection of $A$ onto the four subspaces of $A$ is given by

   $$ P_{R(A)} = I, $$
   
   $$ P_{R(A^T)} = A^T(AA^T)^{-1}A, $$
   
   $$ P_{N(A)} = I - A^T(AA^T)^{-1}A, $$
   
   $$ P_{N(A^T)} = I - AA^T. $$

4. When $A$ is not of full rank, the matrix $\tilde{A}$ satisfies the requirements for a projection matrix. The matrix $\tilde{A}$ is the coefficient matrix of the system of equations $x_+ = \tilde{A}b$, generated by the least squares problem $\min \| b - Ax \|_2^2$. Thus,

   $$ P_{R(A)} = \tilde{A}\tilde{A}, $$
   
   $$ P_{R(A^T)} = \tilde{A}\tilde{A}, $$
   
   $$ P_{N(A)} = I - \tilde{A}\tilde{A}, $$
   
   $$ P_{N(A^T)} = I - \tilde{A}\tilde{A}. $$

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5. A matrix \( B \in \mathbb{R}^{n \times n} \) is called similar to a matrix \( A \in \mathbb{R}^{n \times n} \) if \( B = T^{-1}AT \) for some nonsingular matrix \( T \).

6. If \( B \) is similar to \( A \), then \( B \) has the same eigenvalues as \( A \).

7. If \( B \) is similar to \( A \) and if \( x \) is an eigenvector of \( A \), then \( y = T^{-1}x \) is an eigenvector of \( B \) corresponding to the same eigenvalue.

### 2.5.10 TRACES

1. The trace of an \( n \times n \) matrix \( A \), usually denoted as \( \text{tr}(A) \), is defined as the sum of the \( n \) diagonal components of \( A \).

2. The trace of an \( n \times n \) matrix \( A \) equals the sum of the \( n \) eigenvalues of \( A \), i.e.,
   \[
   \text{tr}A = a_{11} + a_{22} + \cdots + a_{nn} = \lambda_1 + \lambda_2 + \cdots + \lambda_n.
   \]

3. The trace of a 1 \( \times \) 1 matrix, a scalar, is itself.

4. If \( A \in \mathbb{R}^{m \times k} \) and \( B \in \mathbb{R}^{k \times m} \), then \( \text{tr}(AB) = \text{tr}(BA) \).

5. If \( A \in \mathbb{R}^{m \times k} \), \( B \in \mathbb{R}^{k \times r} \), and \( C \in \mathbb{R}^{r \times m} \), then \( \text{tr}(ABC) = \text{tr}(BCA) = \text{tr}(CAB) \). For example, if \( B = b \) is a column vector and \( C = c^T \) is a row vector, then \( \text{tr}(Abe^T) = \text{tr}(bc^TA) = \text{tr}(c^TAb) \).

6. \( \text{tr}(A + \gamma B) = \text{tr}(A) + \gamma \text{tr}(B) \), where \( \gamma \) is a scalar.

7. \( \text{tr}(AB) = (\text{Vec} A^T)^T \text{Vec} B \) (see Section 2.5.20).

### 2.5.11 GENERALIZED INVERSES

1. Every matrix \( A \) (singular or nonsingular, rectangular or square) has a generalized inverse, or pseudoinverse, \( A^+ \) defined by the Moore–Penrose conditions

\[
\begin{align*}
AA^+A &= A, \\
A^+AA^+ &= A^+, \\
(AA^+)^T &= AA^+, \\
(A^+A)^T &= A^+A. 
\end{align*}
\]

(2.5.26)

2. Only if \( A \) is square and nonsingular, \( A^+ \) will be unique and \( A^+ = A^{-1} \). Otherwise, there will exist infinitely many matrices \( A^+ \) that will satisfy the defining relations.

3. If \( A \) is a rectangular \( m \times n \) matrix of rank \( n \), with \( m > n \), then \( A^+ \) is of order \( n \times m \) and \( A^+A = I \in \mathbb{R}^{n \times n} \). In this case \( A^+ \) is called a left inverse, and \( AA^+ \neq I \).

4. If \( A \) is a rectangular \( m \times n \) matrix of rank \( m \), with \( m < n \), then \( A^+ \) is of order \( n \times m \) and \( AA^+ = I \in \mathbb{R}^{m \times m} \). In this case \( A^+ \) is called a right inverse, and \( A^+A \neq I \).

5. For a square singular matrix \( A \), \( AA^+ \neq I \), and \( A^+A \neq I \).
6. The least squares problem is to find the \(x\) that minimizes \(\|y - Ax\|\). The \(x\) of least norm is \(x = A^+y\).

7. A square matrix is called \textit{idempotent} if \(AA = A^2 = A\).

8. \(AA^+\) and \(A^+A\) are idempotent.

9. Let \(A\) be of rank \(r\); and select \(r\) rows and \(r\) columns which form a basis of \(A\). Then, a pseudoinverse of \(A\) can be obtained as follows: invert the \(r \times r\) matrix, place the inverse (without transposing) into the \(r\) rows corresponding to the column numbers and the \(r\) columns corresponding to the row numbers of the basis, and place zero into the remaining component positions. Thus, if \(A\) is of order 5 \(\times\) 4 and rank 3, for example, and if rows 1, 2, 4 and columns 2, 3, 4 are selected as a basis, \(A^+\) of order 4 \(\times\) 5 will contain the inverse components of the basis in rows 2, 3, 4 and column 1, 2, 4, and zeros elsewhere.

10. A pseudoinverse can also be computed for a general matrix \(A_{m \times n}\) using the singular value decomposition \(A = U\Sigma V^T\), where \(U_{m \times m}\) and \(V_{n \times n}\) are orthogonal matrices. When \(A\) is of rank \(r > 0\), \(\Sigma_{m \times n}\) will have exactly \(r\) positive singular values \((\sigma_i)\) along the main diagonal extending from the upper left-hand corner. The remaining components of \(\Sigma\) are zero. The pseudoinverse of \(A\) is then given by \(A^+ = (U\Sigma V^T)^+ = (V^T\Sigma^+)^+U^T\), with \((V^T)^+ = V\) and \(U^+ = U^T\) because of the orthogonality of \(V\) and \(U\). The components \(\sigma_i^+\) of the pseudoinverse of \(\Sigma\) (rectangular, in this case) are given by

\[
\sigma_i^+ = \begin{cases} 
1/\sigma_i, & \text{if } \sigma_i \neq 0; \\
0, & \text{if } \sigma_i = 0.
\end{cases}
\tag{2.5.27}
\]

11. The pseudoinverse is ill-conditioned with respect to rank changing perturbations. For example

\[
\left(\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} + \epsilon \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)^+ = \frac{1}{\epsilon^2} \begin{bmatrix} -1 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix} + \frac{1}{\epsilon} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.
\]

2.5.12 \textbf{EIGENSTRUCTURE}

1. If \(A\) is a square \(n \times n\) matrix, then the \(n\)th degree polynomial defined by \(\det(A - \lambda I) = 0\) is called the \textit{characteristic polynomial}, or \textit{characteristic equation} of \(A\).

2. The \(n\) roots (not necessarily distinct) of the characteristic polynomial are called the \textit{eigenvalues} (or characteristic roots) of \(A\). Therefore, the values, \(\lambda_i, i = 1, \ldots, n\), are eigenvalues if, and only if, \(|A - \lambda_i I| = 0\).
3. The characteristic polynomial det \((A - \lambda I) = \sum_{i=0}^{n} r_i \lambda^i\) has the properties
\[
\begin{align*}
    r_n &= (-1)^n \\
    r_{n-1} &= -r_n \, \text{tr} \,(A) \\
    r_{n-2} &= -\frac{1}{2} \left[ r_{n-1} \, \text{tr} \,(A) + r_n \, \text{tr} \,(A^2) \right] \\
    r_{n-3} &= -\frac{1}{3} \left[ r_{n-2} \, \text{tr} \,(A) + r_{n-1} \, \text{tr} \,(A^2) + r_n \, \text{tr} \,(A^3) \right] \\
    &\vdots \\
    r_0 &= -\frac{1}{n} \left[ \sum_{j=1}^{n-1} r_{n-j} \, \text{tr} \,(A^j) \right]
\end{align*}
\]

4. Each eigenvalue \(\lambda\) has a corresponding eigenvector \(x\) that solves the system
\[\begin{align*}
    Ax &= \lambda x, \\
    (A - \lambda I)x &= 0.
\end{align*}\]

5. If \(x\) solves \(Ax = \lambda x\), then so does \(\gamma x\), where \(\gamma\) is an arbitrary scalar.


7. The eigenvalues of a triangular (or diagonal) matrix are the diagonal components of the matrix.

8. The eigenvalues of idempotent matrices are either zero or one.

9. If \(A\) is a real matrix with positive eigenvalues, then
\[
    \lambda_{\min}(AA^T) \leq [\lambda_{\min}(A)]^2 \leq [\lambda_{\max}(A)]^2 \leq \lambda_{\max}(AA^T),
\]
where \(\lambda_{\min}\) denotes the smallest and \(\lambda_{\max}\) the largest eigenvalue.

10. If all the eigenvalues of a real symmetric matrix are distinct, then their associated eigenvectors are also distinct (linearly independent).

11. Symmetric and Hermitian matrices have real eigenvalues.

12. For an \(n \times n\) matrix \(A\) with eigenvalues \(\lambda_1, \lambda_2, \ldots, \lambda_n\), the product \(\lambda_1 \lambda_2 \cdots \lambda_n = \det (A)\).

### 2.5.13 EIGENVALUE DIAGONALIZATION

1. If \(A \in \mathbb{R}^{n \times n}\) possesses \(n\) linearly independent eigenvectors \(x_1, \ldots, x_n\), then \(A\) can be diagonalized as \(S^{-1}AS = \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)\), where the eigenvectors of \(A\) are chosen to comprise the columns of \(S\).

2. If \(A \in \mathbb{R}^{n \times n}\) can be diagonalized into \(S^{-1}AS = \Lambda\), then \(A^k = S \Lambda^k S^{-1}\), or \(\Lambda^k = S^{-1}A^k S\).

3. Spectral decomposition: Any real symmetric matrix \(A \in \mathbb{R}^{n \times n}\) can be diagonalized into the form \(A = U \Lambda U^T\), where \(\Lambda\) is the diagonal matrix of ordered eigenvalues of \(A\) such that \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\), and the columns of \(U\) are the corresponding \(n\) orthonormal eigenvectors of \(A\).
4. The *spectral radius* of a real symmetric matrix \( A \), commonly denoted by \( \rho(A) \), is defined as \( \rho(A) = \max_{1 \leq i \leq n} |\lambda_i(A)| \).

5. If \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times n} \) are diagonalizable, then they share a common eigenvector matrix \( S \) if and only if \( AB = BA \). (Not every eigenvector of \( A \) need be an eigenvector for \( B \), e.g., \( A = I \).)

6. Schur decomposition: If \( A \in \mathbb{C}^{n \times n} \), then a unitary matrix \( Q \in \mathbb{C}^{n \times n} \) exists such that \( Q^H A Q = D + N \), where \( D = \text{diag}(\lambda_1, \ldots, \lambda_n) \) and \( N \in \mathbb{C}^{n \times n} \) is strictly upper triangular. The matrix \( Q \) can be chosen so that the eigenvalues \( \lambda_i \) appear in any order along the diagonal.

7. If \( A \in \mathbb{R}^{n \times n} \) and symmetric, then a real orthogonal matrix \( Q \) exists such that \( Q^T A Q = \text{diag}(\lambda_1, \ldots, \lambda_n) \).

8. If \( A \in \mathbb{R}^{n \times n} \) possesses \( s \leq n \) linearly independent eigenvectors, it is similar to a matrix with \( s \) *Jordan blocks*

\[
\mathbf{J} = M^{-1} AM = \begin{bmatrix}
\mathbf{J}_1 & 0 \\
0 & \mathbf{J}_s
\end{bmatrix},
\]

where each Jordan block \( \mathbf{J}_i \) is an upper triangular matrix with only a single eigenvalue \( \lambda_i \) and a single eigenvector:

\[
\mathbf{J}_i = \begin{bmatrix}
\lambda_i & 1 & 0 \\
0 & \lambda_i & 1 \\
0 & 0 & \lambda_i
\end{bmatrix}
\]  

(2.5.29)

If \( \mathbf{J}_i \) has \( m > 1 \), then \( \lambda_i \) repeats \( m \) times along the main diagonal, with \((m-1)\) 1’s appearing above the diagonal entries, and all other components are equal to zero.

### 2.5.14 MATRIX EXPONENTIALS

1. Matrix exponentiation is defined as (the series always converges):

\[
e^{A t} = I + A t + \frac{(A t)^2}{2!} + \frac{(A t)^3}{3!} + \cdots.
\]

2. Common properties of matrix exponentials are

   (a) \( \left( e^{A t} \right) \left( e^{B t} \right) = e^{(A + B) t} \),

   (b) \( \left( e^{A t} \right) \left( e^{-A t} \right) = I \),

   (c) \( \frac{d}{dt} e^{A t} = A e^{A t} \), and
when \( A \) and \( B \) are square matrices, the commutator of \( A \) and \( B \) is 
\[
C = [B, A] = BA - AB.
\]
Then \( e^{(A+B)} = e^A e^B e^{C/2} \) provided that \([C, A] = [C, B] = 0\) (i.e., each of \( A \) and \( B \) commute with their commutator).

3. For a matrix \( A \in \mathbb{R}^{n \times n} \), the determinant of \( e^A \) is given by
\[
\det (e^A) = e^{\text{trace}(A)}.
\]

4. The diagonalization of \( e^A \) is given by \( e^A = S e^D S^{-1} \) where the columns of \( S \) consist of the eigenvectors of \( A \), and the entries of the diagonal matrix \( D \) are the corresponding eigenvalues of \( A \), that is, \( A = SDS^{-1} \).

5. If \( A \) is skew-symmetric, then \( e^A \) is an orthogonal matrix.

### 2.5.15 Quadratic Forms

1. For a symmetric matrix \( A \), the product \( x^T A x \) is called a pure quadratic form. It has the form
\[
x^T A x = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j = a_{11} x_1^2 + a_{12} x_1 x_2 + a_{21} x_2 x_1 + \cdots + a_{nn} x_n^2.
\]

2. For \( A \) symmetric, the gradient of \( x^T A x / x^T x \) equals zero if, and only if, \( x \) is an eigenvector of \( A \). Thus, the stationary values for this expression are the eigenvalues of \( A \).

3. The ratio of two quadratic forms (\( B \) nonsingular) \( u(x) = (x^T A x) / (x^T B x) \) attains stationary values at the eigenvalues of \( B^{-1} A \). In particular,
\[
\lambda_{\max} = \lambda_{\max}(B^{-1}A), \quad \text{and} \quad \lambda_{\min} = \lambda_{\min}(B^{-1}A).
\]

4. A matrix \( A \) is positive definite if \( x^T A x > 0 \) for all \( x \neq 0 \).

5. A matrix \( A \) is positive semidefinite if \( x^T A x \geq 0 \) for all \( x \).

6. For a real, symmetric matrix \( A \in \mathbb{R}^{n \times n} \), the following are necessary and sufficient conditions to establish the positive definiteness of \( A \):
   (a) All eigenvalues, or characteristic roots, of \( A \) have \( \lambda_i > 0 \), for \( i = 1, \ldots, n \), and
   (b) The upper-left submatrices of \( A \), defined by
\[
A_1 = [a_{11}], \quad A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \ldots,
\]
\[
A_n = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},
\]
have \( \det A_k > 0 \), for all \( k = 1, \ldots, n \).

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7. If \( A \) is positive definite, then all of the principal submatrices of \( A \) are also positive definite. Additionally, all diagonal entries of \( A \) are positive.

8. For a real, symmetric matrix \( A \in \mathbb{R}^{n \times n} \), the following are necessary and sufficient conditions to establish the positive semidefiniteness of \( A \):
   - All eigenvalues, or characteristic roots, of \( A \) have \( \lambda_i \geq 0 \), for \( i = 1, \ldots, n \), and
   - The principal submatrices of \( A \) have \( \det A_k \geq 0 \), for all \( k = 1, \ldots, n \).

9. If \( A \) is positive definite, then all of the principal submatrices of \( A \) are also positive definite. Additionally, all diagonal entries of \( A \) are nonnegative.

10. If \( x^T \) denotes the radius vector (running coordinates \( [x, y, z] \)), and if a matrix \( Q \) is positive definite, then \( (x^T - x_0^T)Q^{-1}(x - x_0) = 1 \) is the equation of an ellipsoid with its center at \( [x_0, y_0, z_0] = x_0^T \) and semiaxes equal to the square roots of the eigenvalues of \( Q \).

### 2.5.16 MATRIX FACTORIZATIONS

1. Singular value decomposition (SVD): Any \( m \times n \) matrix \( A \) can be written as the product \( A = U\Sigma V^T \), where \( U \) is an \( m \times m \) orthogonal matrix, \( V \) is an \( n \times n \) orthogonal matrix, and \( \Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_p) \), with \( p = \min(m, n) \) and \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0 \). The values \( \sigma_i \), \( i = 1, \ldots, p \), are called the singular values of \( A \).

2. When \( \text{rank}(A) = r > 0 \), \( A \) has exactly \( r \) positive singular values, and \( \sigma_{r+1} = \cdots = \sigma_p = 0 \).

3. When \( A \) is a symmetric \( n \times n \) matrix, then \( \sigma_1 = |\lambda_1|, \ldots, \sigma_n = |\lambda_n| \), where \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the eigenvalues of \( A \).

4. When \( A \) is an \( m \times n \) matrix, if \( m \geq n \) then the singular values of \( A \) are the square roots of the eigenvalues of \( A^T A \). Otherwise, they are the square roots of the eigenvalues of \( A A^T \).

5. Any \( m \times n \) matrix \( A \) can be factored as \( PA = LU \), where \( P \) is a permutation matrix, \( L \) is lower triangular, and \( U \) is an \( m \times n \) matrix in echelon form.

6. QR factorization: If all the columns of \( A \in \mathbb{R}^{m \times n} \) are linearly independent, then \( A \) can be factored as \( A = QR \), where \( Q \in \mathbb{R}^{m \times n} \) has orthonormal columns and \( R \in \mathbb{R}^{n \times n} \) is upper triangular and nonsingular.

7. If \( A \in \mathbb{R}^{n \times n} \) is symmetric positive definite, then
\[
A = LDL^T = LD^{1/2}D^{1/2}L^T = (LD^{1/2})(LD^{1/2})^T = GG^T
\]
where \( L \) is a lower triangular matrix and \( D \) is a diagonal matrix. The factorization \( A = GG^T \) is called the Cholesky factorization, and the matrix \( G \) is commonly referred to as the Cholesky triangle.
2.5.17 THEOREMS

1. **Frobenius–Perron theorem:** If $A > 0$, then there exists a $\lambda_0 > 0$ and $x_0 > 0$ such that
   
   (a) $Ax_0 = \lambda_0 x_0$,
   
   (b) if $\lambda$ is any other eigenvalue of $A$, $\lambda \neq \lambda_0$, then $|\lambda| < \lambda_0$, and
   
   (c) $\lambda_0$ is an eigenvalue with geometric and algebraic multiplicity equal to one.

2. If $A \geq 0$, and $A^k > 0$ for some positive integer $k$, then the results of the Frobenius–Perron theorem apply to $A$.

3. **Courant–Fischer minimax theorem:** If $\lambda_i(A)$ denotes the $i^{\text{th}}$ largest eigenvalue of a matrix $A = A^T \in \mathbb{R}^{n \times n}$, then
   
   $\lambda_j(A) = \max_{S_j} \min_{x \neq 0 \in S_j} \frac{x^T Ax}{x^T x} \quad j = 1, \ldots, n$

   where $x \in \mathbb{R}^n$ and $S_j$ is a $j$-dimensional subspace.

4. **Cramer’s rule:** The $j^{\text{th}}$ component of $x = A^{-1} b$ is given by
   
   $x_j = \frac{\det B_j}{\det A}$, \quad where \quad $B_j = \begin{bmatrix} a_{11} & a_{12} & b_1 & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & b_n & a_{nn} \end{bmatrix}$.

   The vector $b = (b_1, \ldots, b_n)^T$ replaces the $j^{\text{th}}$ column of the matrix $A$ to form the matrix $B_j$.

5. **Sylvester’s law of inertia:** For a symmetric matrix $A \in \mathbb{R}^{n \times n}$, the congruence $A \Rightarrow C^T AC$, for $C$ nonsingular, has the same number of positive, negative, and zero eigenvalues.

6. **Raleigh’s principle:** The quotient $R(x) = x^T Ax / x^T x$ is minimized by the eigenvector $x = x_1$ corresponding to the smallest eigenvalue $\lambda_1$ of $A$. The minimum of $R(x)$ is $\lambda_1$, that is,

   $\min R(x) = \min \frac{x^T Ax}{x^T x} = R(x_1) = \frac{x_1^T Ax_1}{x_1^T x_1} = \frac{x_1^T \lambda_1 x_1}{x_1^T x_1} = \lambda_1$.

2.5.18 KRONECKER PRODUCTS OR TENSOR PRODUCTS

If the matrix $A = (a_{ij})$ has size $m \times n$, and matrix $B = (b_{ij})$ has size $r \times s$, then the Kronecker product of these matrices, denoted $A \otimes B$, is defined as the partitioned matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \ldots & a_{1n}B \\ a_{21}B & a_{22}B & \ldots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \ldots & a_{mn}B \end{bmatrix}.$$
Hence, the \( A \otimes B \) matrix has size \( mr \times ns \).

For example, if
\[
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},
\]
then
\[
A \otimes B = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & \cdots & a_{12}b_{21} & a_{12}b_{22} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{21}b_{11} & a_{21}b_{12} & \cdots & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & \cdots & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}.
\]

The Kronecker product has the following properties:

1. If \( z \) and \( w \) are vectors of appropriate dimensions, then
\[
A z \otimes B w = (A \otimes B)(z \otimes w).
\]
2. If \( \alpha \) is a scalar, then
\[
A \otimes (\alpha B) = \alpha (A \otimes B).
\]
3. The Kronecker product is distributive with respect to addition:
   - \( (A + B) \otimes C = A \otimes C + B \otimes C \), and
   - \( A \otimes (B + C) = A \otimes B + A \otimes C \).
4. The Kronecker product is associative: \( A \otimes (B \otimes C) = (A \otimes B) \otimes C \).
5. \( (A \otimes B)^T = A^T \otimes B^T \).
6. The mixed product rule: If the dimensions of the matrices are such that the expressions following exist, then \( (A \otimes B)(C \otimes D) = AC \otimes BD \).
7. If the inverses exist, then \( (A \otimes B)^{-1} = A^{-1} \otimes B^{-1} \).
8. If \( \{\lambda_i\} \) and \( \{x_i\} \) are the eigenvalues and the corresponding eigenvectors for \( A \), and \( \{\mu_j\} \) and \( \{y_j\} \) are the eigenvalues and the corresponding eigenvectors for \( B \), then \( A \otimes B \) has eigenvalues \( \{\lambda_i \mu_j\} \) with corresponding eigenvectors \( \{x_i \otimes y_j\} \).
9. If matrix \( A \) has size \( n \times n \) and \( B \) has size \( m \times m \), then \( \det(A \otimes B) = (\det A)^m (\det B)^n \).
10. If \( f(z) \) is an analytic function and \( A \) has size \( n \times n \), then
   - \( f(I_n \otimes A) = I_n \otimes f(A) \), and
   - \( f(A \otimes I_n) = f(A) \otimes I_n \).
11. \( \text{tr} (A \otimes B) = (\text{tr} A)(\text{tr} B) \).
12. If \( A, B, C, \) and \( D \) are matrices with \( A \) similar to \( C \) and \( B \) similar to \( D \), then \( A \otimes B \) is similar to \( C \otimes D \).
13. If \( C(t) = A(t) \otimes B(t) \), then
\[
\frac{dC}{dt} = \frac{dA}{dt} \otimes B + A \otimes \frac{dB}{dt}.
\]
2.5.19 KRONECKER SUMS

If the matrix $A = (a_{ij})$ has size $n \times n$ and matrix $B = (b_{ij})$ has size $m \times m$, then the Kronecker sum of these matrices, denoted $A \oplus B$, is defined\(^2\) as $A \oplus B = A \otimes I_m + I_n \otimes B$.

The Kronecker sum has the following properties:

1. If $A$ has eigenvalues $\{\lambda_i\}$ and $B$ has eigenvalues $\{\mu_j\}$, then $A \oplus B$ has eigenvalues $\{\lambda_i + \mu_j\}$.
2. The matrix equation $AX + XB = C$ may be equivalently written as $(B^T \oplus A) \text{Vec } X = \text{Vec } C$, where $\text{Vec}$ is defined in Section 2.5.20.
3. $e^{A \oplus B} = e^A \otimes e^B$.

2.5.20 THE VECTOR OPERATION

The matrix $A_{m \times n}$ can be represented as a collection of $m \times 1$ column vectors: $A = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix}$. Define $\text{Vec } A$ as the matrix of size $nm \times 1$ by

$$\text{Vec } A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}. \quad (2.5.35)$$

This operator has the following properties:

1. $\text{tr } AB = (\text{Vec } A^T)^T \text{Vec } B$.
2. The permutation matrix $U$ that associates $\text{Vec } X$ and $\text{Vec } X^T$ (that is, $\text{Vec } X^T = U \text{Vec } X$) is given by:

$$U = [\text{Vec } E^T_{11} \quad \text{Vec } E^T_{21} \quad \ldots \quad \text{Vec } E^T_{n1}] = \sum_{r,s} E_{rs} \otimes E^T_{rs}. \quad (2.5.36)$$

3. $\text{Vec } (AYB) = (B^T \otimes A) \text{Vec } Y$.
4. If $A$ and $B$ are both of size $n \times n$, then
   (a) $\text{Vec } AB = (I_n \otimes A) \text{Vec } B$.
   (b) $\text{Vec } AB = (B^T \otimes A) \text{Vec } I_n$.

\(^{2}\)Note that $A \otimes B$ is also used to denote the $(m + n) \times (m + n)$ matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. 

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2.6 ABSTRACT ALGEBRA

2.6.1 BASIC CONCEPTS

Definitions

1. A binary operation on a set $S$ is a function $\star : S \times S \rightarrow S$.

2. An algebraic structure $(S, \star_1, \ldots, \star_n)$ consists of a nonempty set $S$ with one or more binary operations $\star_i$, defined on $S$. If the operations are understood, then the binary operations need not be mentioned explicitly.

3. The order of an algebraic structure $S$ is the number of elements in $S$, written $|S|$.

4. Properties that a binary operation $\star$ on an algebraic structure $(S, \star)$ may have are
   - Associate: $a \star (b \star c) = (a \star b) \star c$ for all $a, b, c \in S$.
   - Identity: there exists an element $e \in S$ (identity element of $S$) such that $e \star a = a \star e = a$ for all $a \in S$.
   - Inverse: $a^{-1} \in S$ is an inverse of $a$ if $a \star a^{-1} = a^{-1} \star a = e$.
   - Commutative (or abelian): if $a \star b = b \star a$ for all $a, b \in S$.

5. A semigroup $(S, \star)$ consists of a nonempty set $S$ and an associative binary operation $\star$ on $S$.

6. A monoid $(S, \star)$ consists of a nonempty set $S$ with an identity element and an associative binary operation $\star$.

Examples of semigroups and monoids

1. The sets $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ (natural numbers), $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ (integers), $\mathbb{Q}$ (rational numbers), $\mathbb{R}$ (real numbers), and $\mathbb{C}$ (complex numbers) where $\star$ is either addition or multiplication are semigroups and monoids.

2. The set of positive integers under addition is a semigroup but not a monoid.

3. If $A$ is any nonempty set, then the set of all functions $f : A \rightarrow A$ where $\star$ is the composition of functions is a semigroup and a monoid.

4. Given a set $S$, the set of all strings of elements of $S$, where $\star$ is concatenation of strings, is a monoid (the identity is $\lambda$, the empty string).

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2.6.2 GROUPS

1. A group \((G, \star)\) consists of a set \(G\) with a binary operation \(\star\) defined on \(G\) such that \(\star\) satisfies the associative, identity, and inverse laws. Note: The operation \(\star\) is often written as + (an additive group) or as \(\cdot\) or \(\times\) (a multiplicative group). If + is used, the identity is written 0 and the inverse of \(a\) is written \(-a\). If multiplicative notation is used, \(a \star b\) is often written \(ab\) and the identity is often written 1.

2. The order of \(a \in G\) is the smallest positive integer \(n\) such that \(a^n = 1\) where \(a^n = a \cdot a \cdots a\) (\(n\) times) (or \(a + a + \cdots + a = 0\) if \(G\) is written additively). If there is no such integer, the element has infinite order. In a finite group of order \(n\), each element has some order \(k\) (depending on the particular element) and it must be that \(k\) divides \(n\).

3. \((H, \star)\) is a subgroup of \((G, \star)\) if \(H \subseteq G\) and \((H, \star)\) is a group (using the same binary operation used in \((G, \star)\)).

4. The cyclic subgroup \(<a>\) generated by \(a \in G\) is the subgroup \(\{a^n \mid n \in \mathbb{Z}\} = \{\ldots, a^{-2} = (a^{-1})^2, a^{-1}, a^0 = e, a, a^2, \ldots\}\). The element \(a\) is a generator of \(<a>\). A group \(G\) is cyclic if there is \(a \in G\) such that \(G = <a>\).

5. If \(H\) is a subgroup of a group \(G\), then a left [right] coset of \(H\) in \(G\) is the set \(aH = \{ah \mid h \in H\}\) [\(Ha = \{ha \mid h \in H\}\)].

6. A normal subgroup of a group \(G\) is a subgroup \(H\) such that \(aH = Ha\) for all \(a \in G\).

7. A simple group is a group \(G \neq \{e\}\) with only \(G\) and \(\{e\}\) as normal subgroups.

8. If \(H\) is a normal subgroup of \(G\), then the quotient group (or factor group) of \(G\) modulo \(H\) is the group \(G/H = \{aH \mid a \in G\}\), with binary operation \(aH \cdot bH = (ab)H\).

9. A finite group \(G\) is solvable if there is a sequence of subgroups \(G_1 = G, G_2, \ldots, G_{k-1}, G_k = \{e\}\) such that each \(G_{i+1}\) is a normal subgroup of \(G_i\) and \(G_i/G_{i+1}\) is abelian.

Facts about groups

1. The identity element is unique.
2. Each element has exactly one inverse.
3. Each of the equations \(a \star x = b\) and \(x \star a = b\) has exactly one solution, \(x = a^{-1} \star b\) and \(x = b \star a^{-1}\).
4. \((a^{-1})^{-1} = a\).
5. \((a \star b)^{-1} = b^{-1} \star a^{-1}\).
6. The left [right] cancellation law holds in all groups: If \(a \star b = a \star c\) then \(b = c\) [if \(b \star a = c \star a\) then \(b = c\)].
7. Lagrange’s theorem: If \(G\) is a finite group and \(H\) is a subgroup of \(G\), then the order of \(H\) divides the order of \(G\).
8. Every group of prime order is simple.
9. Every abelian group is solvable.
10. Feit–Thompson theorem: All groups of odd order are solvable. Hence, all finite non-Abelian simple groups have even order.
11. Finite simple groups are of the following types:
   - $\mathbb{Z}_p$ ($p$ prime)
   - A group of Lie type
   - $A_n$ ($n \geq 5$)
   - Sporadic groups (see table)

Examples of groups
1. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, with $\star$ as addition of numbers, are additive groups.
2. For $n$ a positive integer, $n\mathbb{Z} = \{nz | z \in \mathbb{Z}\}$ is an additive group.
3. $\mathbb{Q} - \{0\} = \mathbb{Q}^*$, $\mathbb{R} - \{0\} = \mathbb{R}^*$, $\mathbb{C} - \{0\} = \mathbb{C}^*$, with $\star$ as multiplication of numbers, are multiplicative groups.
4. $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} = \{0, 1, 2, \ldots, n-1\}$ is a group where $\star$ is addition modulo $n$.
5. $\mathbb{Z}_n^* = \{k | k \in \mathbb{Z}_n, k \text{ has a multiplicative inverse (under multiplication modulo } n)\}$ in $\mathbb{Z}_n$ is a group under multiplication modulo $n$. If $p$ is prime, $\mathbb{Z}_p^*$ is cyclic. If $p$ is prime and $a \in \mathbb{Z}_p^*$ has order (index) $p - 1$, then $a$ is a primitive root modulo $p$. See the tables on pages 78 and 79 for primitive roots and power residues.
6. If $(G_1, \star_1), (G_2, \star_2), \ldots, (G_n, \star_n)$ are groups, the (direct) product group is $(G_1 \times G_2 \times \cdots \times G_n, \star) = \{(a_1, a_2, \ldots, a_n) | a_i \in G_i, i = 1, 2, \ldots, n\}$ where $\star$ is defined by $(a_1, a_2, \ldots, a_n) \star (b_1, b_2, \ldots, b_n) = (a_1 \star_1 b_1, a_2 \star_2 b_2, \ldots, a_n \star_n b_n)$.
7. All $m \times n$ matrices with real entries form a group under addition of matrices.
8. All $n \times n$ matrices with real entries and nonzero determinants form a group under matrix multiplication.
9. All 1–1, onto functions $f : S \rightarrow S$ (permutations of $S$), where $S$ is any nonempty set, form a group under composition of functions. In particular, if $S = \{1, 2, 3, \ldots, n\}$, the group of permutations of $S$ is called the symmetric group $S_n$. Each permutation can be written as a product of cycles. A cycle is a permutation $\sigma = (i_1, i_2, \ldots, i_k)$, where $\sigma(i_1) = i_2$, $\sigma(i_2) = i_3$, $\ldots$, $\sigma(i_k) = i_1$. Each cycle of length greater than 1 can be written as a product of transpositions (cycles of length 2). A permutation is even (odd) if it can be written as the product of an even (odd) number of transpositions. (Every permutation is either even or odd.) The set of all even permutations in $S_n$ is a subgroup $A_n$ of $S_n$. The group $A_n$ is called the alternating group on $n$ letters.
10. Given a regular $n$-gon, the dihedral group $D_n$ is the group of all symmetries of the $n$-gon, that is, the group generated by the set of all rotations around the center.
of the \( n \)-gon through angles of \( 360k/n \) degrees (where \( k = 0, 1, 2, \ldots, n - 1 \)),
together with all reflections in lines passing through a vertex and the center of
the \( n \)-gon, using composition of functions. Alternately, \( D_n = \{ a^i b^j \mid i = 0, 1; j = 0, 1, \ldots, n - 1; aba^{-1} = b^{-1} \} \).

11. The group \( T \) is of order 12 where \( T = \{ a^i b^j \mid i = 0, 1, \ldots, 5; j = 0, 1; b^2 = a^3, ba = a^{-1}b \} \).

12. The quaternion group \( Q_8 \) is the set \( \{ 1, -1, i, -i, j, -j, k, -k \} \) where multiplication is defined by the following relationships:

\[
i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j
\]

with 1 as the identity. These relations yield the following multiplication table:

\[
\begin{array}{cccccc}
\times & 1 & -1 & i & -i & j & -j \\
1 & 1 & -1 & i & -i & j & -j \\
-1 & -1 & 1 & -i & i & -j & j \\
i & i & -i & -1 & 1 & k & -k \\
-i & i & 1 & -1 & k & j & -j \\
-1 & j & j & -k & 1 & k & -j \\
i & j & -j & k & 1 & i & -i \\
-1 & k & -k & j & -j & i & 1 \\
-1 & -k & k & j & i & -i & 1 \\
\end{array}
\]

The quaternion group can also be defined as the following group of 8 matrices:

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix}, \begin{bmatrix}
-i & 0 \\
0 & i
\end{bmatrix}, \begin{bmatrix}
-i & 0 \\
0 & -i
\end{bmatrix}, \begin{bmatrix}
i & 0 \\
0 & i
\end{bmatrix}, \begin{bmatrix}
i & 0 \\
-1 & 0
\end{bmatrix}, \begin{bmatrix}
0 & -i \\
-i & 0
\end{bmatrix}, \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}, \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\]

where \( i^2 = -1 \) and matrix multiplication is the group operation.

### 2.6.3 RINGS

**Definitions**

1. A **ring** \( (R, +, \cdot) \) consists of a nonempty set \( R \) and two binary operations, + and \( \cdot \), such that \( (R, +) \) is an abelian group, the operation \( \cdot \) is associative, and the **left distributive law** \( a(b + c) = (ab) + (ac) \) and the **right distributive law** \( (a + b)c = (ac) + (bc) \) hold for all \( a, b, c \in R \).

2. A subset \( S \) of a ring \( R \) is a **subring** of \( R \) if \( S \) is a ring using the same operations used in \( R \).

3. A ring \( R \) is a **commutative** ring if the multiplication operation is commutative: \( ab = ba \) for all \( a, b \in R \).

4. A ring \( R \) is a **ring with unity** if there is an element 1 (called unity) such that \( a1 = 1a = a \) for all \( a \in R \).
5. A unit in a ring with unity is an element $a$ with a multiplicative inverse $a^{-1}$ ($aa^{-1} = a^{-1}a = 1$).

6. If $a \neq 0$, $b \neq 0$, and $ab = 0$, then $a$ is a left divisor of zero and $b$ is a right divisor of zero.

7. A subset $I$ of a ring $(R, +, \cdot)$ is an ideal of $R$ if $(I, +)$ is a subgroup of $(R, +)$ and $I$ is closed under left and right multiplication by elements of $R$ (if $x \in I$ and $r \in R$, then $rx \in I$ and $xr \in I$).

8. An ideal $I \subseteq R$ is
   - Proper: if $I \neq \{0\}$ and $I \neq R$
   - Maximal: if $I$ is proper there is no proper ideal properly containing $I$
   - Prime: if $ab \in I$ implies that $a$ or $b \in I$
   - Principal: if there is $a \in R$ such that $I$ is the intersection of all ideals containing $a$.

9. If $I$ is an ideal in a ring $R$, then a coset is a set $r + I = \{r + a \mid a \in I\}$.

10. If $I$ is an ideal in a ring $R$, then the quotient ring is the ring $R/I = \{r + I \mid r \in R\}$, where $(r + I) + (s + I) = (r + s) + I$ and $(r + I)(s + I) = (rs) + I$.

11. An integral domain $(R, +, \cdot)$ consists of a nonempty set $R$ and two binary operations, $+$ and $\cdot$, such that $(R, +, \cdot)$ is a commutative ring with unity, and the left [right] cancellation laws hold: if $ab = ac$ then $b = c$ [if $ba = ca$ then $b = c$] for all $a, b, c \in R$, where $a \neq 0$. (Equivalently, an integral domain is a commutative ring with unity that has no divisors of zero.)

12. If $R$ is an integral domain, then a nonzero element $r \in R$ that is not a unit is irreducible if $r = ab$ implies that either $a$ or $b$ is a unit.

13. If $R$ is an integral domain, a nonzero element $r \in R$ that is not a unit is a prime if, whenever $r|ab$, then either $r|a$ or $r|b$ ($x|y$ means that there is an element $z \in R$ such that $y = zx$).

14. A unique factorization domain (UFD) is an integral domain such that every nonzero element that is not a unit can be written uniquely as the product of irreducible elements (except for factors that are units and except for the order in which the factor appears).

15. A principal ideal domain (PID) is an integral domain in which every ideal is a principal ideal.

16. A division ring is a ring in which every nonzero element has a multiplicative inverse (that is, every nonzero element is a unit). (Equivalently, a division ring is a ring in which the nonzero elements form a multiplicative group.) A noncommutative division ring is called a skew field.

**Examples**

1. $\mathbb{Z}$ (integers), $\mathbb{Q}$ (rational numbers), $\mathbb{R}$ (real numbers), and $\mathbb{C}$ (complex numbers) are rings, with ordinary addition and multiplication of numbers.
2. \( \mathbb{Z}_n \) is a ring, with addition and multiplication modulo \( n \).

3. If \( \sqrt{n} \) is not an integer, then \( \mathbb{Z}[\sqrt{n}] = \{a + b\sqrt{n} \mid a, b \in \mathbb{Z}\} \), where \((a + b\sqrt{n}) + (c + d\sqrt{n}) = (a + c) + (b + d)\sqrt{n}\) and \((a + b\sqrt{n})(c + d\sqrt{n}) = (ac + nbd) + (ad + bc)\sqrt{n}\) is a ring.

4. The set of Gaussian integers \( \mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\} \) is a ring, with the usual definitions of addition and multiplication of complex numbers.

5. The polynomial ring in one variable over a ring \( R \) is the ring \( R[x] = \{a_nx^n + \cdots + a_1x + a_0 \mid a_i \in R; i = 0, 1, \ldots, n; n \in \mathbb{N}\} \) (elements of \( R[x] \) are added and multiplied using the usual rules for addition and multiplication of polynomials). The degree of a polynomial \( a_nx^n + \cdots + a_1x + a_0 \) with \( a_n \neq 0 \) is \( n \). A polynomial is monic if \( a_n = 1 \). A polynomial \( f(x) \) is irreducible over \( R \) if \( f(x) \) cannot be factored as a product of polynomials in \( R[x] \) of degree less than the degree of \( f(x) \). A monic irreducible polynomial \( f(x) \) of degree \( k \) in \( \mathbb{Z}_p[x] \) ( \( p \) prime) is primitive if the order of \( x \) in \( \mathbb{Z}_p[x]/(f(x)) \) is \( p^k - 1 \), where \( (f(x)) = \langle f(x)g(x) \mid g(x) \in \mathbb{Z}_p[x] \rangle \) (the ideal generated by \( f(x) \)). For example, the polynomial \( x^2 + 1 \) is
  - Irreducible in \( \mathbb{R}[x] \) because \( x^2 + 1 \) has no real root
  - Reducible in \( \mathbb{C}[x] \) because \( x^2 + 1 = (x - i)(x + i) \)
  - Reducible in \( \mathbb{Z}_2[x] \) because \( x^2 + 1 = (x + 1)^2 \)
  - Reducible in \( \mathbb{Z}_3[x] \) because \( x^2 + 1 = (x + 2)(x + 3) \)

6. The division ring of quaternions is the ring \((\{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}, +, \cdot)\), where operations are carried out using the rules for polynomial addition and multiplication and the defining relations for the quaternion group \( Q_8 \).

**Facts**

1. The set of all units of a ring is a group under the multiplication defined on the ring.

2. Every principal ideal domain is a unique factorization domain.

3. If \( R \) is a commutative ring with unity, then every maximal ideal is a prime ideal.

4. If \( R \) is a commutative ring with unity, then \( R \) is a field if and only if the only ideals of \( R \) are \( R \) and \( \{0\} \).

5. If \( R \) is a commutative ring with unity and \( I \neq R \) is an ideal, then \( R/I \) is an integral domain if and only if \( I \) is a prime ideal.

6. If \( R \) is a commutative ring with unity, then \( I \) is a maximal ideal if and only if \( R/I \) is a field.

7. If \( f(x) \in F[x] \) (\( F \) a field) and the ideal \( (f(x)) \neq \{0\} \), then the ideal \( (f(x)) \) is maximal if and only if \( f(x) \) is irreducible over \( F \).
2.6.4 FIELDS

Definitions

1. A field \((F, +, \cdot)\) consists of a commutative ring with unity such that each nonzero element of \(F\) has a multiplicative inverse (equivalently, a field is a commutative division ring).

2. The characteristic of a field is the smallest positive integer \(n\) such that \(1 + 1 + \cdots + 1 = 0\) (\(n\) summands). If no such \(n\) exists, the field has characteristic 0 (or characteristic \(\infty\)).

3. Field \(K\) is an extension field of the field \(F\) if \(F\) is a subfield of \(K\) (i.e., \(F \subseteq K\), and \(F\) is a field using the same operations used in \(K\)).

Examples

1. The sets \(\mathbb{Q}\), \(\mathbb{R}\), and \(\mathbb{C}\) with ordinary addition and multiplication are fields.

2. \(\mathbb{Z}_p\) (\(p\) a prime) is a field under addition and multiplication modulo \(p\).

3. \(F[x]/(f(x))\) is a field, provided that \(F\) is a field and \(f(x)\) is a nonconstant polynomial irreducible in \(F[x]\).

2.6.5 FINITE FIELDS

Facts

1. If \(p\) is prime, then the ring \(\mathbb{Z}_p\) is a finite field.

2. If \(p\) is prime and \(n\) is a positive integer, then there is exactly one field (up to isomorphism) with \(p^n\) elements. This field is denoted \(GF(p^n)\) or \(F_{p^n}\) and is called a Galois field. (See the tables beginning on page 155.)

3. For \(F\) a finite field, there is a prime \(p\) and a positive integer \(n\) such that \(F\) has \(p^n\) elements. The prime number \(p\) is the characteristic of \(F\). The field \(F\) is a finite extension of \(\mathbb{Z}_p\), that is, \(F\) is a vector space over \(\mathbb{Z}_p\).

4. If \(F\) is a finite field, then the set of nonzero elements of \(F\) under multiplication is a cyclic group. A generator of this group is a primitive element.

5. There are \(\phi(p^n - 1)/n\) primitive polynomials of degree \(n\) (\(n > 1\)) over \(GF(p)\), where \(\phi\) is the Euler \(\phi\)-function. (See table on page 154.)

6. There are \(\sum_{j=0}^{n} 1 / k^{\mu(j)}\) irreducible polynomials of degree \(k\) over \(GF(p^n)\), where \(\mu\) is the M"obius function.

7. If \(F\) is a finite field where \(|F| = k\) and \(p(x)\) is a polynomial of degree \(n\) irreducible over \(F\), then the field \(F[x]/(p(x))\) has order \(k^n\). If \(\alpha\) is a root of \(p(x) \in F[x]\) of degree \(n \geq 1\), then \(F[x]/(p(x)) = \{c_{n-1}\alpha^{n-1} + \cdots + c_1\alpha + c_0 | c_i \in F\ \text{for all} \ i\}.

8. When \(q\) is a power of a prime, \(F_{q^n}\) can be viewed as a vector space of dimension \(n\) over \(F_q\). A basis of \(F_{q^n}\) of the form \(\{\alpha, \alpha^q, \alpha^{q^2}, \ldots, \alpha^{q^{n-1}}\}\) is called a normal
basis. If \( \alpha \) is a primitive element of \( F_{q^n} \), then the basis is said to be a primitive normal basis. Such an \( \alpha \) satisfies a primitive normal polynomial of degree \( n \) over \( F_q \).

<table>
<thead>
<tr>
<th>Degree ( n )</th>
<th>Primitive normal polynomials ( q = 2 )</th>
<th>Primitive normal polynomials ( q = 3 )</th>
<th>Primitive normal polynomials ( q = 5 )</th>
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</thead>
<tbody>
<tr>
<td>2</td>
<td>( x^2 + x + 1 )</td>
<td>( x^2 + x + 2 )</td>
<td>( x^2 + x + 2 )</td>
</tr>
<tr>
<td>3</td>
<td>( x^3 + x^2 + 1 )</td>
<td>( x^3 + 2x^2 + 1 )</td>
<td>( x^3 + x^2 + 2 )</td>
</tr>
<tr>
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<td>( x^4 + x^3 + 2 )</td>
<td>( x^4 + x^3 + 4x + 2 )</td>
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<tr>
<td>5</td>
<td>( x^5 + x^4 + x^2 + x + 1 )</td>
<td>( x^5 + 2x^4 + 1 )</td>
<td>( x^5 + 2x^4 + 3 )</td>
</tr>
<tr>
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<td>( x^6 + x^5 + 1 )</td>
<td>( x^6 + x^5 + x^3 + 2 )</td>
<td>( x^6 + x^5 + 2 )</td>
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<tr>
<td>7</td>
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<td>( x^7 + x^6 + x^2 + 1 )</td>
<td>( x^7 + x^6 + 2 )</td>
</tr>
</tbody>
</table>

### 2.6.6 HOMOMORPHISMS AND ISOMORPHISMS

#### Definitions

1. A group homomorphism from group \( G_1 \) to group \( G_2 \) is a function \( \varphi : G_1 \rightarrow G_2 \) such that \( \varphi(ab) = \varphi(a)\varphi(b) \) for all \( a, b \in G_1 \). Note: \( a\varphi \) is often written instead of \( \varphi(a) \).

2. A character of a group \( G \) is a group homomorphism \( \chi : G \rightarrow \mathbb{C}^\ast \) (nonzero complex numbers under multiplication). (See table on page 149.)

3. A ring homomorphism from ring \( R_1 \) to ring \( R_2 \) is a function \( \varphi : R_1 \rightarrow R_2 \) such that \( \varphi(a + b) = \varphi(a) + \varphi(b) \) and \( \varphi(ab) = \varphi(a)\varphi(b) \) for all \( a, b \in R_1 \).

4. An isomorphism from group (ring) \( S_1 \) to group (ring) \( S_2 \) is a group (ring) homomorphism \( \varphi : S_1 \rightarrow S_2 \) that is 1-1 and onto \( S_2 \). If an isomorphism exists, then \( S_1 \) is said to be isomorphic to \( S_2 \). Write \( S_1 \cong S_2 \). (See the table on page 150 for numbers of nonisomorphic groups and the table on page 149 for examples of groups of orders less than 16.)

5. An automorphism of \( S \) is an isomorphism \( \varphi : S \rightarrow S \).

6. The kernel of a group homomorphism \( \varphi : G_1 \rightarrow G_2 \) is \( \varphi^{-1}(e) = \{ g \in G_1 \mid \varphi(g) = e \} \). The kernel of a ring homomorphism \( \varphi : R_1 \rightarrow R_2 \) is \( \varphi^{-1}(0) = \{ r \in R_1 \mid \varphi(r) = 0 \} \).

#### Facts

1. If \( \varphi : G_1 \rightarrow G_2 \) is a group homomorphism, then \( \varphi(G_1) \) is a subgroup of \( G_2 \).
2. Fundamental homomorphism theorem for groups: If \( \varphi : G_1 \rightarrow G_2 \) is a group homomorphism with kernel \( K \), then \( K \) is a normal subgroup of \( G_1 \) and \( G_1/K \cong \varphi(G_1) \).
3. If \( G \) is a cyclic group of infinite order, then \( G \cong (\mathbb{Z}, +) \).
4. If \( G \) is a cyclic group of order \( n \), then \( G \cong (\mathbb{Z}_n, +) \).
5. If \( p \) is prime, then there is only one group (up to isomorphism) of order \( p \), the group \((\mathbb{Z}_p, +)\).
6. **Cayley’s theorem**: If \( G \) is a finite group of order \( n \), then \( G \) is isomorphic to some subgroup of the group of permutations on \( n \) objects.

7. \( \mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn} \) if, and only if, \( m \) and \( n \) are relatively prime.

8. If \( n = n_1 \cdot n_2 \cdot \ldots \cdot n_k \) where each \( n_i \) is a power of a different prime, then \( \mathbb{Z}_n \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k} \).

9. **Fundamental theorem of finite abelian groups**: Every finite abelian group \( G \) (order \( \geq 2 \)) is isomorphic to a product of cyclic groups where each cyclic group has order a power of a prime, that is, there is a unique set \( \{n_1, \ldots, n_k\} \) where each \( n_i \) is a power of some prime such that \( G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k} \).

10. **Fundamental theorem of finitely generated abelian groups**: If \( G \) is a finitely generated abelian group, then there is a unique integer \( n \geq 0 \) and a unique set \( \{n_1, \ldots, n_k\} \) where each \( n_i \) is a power of some prime such that \( G \cong \mathbb{Z}^n \times \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k} \) (\( G \) is finitely generated if there are \( a_1, a_2, \ldots, a_n \in G \) such that every element of \( G \) can be written as \( a_1^{k_1}a_2^{k_2}\cdots a_n^{k_n} \) where \( k_i \in \{1, \ldots, n\} \) (the \( k_i \) are not necessarily distinct) and \( \epsilon_i \in \{1, -1\} \)).

11. **Fundamental homomorphism theorem for rings**: If \( \varphi : R_1 \rightarrow R_2 \) is a ring homomorphism with kernel \( K \), then \( K \) is an ideal in \( R_1 \) and \( R_1/K \cong \varphi(R_1) \).

### 2.6.7 MATRIX CLASSES THAT ARE GROUPS

In the following examples, the group operation is ordinary matrix multiplication:

- **\( GL(n, \mathbb{C}) \)** all complex non-singular \( n \times n \) matrices
- **\( GL(n, \mathbb{R}) \)** all real non-singular \( n \times n \) matrices
- **\( O(n) \)** all \( n \times n \) matrices \( A \) with \( AA^T = I \), also called the *orthogonal group*
- **\( SL(n, \mathbb{C}) \)** all complex \( n \times n \) matrices of determinant 1, also called the *unimodular group* or the *special linear group*
- **\( SL(n, \mathbb{R}) \)** all real \( n \times n \) matrices of determinant 1
- **\( SO(2) \)** rotations of the plane: matrices of the form
  \[
  A(\theta) = \begin{bmatrix}
  \cos \theta & -\sin \theta \\
  \sin \theta & \cos \theta 
  \end{bmatrix}
  \]
- **\( SO(n) \)** rotations of \( n \)-dimensional space
- **\( SU(n) \)** all \( n \times n \) unitary matrices of determinant 1
- **\( U(n) \)** all \( n \times n \) unitary matrices with \( UU^\dagger = I \)
### 2.6.8 PERMUTATION GROUPS

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
<th>Order</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symmetric group</td>
<td>$S_p$</td>
<td>$p!$</td>
<td>All permutations on {1, 2, \ldots, p}</td>
</tr>
<tr>
<td>Alternating group</td>
<td>$A_p$</td>
<td>$p!/2$</td>
<td>All even permutations on {1, 2, \ldots, p}</td>
</tr>
<tr>
<td>Cyclic group</td>
<td>$C_p$</td>
<td>$p$</td>
<td>Generated by $(12\ldots p)$</td>
</tr>
<tr>
<td>Dihedral group</td>
<td>$D_p$</td>
<td>$2p$</td>
<td>Generated by $(12\ldots p)$ and $(1p)(2p-1)$</td>
</tr>
<tr>
<td>Identity group</td>
<td>$E_p$</td>
<td>$1$</td>
<td>$(1)(2)\ldots(p)$ is the only permutation</td>
</tr>
</tbody>
</table>

For example, with $p = 3$ elements

$$
A_3 = \{(123), (231), (312)\},
C_3 = \{(123), (231), (312)\},
D_3 = \{(231), (213), (132), (321), (312), (123)\},
E_3 = \{(123)\} \text{ and }
S_3 = \{(231), (213), (132), (321), (312), (123)\}.
$$

**Creating new permutation groups**

Let $A$ have permutations $\{X_i\}$, order $n$, degree $d$, let $B$ have permutations $\{Y_j\}$, order $m$, degree $e$, and let $C$ (a function of $A$ and $B$) have permutations $\{W_k\}$, order $p$, degree $f$.

<table>
<thead>
<tr>
<th>Name</th>
<th>Definition</th>
<th>Permutation</th>
<th>Order</th>
<th>Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sum</td>
<td>$C = A + B$</td>
<td>$W = X \cup Y$</td>
<td>$p = mn$</td>
<td>$f = d + e$</td>
</tr>
<tr>
<td>Product</td>
<td>$C = A \times B$</td>
<td>$W = X \times Y$</td>
<td>$p = mn$</td>
<td>$f = de$</td>
</tr>
<tr>
<td>Composition</td>
<td>$C = A[B]$</td>
<td>$W = X \times Y$</td>
<td>$p = mn^d$</td>
<td>$f = de$</td>
</tr>
<tr>
<td>Power</td>
<td>$C = B^A$</td>
<td>$W = Y^X$</td>
<td>$p = mn$</td>
<td>$f = e^d$</td>
</tr>
</tbody>
</table>

**Polya theory**

Define $\text{Inv}(\pi)$ to be the number of invariant elements (i.e., mapped to themselves) of the permutation $\pi$. Define $\text{cyc}(\pi)$ as the number of cycles in $\pi$.

1. *Burnside’s Lemma*: Let $G$ be a group of permutations of a set $A$, and let $S$ be the equivalence relation on $A$ induced by $G$. Then the number of equivalence classes in $A$ is given by $\frac{1}{|G|} \sum_{\pi \in G} \text{Inv}(\pi)$.

2. *Special case of Polya’s theorem*: Let $R$ be an $m$ element set of colors. Let $G$ be a group of permutations $\{\pi_1, \pi_2, \ldots\}$ of the set $A$. Let $C(A, R)$ be the set of colorings of the elements of $A$ using colors in $R$. Then the number of distinct colorings in $C(A, R)$ is given by

$$
\frac{1}{|G|} \left[ m^{\text{cyc}(\pi_1)} + m^{\text{cyc}(\pi_2)} + \ldots \right].
$$

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3. *Polya’s theorem:* Let $G$ be a group of permutations on a set $A$ with cycle index $P_G(x_1, x_2, \ldots, x_k)$. Let $C(A, R)$ be the collection of all colorings of $A$ using colors in $R$. If $w$ is a weight assignment on $R$, then the pattern inventory of colorings in $C(A, R)$ is given by

$$P_G \left( \sum_{r \in R} w(r), \sum_{r \in R} w^2(r), \ldots, \sum_{r \in R} w^k(r) \right).$$

For example, consider necklaces made of $2k$ beads. Allowing a necklace reversal results in the permutation group $G = \{\pi_1, \pi_2\}$ with $\pi_1 = (1)(2)\ldots(2k)$ and $\pi_2 = (1 \ 2k)(2 \ 2k-1)(3 \ 2k-2)\ldots(k \ k+1)$. Hence, cyc($\pi_1$) = 2k, cyc($\pi_2$) = k, and the cycle index is $P_G(x_1, x_2) = \left( x_1^{2k} + x_2^k \right)/2$. Using $r$ colors, the number of distinct necklaces is $(r^{2k} + r^k)/2$.

For a 4 bead necklace ($k = 2$) using $r = 2$ colors (say $b$ and $g$), the $(2^4 + 2^2)/2 = 10$ different necklaces are \{bbbh, bbbg, bbgb, bbgg, bggb, bgbg, gbbg, [gbbg], [gbgg], and [ggbg]. The pattern inventory of colorings, $P_G(\sum w, \sum w^2) = ((b + g)^4 + (b^2 + g^2)^2)/2 = b^4 + 2b^3g + 4b^2g^2 + 2bg^3 + g^4$, tells how many colorings of each type there are.
2.6.9 TABLES

Groups of Small Order

<table>
<thead>
<tr>
<th>Order ( n )</th>
<th>Distinct groups of order ( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( {e} )</td>
</tr>
<tr>
<td>2</td>
<td>( \mathbb{Z}_2 )</td>
</tr>
<tr>
<td>3</td>
<td>( \mathbb{Z}_3 )</td>
</tr>
<tr>
<td>4</td>
<td>( \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4 )</td>
</tr>
<tr>
<td>5</td>
<td>( \mathbb{Z}_5 )</td>
</tr>
<tr>
<td>6</td>
<td>( \mathbb{Z}_6, D_3 )</td>
</tr>
<tr>
<td>7</td>
<td>( \mathbb{Z}_7 )</td>
</tr>
<tr>
<td>8</td>
<td>( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_8, Q_8, D_4 )</td>
</tr>
<tr>
<td>9</td>
<td>( \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_9 )</td>
</tr>
<tr>
<td>10</td>
<td>( \mathbb{Z}_{10}, D_5 )</td>
</tr>
<tr>
<td>11</td>
<td>( \mathbb{Z}_{11} )</td>
</tr>
<tr>
<td>12</td>
<td>( \mathbb{Z}_2 \times \mathbb{Z}<em>6, \mathbb{Z}</em>{12}, A_4, D_6, T )</td>
</tr>
<tr>
<td>13</td>
<td>( \mathbb{Z}_{13} )</td>
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<tr>
<td>14</td>
<td>( \mathbb{Z}_{14}, D_7 )</td>
</tr>
<tr>
<td>15</td>
<td>( \mathbb{Z}_{15} )</td>
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</tbody>
</table>

Characters for Some Families of Groups

- **\( \mathbb{Z}_n \)**: For \( m = 0, 1, \ldots, n - 1 \), \( \chi_m : 1 \mapsto e^{2\pi im/n} \)
- **\( G \) finite Abelian**:
  
  \[ G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_j} \]
  
  with each \( n_i \) a power of a prime. For \( m_j = 0, 1, \ldots, n_j - 1 \)
  
  and \( g_j = (0, 0, \ldots, 0, 1, 0, \ldots, 0) \chi_{m_1,m_2,\ldots,m_n} : g_j \mapsto e^{2\pi im_j/n_j} \)

- **\( D_n \) dihedral**:
  
  For \( x = \pm 1, y = \begin{cases} 
  \pm 1 & \text{if } n \text{ even} \\
  1 & \text{if } n \text{ odd} 
  \end{cases} \)
  
  \( \chi_{x,y} : a \mapsto x, b \mapsto y \). (See definition of \( D_n \).)

- **Quaternions**:
  
  For \( x, y = \pm 1 \) or \( x, y = \pm i \),
  
  \( \chi_{x,y} : \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mapsto x, \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \mapsto y \)
<table>
<thead>
<tr>
<th>Order</th>
<th>No. groups</th>
<th>No. Abelian groups</th>
<th>Order</th>
<th>No. groups</th>
<th>No. Abelian groups</th>
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List of All Sporadic Simple Groups

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<th>Index</th>
<th>Power residues</th>
<th>Group</th>
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Power Residues in $\mathbb{Z}_p$

For prime $p < 40$, the following table lists the minimal primitive root $a$ and the power residues of $a$. These can be used to find $a^m \pmod{p}$ for any $(a, p) = 1$. For example, to find $3^7 \pmod{11} (a = 3, m = 7)$, look in row $p = 11$ until the power of $a$ that is equal to 3 is found. In this case $2^8 \equiv 3 \pmod{11}$. This means that $3^7 = (2^8)^7 \equiv 2^{56} = (2^{10})^5 \cdot 2^6 \equiv 2^6 = 9 \pmod{11}$.

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Table of Primitive Monic Polynomials

In the table below, the elements in each string are the coefficients of the polynomial after the highest power of $x$. (For example, 564 represents $x^3 + 5x^2 + 6x + 4$.)

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Small finite fields

In the following, the entries under $\alpha^i$ denote the coefficient of powers of $\alpha$. For example, the last entry of the $p(x) = x^3 + x^2 + 1$ table is 1 1 0. That is: $\alpha^6 \equiv 1\alpha^5 + 1\alpha^4 + 0\alpha^0 \bmod p(\alpha)$, where the coefficients are taken modulo 2.
Addition and multiplication tables for $F_2$, $F_3$, $F_4$, and $F_8$:

$F_2$ addition and multiplication:

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$F_3$ addition and multiplication:

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$F_4$ addition and multiplication (using $\beta = \alpha + 1$):

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$F_8$ addition and multiplication (using strings of 0s and 1s to represent the polynomials: $0 = 000, 1 = 001, \alpha = 010, \alpha + 1 = 011, \alpha^2 = 100, \alpha^2 + \alpha = 110, \alpha^2 + 1 = 101, \alpha^2 + \alpha + 1 = 111$):
Table of primitive roots

As noted on page 106, the number of integers not exceeding and relatively prime to a fixed integer \( n \) is represented by \( \phi(n) \). These integers form a group; the group is cyclic if, and only if, \( n = 1, 2, 4 \) or \( n \) is of the form \( p^k \) or \( 2p^k \), where \( p \) is an odd prime. We refer to \( g \) as a primitive root of \( n \) if it generates that group, i.e., if \( \{ g, g^2, \ldots, g^{\phi(n)} \} \) are distinct modulo \( p \). There are \( \phi(\phi(n)) \) primitive roots of \( n \). If \( g \) is a primitive root of \( p \) and \( g^{p-1} \not\equiv 1 \pmod{p^2} \), then \( g \) is a primitive root of \( p^k \) for all \( k \).

If \( g \) is a primitive root of \( p^{2k} \), then either \( g \) or \( g + p^k \), whichever is odd, is a primitive root of \( 2p^k \).

If \( g \) is a primitive root of \( n \), then \( g^k \) is a primitive root of \( n \) if, and only if, \( k \) and \( \phi(n) \) are relatively prime, and each primitive root of \( n \) is of this form, i.e., \( (\phi(n), k) = 1 \).

In the table below,

- \( g \) denotes the least primitive root of \( p \)
- \( G \) denotes the least negative primitive root of \( p \)
- \( \epsilon \) denotes whether 10, \(-10\), or both, are primitive roots of \( p \)

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<td>$2^5 \cdot 5 \cdot 13$</td>
<td>3</td>
<td>$-3$</td>
<td>$-$</td>
</tr>
<tr>
<td>2083</td>
<td>$2 \cdot 3 \cdot 347$</td>
<td>2</td>
<td>$-4$</td>
<td>$-10$</td>
<td>2087</td>
<td>$2 \cdot 7 \cdot 149$</td>
<td>5</td>
<td>$-2$</td>
<td>$-$</td>
</tr>
<tr>
<td>2089</td>
<td>$2^3 \cdot 3^2 \cdot 29$</td>
<td>7</td>
<td>$-7$</td>
<td>$-$</td>
<td>2099</td>
<td>$2 \cdot 1049$</td>
<td>2</td>
<td>$-3$</td>
<td>$10$</td>
</tr>
<tr>
<td>2111</td>
<td>$2 \cdot 5 \cdot 211$</td>
<td>7</td>
<td>$-2$</td>
<td>$-10$</td>
<td>2113</td>
<td>$2^6 \cdot 3 \cdot 11$</td>
<td>5</td>
<td>$-5$</td>
<td>$\pm 10$</td>
</tr>
<tr>
<td>2129</td>
<td>$2^4 \cdot 7 \cdot 19$</td>
<td>3</td>
<td>$-3$</td>
<td>$-$</td>
<td>2131</td>
<td>$2 \cdot 3 \cdot 5 \cdot 71$</td>
<td>2</td>
<td>$-4$</td>
<td>$-$</td>
</tr>
<tr>
<td>2137</td>
<td>$2^3 \cdot 3 \cdot 89$</td>
<td>10</td>
<td>$-10$</td>
<td>$\pm 10$</td>
<td>2141</td>
<td>$2^2 \cdot 5 \cdot 107$</td>
<td>2</td>
<td>$-2$</td>
<td>$\pm 10$</td>
</tr>
<tr>
<td>2143</td>
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<td>3</td>
<td>$-9$</td>
<td>$10$</td>
<td>2153</td>
<td>$2^4 \cdot 3 \cdot 269$</td>
<td>3</td>
<td>$-3$</td>
<td>$\pm 10$</td>
</tr>
<tr>
<td>2161</td>
<td>$2^4 \cdot 3^2 \cdot 5$</td>
<td>23</td>
<td>$-23$</td>
<td>$-$</td>
<td>2179</td>
<td>$2 \cdot 3 \cdot 11^2$</td>
<td>7</td>
<td>$-5$</td>
<td>$10$</td>
</tr>
<tr>
<td>2203</td>
<td>$2 \cdot 3 \cdot 367$</td>
<td>5</td>
<td>$-7$</td>
<td>$-10$</td>
<td>2207</td>
<td>$2 \cdot 1103$</td>
<td>5</td>
<td>$-2$</td>
<td>$10$</td>
</tr>
<tr>
<td>2213</td>
<td>$2^2 \cdot 7 \cdot 79$</td>
<td>2</td>
<td>$-2$</td>
<td>$-$</td>
<td>2221</td>
<td>$2^2 \cdot 3 \cdot 5 \cdot 37$</td>
<td>2</td>
<td>$-2$</td>
<td>$\pm 10$</td>
</tr>
<tr>
<td>2237</td>
<td>$2^2 \cdot 13 \cdot 43$</td>
<td>2</td>
<td>$-2$</td>
<td>$-$</td>
<td>2239</td>
<td>$2 \cdot 3 \cdot 373$</td>
<td>3</td>
<td>$-2$</td>
<td>$-10$</td>
</tr>
<tr>
<td>2243</td>
<td>$2 \cdot 19 \cdot 59$</td>
<td>2</td>
<td>$-3$</td>
<td>$-10$</td>
<td>2251</td>
<td>$2 \cdot 3^2 \cdot 5^3$</td>
<td>7</td>
<td>$-5$</td>
<td>$10$</td>
</tr>
<tr>
<td>2267</td>
<td>$2 \cdot 11 \cdot 103$</td>
<td>2</td>
<td>$-3$</td>
<td>$-10$</td>
<td>2269</td>
<td>$2 \cdot 3^3 \cdot 7$</td>
<td>2</td>
<td>$-2$</td>
<td>$\pm 10$</td>
</tr>
<tr>
<td>2273</td>
<td>$2^5 \cdot 71$</td>
<td>3</td>
<td>$-3$</td>
<td>$\pm 10$</td>
<td>2281</td>
<td>$2^3 \cdot 3 \cdot 5 \cdot 19$</td>
<td>7</td>
<td>$-7$</td>
<td>$-$</td>
</tr>
<tr>
<td>2287</td>
<td>$2 \cdot 3^2 \cdot 127$</td>
<td>19</td>
<td>$-7$</td>
<td>$-$</td>
<td>2293</td>
<td>$2^2 \cdot 3 \cdot 191$</td>
<td>2</td>
<td>$-2$</td>
<td>$-$</td>
</tr>
</tbody>
</table>
Chapter 3

Discrete Mathematics

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3.1 SET THEORY

3.1.1 PROPOSITIONAL CALCULUS

Propositional calculus is the study of statements: how they are combined and how to determine their truth. Statements (or propositions) are combined by means of connectives such as and \((\land)\) or \((\lor)\), not \((\neg)\), or sometimes \(\sim\), implies \((\rightarrow)\), and if and only if \((\leftrightarrow)\). Propositions are assigned letters \(\{p, q, r, \ldots\}\). For example, if \(p\) is the statement “\(x = 3\),” and \(q\) the statement “\(y = 4\),” then \(p \lor \neg q\) would be interpreted as “\(x = 3\) or \(y \neq 4\).” To determine the truth of a statement, truth tables are used. Using T (for true) and F (for false), the truth tables for these connectives are as follows:

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(p)</td>
<td>(q)</td>
<td>(p \land q)</td>
<td>(p \lor q)</td>
<td>(p \rightarrow q)</td>
<td>(p \leftrightarrow q)</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

The proposition \(p \rightarrow q\) can be read “If \(p\) then \(q\)” or, less often, “\(q\) if \(p\).” The table shows that \(p \lor q\) is an inclusive or because it is true even when \(p\) and \(q\) are both true. Thus, the statement “I’m watching TV or I’m doing homework” is a true statement if the narrator happens to be both watching TV and doing homework. Note that \(p \rightarrow q\) is false only when \(p\) is true and \(q\) is false. Thus, a false statement implies any statement and a true statement is implied by any statement.

3.1.2 TAUTOLOGIES

A statement such as \((p \rightarrow (q \land r)) \lor \neg p\) is a compound statement composed of the atomic propositions \(p, q,\) and \(r\). The letters \(P, Q,\) and \(R\) are used to designate compound statements. A tautology is a compound statement which always is true, regardless of the truth values of the atomic statements used to define it. For example, a simple tautology is \((\neg \neg p) \leftrightarrow p\). Tautologies are logical truths. Some examples are as follows:
Law of the excluded middle \( p \lor \neg p, \)

De Morgan’s laws \( \neg(p \lor q) \iff (\neg p \land \neg q), \)
\( \neg(p \land q) \iff (\neg p \lor \neg q), \)

Modus ponens \( (p \land (p \rightarrow q)) \rightarrow q, \)

Contrapositive law \( (p \rightarrow q) \iff (\neg q \rightarrow \neg p), \)

Reductio ad absurdum \( (\neg p \rightarrow p) \rightarrow p, \)

Elimination of cases \( ((p \lor q) \land \neg p) \rightarrow q, \)

Transitivity of implication \( ((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r), \)

Proof by cases \( ((p \rightarrow q) \land (\neg p \rightarrow q)) \rightarrow q, \)

Idempotent laws \( p \land p \iff p; \ p \lor p \iff p, \)

Commutative laws \( (p \land q) \iff (q \land p); \ (p \lor q) \iff (q \lor p), \)

Associative laws \( (p \land (q \land r)) \iff ((p \land q) \land r), \) and \( (p \lor (q \lor r)) \iff ((p \lor q) \lor r). \)

### 3.1.3 TRUTH TABLES AS FUNCTIONS

If we assign the value 1 to T, and 0 to F, then the truth table for \( p \land q \) is simply the value \( pq \). This can be done with all the connectives as follows:

<table>
<thead>
<tr>
<th>Connective</th>
<th>Arithmetic function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p \land q )</td>
<td>( pq )</td>
</tr>
<tr>
<td>( p \lor q )</td>
<td>( p + q - pq )</td>
</tr>
<tr>
<td>( p \rightarrow q )</td>
<td>( 1 - p + pq )</td>
</tr>
<tr>
<td>( p \leftrightarrow q )</td>
<td>( 1 - p - q + 2pq )</td>
</tr>
<tr>
<td>( \neg p )</td>
<td>( 1 - p )</td>
</tr>
</tbody>
</table>

These formulas may be used to verify tautologies, because, from this point of view, a tautology is a function whose value is identically 1. In using them, it is useful to remember that \( pp = p^2 = p, \) since \( p = 0 \) or \( p = 1. \)

### 3.1.4 RULES OF INFERENCE

A rule of inference in propositional calculus is a method of arriving at a valid (true) conclusion, given certain statements, assumed to be true, which are called the hypotheses. For example, suppose that \( P \) and \( Q \) are compound statements. Then if \( P \) and \( P \rightarrow Q \) are true, then \( Q \) must necessarily be true. This follows from the modus ponens tautology in the above list of tautologies. We write this rule of inference \( P, \ P \rightarrow Q \Rightarrow Q. \) It is also classically written

\[
\begin{array}{c}
P \\
\hline
P \rightarrow Q \\
\hline
Q
\end{array}
\]

Some examples of rules of inferences follow, all derived from the above list of tautologies:
3.1.5 DEDUCTIONS

A **deduction** from hypotheses is a list of statements, each one of which is either one of the hypotheses, a tautology, or follows from previous statements in the list by a valid rule of inference. It follows that if the hypotheses are true, then the conclusion must be true. Suppose for example, that we are given hypotheses $\neg q \rightarrow p$, $q \rightarrow \neg r$, $r$; it is required to deduce the conclusion $p$. A deduction showing this, with reasons for each step is as follows:

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $q \rightarrow \neg r$</td>
<td>Hypothesis</td>
</tr>
<tr>
<td>2. $r$</td>
<td>Hypothesis</td>
</tr>
<tr>
<td>3. $\neg q$</td>
<td>Modus tollens (1,2)</td>
</tr>
<tr>
<td>4. $\neg q \rightarrow p$</td>
<td>Hypothesis</td>
</tr>
<tr>
<td>5. $p$</td>
<td>Modus ponens (3,4)</td>
</tr>
</tbody>
</table>

3.1.6 SETS

A set is a collection of objects. Some examples of sets are

- The population of Cleveland on January 1, 1995
- The real numbers between 0 and 1 inclusive
- The prime numbers 2, 3, 5, 7, 11, …
- The numbers 1, 2, 3, and 4
- All of the formulas in this book

3.1.7 SET OPERATIONS AND RELATIONS

If $x$ is an object in a set $A$, then we write $x \in A$ (read “$x$ is in $A$.”) If $x$ is not in $A$, we write $x \notin A$. When considering sets, a set $U$, called the universe, is chosen, from which all elements are taken. The **null set or empty set** $\emptyset$ is the set containing no elements. Thus, $x \notin \emptyset$ for all $x \in U$. Some relations on sets are as follows:

<table>
<thead>
<tr>
<th>Relation</th>
<th>Read as</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A \subseteq B$</td>
<td>$A$ is contained in $B$</td>
<td>Any element of $A$ is also an element of $B$</td>
</tr>
<tr>
<td>$A = B$</td>
<td>$A$ equals $B$</td>
<td>$(A \subseteq B) \land (B \subseteq A)$</td>
</tr>
</tbody>
</table>

Some basic operations on sets are as follows:
<table>
<thead>
<tr>
<th>Operation</th>
<th>Read as</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A \cup B$</td>
<td>$A$ union $B$</td>
<td>The elements in $A$ or in $B$</td>
</tr>
<tr>
<td>$A \cap B$</td>
<td>$A$ intersection $B$</td>
<td>The elements in both $A$ and $B$</td>
</tr>
<tr>
<td>$A - B$</td>
<td>$A$ minus $B$</td>
<td>The elements in $A$ which are not in $B$</td>
</tr>
<tr>
<td>$A'$ or $\overline{A}$</td>
<td>Complement of $A$</td>
<td>The elements in $U$ which are not in $A$</td>
</tr>
<tr>
<td>$\mathcal{P}(A)$ or $2^A$</td>
<td>Power set of $A$</td>
<td>The collection of all subsets of $A$</td>
</tr>
</tbody>
</table>

### 3.1.8 VENN DIAGRAMS

The operations and relations on sets can be illustrated by *Venn diagrams*. The diagrams below shows a few possibilities.

![Venn Diagrams](image)

### 3.1.9 PARADOXES AND THEOREMS OF SET THEORY

#### Russell’s paradox

In about 1900, Bertrand Russell presented a paradox, paraphrased as follows: since the elements of sets can be arbitrary, sets can contain sets. Therefore, a set can possibly be a member of itself. (For example, the set of all sets would be a member of itself. Another example is the collection of all sets that can be described in fewer than 50 words.) Now let $A$ be the set of all sets which are *not* members of themselves. Then if $A$ is a member of itself, it is not a member of itself. And if $A$ is not a member of itself, then by definition, $A$ is a member of itself. This paradox leads to a much more careful evaluation of how sets can be defined.

#### Infinite sets and the continuum hypothesis

Georg Cantor showed how infinite sets can be counted, much as finite sets. He used the symbol $\aleph_0$ (*read* aleph null) for the number of integers and introduced larger infinite numbers such as $\aleph_1$, $\aleph_2$, and so on. Cantor introduced a consistent arithmetic on infinite cardinals and a way of comparing infinite cardinals. A few of his results...
were as follows:

\[ \aleph_0 + \aleph_0 = \aleph_0, \quad (\aleph_0)^2 = \aleph_0, \quad 2^{\aleph_0} = \aleph_0^\omega > \aleph_0. \]

Cantor showed that \( c = 2^{\aleph_0} > \aleph_0 \), where \( c \) is the cardinality of real numbers. The continuum hypothesis asked whether or not \( c = \aleph_1 \), the first infinite cardinal greater than \( \aleph_0 \). In 1963, Paul J. Cohen showed that this result is independent of the other axioms of set theory. In his words, “... the truth or falsity of the continuum hypothesis ... cannot be determined by set theory as we know it today.”

### 3.1.10 PREDICATE CALCULUS

Unlike propositional calculus, which may be considered the skeleton of logical discourse, predicate calculus is the language in which most mathematical reasoning takes place. It uses the symbols of propositional calculus, with the exception of the propositional variables \( p, q, \ldots \). Predicate calculus uses the universal quantifier \( \forall \), the existential quantifier \( \exists \), predicates \( P(x) \), \( Q(x, y) \), \ldots , variables \( x, y, \ldots \), and assumes a universe \( U \) from which the variables are taken. The quantifiers are illustrated in the following table.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Read as</th>
<th>Usage</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \exists )</td>
<td>There exists an</td>
<td>( \exists x(x &gt; 10) )</td>
<td>There is an ( x ) such that ( x &gt; 10 )</td>
</tr>
<tr>
<td>( \forall )</td>
<td>For all</td>
<td>( \forall x(x^2 + 1 \neq 0) )</td>
<td>For all ( x ), ( x^2 + 1 \neq 0 )</td>
</tr>
</tbody>
</table>

Predicates are variable statements which may be true or false, depending on the values of its variable. In the above table, “\( x > 10 \)” is a predicate in the one variable \( x \) as is “\( x^2 + 1 \neq 0 \)” Without a given universe, we have no way of deciding whether a statement is true or false. Thus \( \forall x(x^2 + 1 \neq 0) \) is true if the universe \( U \) is the real numbers, but false if \( U \) is the complex numbers. A useful rule for manipulating quantifiers is

\[ \neg \forall x P(x) \iff \exists x \neg P(x). \]

For example, it is not true that all people are mortal if, and only if, there is a person who is immortal. Here the universe \( U \) is the set of people, and \( P(x) \) is the predicate “\( x \) is mortal.” This works with more than one quantifier. Thus,

\[ \neg \forall x \exists y P(x, y) \iff \exists x \forall y \neg P(x, y). \]

For example, if it is not true that every person loves someone, then it follows that there is a person who loves no one.

Fermat’s last theorem, stated in terms of the predicate calculus \( (U = \text{the positive integers}) \), is

\[ \forall n \forall a \forall b \forall c ((n > 2) \to (a^n + b^n \neq c^n)) \]

It was proven in 1995; its proof is extremely complicated. One does not expect a simple deduction, as in the propositional calculus. In 1931, Gödel proved the Gödel Incompleteness Theorem. This states that, in any logical system complex enough to contain arithmetic, it will always be possible to find a true result which is not formally...
provable using predicate logic. This result was especially startling because the notion of truth and provability had been often identified with each other.

### 3.2 COMBINATORICS

#### 3.2.1 SAMPLE SELECTION

There are four ways in which a sample of \( r \) elements can be obtained from a set of \( m \) distinguishable objects.

<table>
<thead>
<tr>
<th>Order counts?</th>
<th>Repetitions allowed?</th>
<th>The sample is called an</th>
<th>Number of ways to choose the sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>No</td>
<td>No</td>
<td>( r )-combination</td>
<td>( C(m, r) )</td>
</tr>
<tr>
<td>Yes</td>
<td>No</td>
<td>( r )-permutation</td>
<td>( P(m, r) )</td>
</tr>
<tr>
<td>No</td>
<td>Yes</td>
<td>( r )-combination with replacement</td>
<td>( C^R(m, r) )</td>
</tr>
<tr>
<td>Yes</td>
<td>Yes</td>
<td>( r )-permutation with replacement</td>
<td>( P^R(m, r) )</td>
</tr>
</tbody>
</table>

where

\[
C(m, r) = \binom{m}{r} = \frac{m!}{r!(m-r)!},
\]

\[
P(m, r) = (m)_r = m^\underline{r} = \frac{m!}{(m-r)!},
\]

\[
C^R(m, r) = C(m + r - 1, r) = \frac{(m + r - 1)!}{r!(m - 1)!}, \text{ and} \]

\[
P^R(m, r) = m^r.
\]

For example, choosing a 2-element sample from the set \( \{a, b\} \):

\begin{align*}
\text{\( r \)-combination} & \quad C(2, 2) = 1 \quad ab \\
\text{\( r \)-permutation} & \quad P(2, 2) = 2 \quad ab \text{ and } ba \\
\text{\( r \)-combination with replacement} & \quad C^R(2, 2) = 3 \quad aa, ab, \text{ and } bb \\
\text{\( r \)-permutation with replacement} & \quad P^R(2, 2) = 4 \quad aa, ab, ba, \text{ and } bb
\end{align*}
3.2.2 BALLS INTO CELLS

There are eight different ways in which \( n \) balls can be placed into \( k \) cells:

<table>
<thead>
<tr>
<th>Distinguish the balls?</th>
<th>Distinguish the cells?</th>
<th>Can cells be empty?</th>
<th>Number of ways to place ( n ) balls into ( k ) cells</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>( k^n )</td>
</tr>
<tr>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>( k! \left{ \begin{array}{c} n \ k \end{array} \right} )</td>
</tr>
<tr>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>( C(k+n-1,n) = \binom{k+n-1}{n} )</td>
</tr>
<tr>
<td>No</td>
<td>Yes</td>
<td>No</td>
<td>( C(n-1,k-1) = \binom{n-1}{k-1} )</td>
</tr>
<tr>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td>( \left{ \begin{array}{l} n \ 1 \end{array} \right} + \left{ \begin{array}{l} n \ 2 \end{array} \right} + \cdots + \left{ \begin{array}{l} n \ k \end{array} \right} )</td>
</tr>
<tr>
<td>Yes</td>
<td>No</td>
<td>No</td>
<td>( p_1(n) + p_2(n) + \cdots + p_k(n) )</td>
</tr>
<tr>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>( p_k(n) )</td>
</tr>
</tbody>
</table>

where \( \left\{ \begin{array}{l} n \\ k \end{array} \right\} \) is the Stirling cycle number (see page 174) and \( p_k(n) \) is the number of partitions of the number \( n \) into exactly \( k \) integer pieces (see page 173).

Given \( n \) distinguishable balls and \( k \) distinguishable cells, the number of ways in which we can place \( n_1 \) balls into cell 1, \( n_2 \) balls into cell 2, \ldots, \( n_k \) balls into cell \( k \), is given by the multinomial coefficient \( \binom{n}{n_1,n_2,\ldots,n_k} \) (see page 171).

3.2.3 BINOMIAL COEFFICIENTS

The binomial coefficient \( \binom{n}{m} \) is the number of ways of choosing \( m \) objects from a collection of \( n \) distinct objects without regard to order:

1. \( \binom{n}{m} = \frac{n!}{m!(n-m)!} = \frac{n(n-1)\cdots(n-m+1)}{m!} = \binom{n}{n-m} \).
2. \( \binom{n}{0} = \binom{n}{n} = 1 \) and \( \binom{n}{1} = n \).
3. \( \binom{2n}{n} = \frac{2n(2n-1)!!}{n!} = \frac{2^n(2n-1)(2n-3)\cdots3\cdot1}{n!} \).
4. Example: For the 5 element set \( \{a, b, c, d, e\} \) there are \( \binom{5}{3} = \frac{5!}{3!2!} = 10 \) subsets containing exactly three elements. They are \( (a,b,c), (a,b,d), (a,b,e), (a,c,d), (a,c,e), (a,d,e), (b,c,d), (b,c,e), (b,d,e), \) and \( (c,d,e) \).
5. The recurrence relation: \( \binom{n+1}{m} = \binom{n}{m} + \binom{n}{m-1} \).

6. Two generating functions for binomial coefficients are \( \sum_{m=0}^{n} \binom{n}{m} x^m = (1 + x)^n \) for \( n = 1, 2, \ldots \), and \( \sum_{m=0}^{\infty} \binom{n}{m} x^{n-m} = (1 - x)^{-m-1} \).

7. The Vandermonde convolution is \( \binom{x + y}{n} = \sum_{k=0}^{n} \binom{x}{k} \binom{y}{n-k} \).

8. The Gaussian binomial coefficient \( \left[ \begin{array}{c} n \\ r \end{array} \right]_q \) is defined to be

\[
\left[ \begin{array}{c} n \\ r \end{array} \right]_q = \begin{cases} \frac{q^n - 1}{q-1} \cdot \frac{q^{n-1} - 1}{q-1} \cdots \frac{q^{n-r+1} - 1}{q-1} & \text{if } 0 < r \leq n \\ 1 & \text{if } r = 0 \\ 0 & \text{if } r < 0 \text{ or } r > n \end{cases}
\]

where \( q \) is a real number. Note that

\[
\left[ \begin{array}{c} n \\ r \end{array} \right]_q = \left[ \begin{array}{c} n \\ n-r \end{array} \right]_q \quad \text{and} \quad \lim_{q \to 1} \left[ \begin{array}{c} n \\ r \end{array} \right]_q = \binom{n}{r}.
\]

Note also that, treated as polynomials, they are reciprocal.

\[
\left[ \begin{array}{c} n \\ r \end{array} \right]_{q \to q^{-1}} = q^{m(n-m)} \left[ \begin{array}{c} n \\ r \end{array} \right]_q.
\]

**Pascal’s triangle**

The binomial coefficients \( \binom{n}{k} \) can be arranged in a triangle

\[
\begin{array}{ccccccc}
1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
\end{array}
\]

in which each number is the sum of the two numbers above it.
The binomial coefficients satisfy

\[
\binom{n}{m} + \binom{n}{m+1} = \binom{n+1}{m+1},
\]
\[
\binom{n}{m} = \binom{n}{n-m},
\]
\[
\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n,
\]
\[
\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n \binom{n}{n} = 0,
\]
\[
\binom{n}{n} + \binom{n+1}{n} + \binom{n+2}{n} + \cdots + \binom{n+m}{n} = \binom{n+m+1}{n+1},
\]
\[
\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = 2^{n-1},
\]
\[
\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1},
\]
\[
\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n},
\]
\[
\binom{m}{0}\binom{n}{p} + \binom{m}{1}\binom{n}{p-1} + \cdots + \binom{m}{p}\binom{n}{0} = \binom{m+n}{p},
\]
\[
1\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} = n2^{n-1}, \text{ and}
\]
\[
1\binom{n}{1} - 2\binom{n}{2} + \cdots + (-1)^{n+1}n\binom{n}{n} = 0.
\]

### 3.2.4 MULTINOMIAL COEFFICIENTS

The multinomial coefficient \(\binom{n}{n_1, n_2, \ldots, n_k}\) (also written \(C(n; n_1, n_2, \ldots, n_k)\)) is the number of ways of choosing \(n_1\) objects, then \(n_2\) objects, \ldots, then \(n_k\) objects from a collection of \(n\) distinct objects without regard to order. This requires that \(\sum_{j=1}^{k} n_j = n\).

The multinomial symbol is numerically evaluated as

\[
\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}.
\]

For example, the number of ways to choose 2 objects, then 1 object, then 1 object from the set \(\{a, b, c, d\}\) is \(\binom{4}{2,1,1} = 12\); they are as follows (vertical bars show the ordered selections):

\[
| ab | c | d |,
| ab | d | c |,
| ac | b | d |,
| ac | d | b |,
| ad | b | c |,
| ad | c | b |,
| bc | a | d |,
| bc | d | a |,
| bd | a | c |,
| bd | c | a |,
| cd | a | b |,
| cd | b | a |.
\]
3.2.5 ARRANGEMENTS AND DERANGEMENTS

The number of ways to arrange \( n \) distinct objects in a row is \( n! \); this is the number of permutations of \( n \) objects. For example, for the three objects \( \{a, b, c\} \), the number of arrangements is \( 3! = 6 \). These permutations are \( \{abc, bac, cab, acb, bca, and cba\} \).

The number of ways to arrange \( n \) objects (assuming that there are \( k \) types of objects and \( n_i \) copies of each object of type \( i \)) is the multinomial coefficient \( \binom{n}{n_1, n_2, ..., n_k} \).

For example, for the set \( \{a, a, b, c\} \) the parameters are \( n = 4, k = 3, n_1 = 2, n_2 = 1, \) and \( n_3 = 1 \). Hence, there are \( \binom{4}{2,1,1} = \frac{4!}{2!1!1!} = 12 \) arrangements; they are

\[
aabc, \ aacb, \ abac, \ abca, \ acab, \ acba, \ baac, \ baca, \ bcaa, \ caab, \ caba, \ cbaa.
\]

A derangement is a permutation of objects, in which object \( i \) is not in the \( i^{th} \) location. For example, all of the derangements of \( \{1, 2, 3, 4\} \) are

\[
2143, \ 2341, \ 2413, \ 3142, \ 3412, \ 3421, \ 4123, \ 4312, \ 4321.
\]

The number of derangements of \( n \) elements, \( D_n \), satisfies the recursion relation, \( D_n = (n - 1) \left( D_{n-1} + D_{n-2} \right) \), with the initial values \( D_1 = 0 \) and \( D_2 = 1 \). Hence,

\[
D_n = n! \left( 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right).
\]

The numbers \( D_n \) are also called subfactorials or rencontres numbers. For large values of \( n \), \( D_n/n! \approx e^{-1} \approx 0.37 \). Hence more than one of every three permutations is a derangement.

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_n )</td>
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<td>1</td>
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<td>44</td>
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</tr>
</tbody>
</table>

3.2.6 CATALAN NUMBERS

The Catalan number is \( C_n = \frac{1}{n-1} \binom{2n-2}{n-1} \). Given the product \( A_1A_2 \ldots A_n \), the number of ways to pair up terms keeping the original order is \( C_n \). For example, with \( n = 4 \), there are the \( C_4 = 5 \) ways to group the terms; they are \( (A_1A_2)(A_3A_4), \ (A_1(A_2A_3))A_4, \ (A_1A_2)(A_3A_4), \ (A_1)(A_2)(A_3)(A_4), \) and \( A_1(A_2(A_3A_4)) \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_n )</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>14</td>
<td>42</td>
<td>132</td>
<td>429</td>
<td>1430</td>
<td>4862</td>
</tr>
</tbody>
</table>

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3.2.7 PARTITIONS

A partition of a number \( n \) is a representation of \( n \) as the sum of any number of positive integral parts (for example: \( 5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1 \)). The number of partitions of \( n \) is denoted \( p(n) \) (for example, \( p(5) = 7 \)). The number of partitions of \( n \) into at most \( m \) parts is equal to the number of partitions of \( n \) into parts which do not exceed \( m \); this is denoted \( p_m(n) \) (for example, \( p_3(5) = 5 \) and \( p_2(5) = 3 \)).

The generating functions for \( p(n) \) and \( p_m(n) \) are

\[
1 + \sum_{n=1}^{\infty} p(n)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots}, \quad \text{and} \quad (3.2.5)
\]

\[
1 + \sum_{n=1}^{\infty} \sum_{m=1}^{n} p_m(n)x^n t^m = \frac{1}{(1-tx)(1-tx^2)(1-tx^3)\cdots}. \quad (3.2.6)
\]

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
p(n) & 1 & 2 & 3 & 5 & 7 & 11 & 15 & 22 & 30 & 42 \\
n & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
p(n) & 56 & 77 & 101 & 135 & 176 & 231 & 297 & 385 & 490 & 627 \\
n & 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 \\
p(n) & 792 & 1002 & 1255 & 1575 & 1958 & 2436 & 3010 & 3718 & 4565 & 5604 \\
n & 31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 & 39 & 40 \\
p(n) & 6842 & 8349 & 10143 & 12310 & 14883 & 37338 & 89134 & 204226 & & \\
\hline
\end{array}
\]

3.2.8 STIRLING NUMBERS

The number \((-1)^{n-m} \binom{n}{m}\) is the number of permutations of \( n \) symbols which have exactly \( m \) cycles. The term \( \binom{n}{m} \) is called a Stirling number (or a Stirling number of the first kind). It can be numerically evaluated as

\[
\binom{n}{m} = \sum_{k=0}^{n-m} (-1)^k \binom{n-1+k}{n-m+k} \binom{2n-m}{n-m-k} \binom{n-m-k}{k}
\]

where \( \binom{n-m-k}{k} \) is a Stirling cycle number.
Example: For the 4 element set \{a, b, c, d\}, there are \[^{4}_{2}\] = 11 permutations containing exactly 2 cycles. They are
\[
\begin{align*}
(1 2 3 4) & = (123)(4), & (1 2 3 4) & = (132)(4), & (1 2 3 4) & = (134)(2), \\
(1 2 3 4) & = (143)(2), & (1 2 3 4) & = (124)(3), & (1 2 3 4) & = (142)(3), \\
(1 2 3 4) & = (234)(1), & (1 2 3 4) & = (243)(1), & (1 2 3 4) & = (12)(34), \\
(1 2 3 4) & = (13)(24), & (1 2 3 4) & = (14)(23).
\end{align*}
\]

• There is the recurrence relation: \[^{n+1}_{m}\] = \[^{n}_{m-1}\] - \[^{n}_{m}\].

• The factorial polynomial is defined as \(x^{(n)} = x(x-1) \ldots (x-n+1)\) with \(x^{(0)} = 1\) by definition. If \(n > 0\), then
\[
x^{(n)} = \left[\frac{n}{1}\right] x + \left[\frac{n}{2}\right] x^2 + \cdots + \left[\frac{n}{n}\right] x^n.
\]

For example: \(x^{(3)} = x(x-1)(x-2) = 2x - 3x^2 + x^3 = \left[\frac{3}{1}\right] x + \left[\frac{3}{2}\right] x^2 + \left[\frac{3}{3}\right] x^3\).

• Stirling numbers satisfy \(\sum_{n=m}^{\infty} \left[\frac{n}{m}\right] \frac{x^n}{n!} = \frac{(\log(1+x))^n}{m!}\) for \(|x| < 1\).

### 3.2.9 STIRLING CYCLE NUMBERS

The Stirling cycle number, \[^{n}_{m}\], is the number of ways to partition \(n\) into \(m\) blocks (this is also called a Stirling number of the second kind). (Equivalently, it is the number of ways that \(n\) distinguishable balls can be placed into \(m\) indistinguishable cells, with no cell empty.) This Stirling number can be numerically evaluated as
\[
[^{n}_{m}] = \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^m.
\]

For example, placing the 4 distinguishable balls \([a, b, c, d]\) into 2 indistinguishable cells, so that no cell is empty can be done in \[^{4}_{2}\] = 7 ways. These are (vertical bars delineate the cells)
\[
| ab | cd |, \quad | ad | bc |, \quad | ac | bd |, \quad | a | bcd |, \\
| b | acd |, \quad | c | abd |, \quad | d | abc |.
\]

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Ordinary powers can be expanded in terms of factorial polynomials. If \( n > 0 \), then
\[
x^n = \sum_{k=1}^{n} \binom{n}{k} x^{(k)}.
\]
For example, \( x^3 = \sum_{k=1}^{3} \binom{3}{k} x^{(k)} \).

### 3.2.10 BELL NUMBERS

The \( n \)th Bell number, \( B_n \), denotes the number of partitions of a set with \( n \) elements. Computationally, the Bell numbers may be written in terms of the Stirling cycle numbers,
\[
B_n = \sum_{m=1}^{n} \frac{n^m}{m!}.
\]
For example, there are \( B_4 = 15 \) different ways to partition the 4 element set \{a, b, c, d\}:

- \{a\}, \{c\}, \{b, d\}, \{a\}, \{d\}, \{b, c\}, \{b\}, \{a, d\}, \{c\}, \{a, c\}, \{c\}, \{d\}, \{a, b\}, \{a, b, c, d\}, \{a, b\}, \{c, d\}, \{a\}, \{b\}, \{c\}, \{d\}.

<table>
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</tr>
</thead>
<tbody>
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</table>

A generating function for Bell numbers is
\[
\sum_{n=0}^{\infty} B_n x^n = \exp (e^x - 1) - 1.
\]
This results in Dobinski’s formula for the \( n \)th Bell number,
\[
B_n = e^{-1} \sum_{m=0}^{\infty} \frac{m^n}{m!}.
\]

For large values of \( n \), \( B_n \sim n^{-1/2} \lambda(n)^{n+1/2} e^{\lambda(n)n - n - 1} \) where \( \lambda(n) \) is defined by the relation: \( \lambda(n) \log \lambda(n) = n \).
### 3.2.11 TABLES

**Permutations $P(n, m)$**

This table contains the number of permutations of $n$ distinct things taken $m$ at a time, given by

$$
P(n, m) = \frac{n!}{(n-m)!} = n(n-1) \cdots (n-m+1).$$

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This table contains the number of combinations of $n$ distinct things taken $m$ at a time, given by

$$C(n, m) = \binom{n}{m} = \frac{n!}{m!(n-m)!}.$$

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©1996 CRC Press LLC
**Combinations** \( C(n, m) = \binom{n}{m} \)

This table contains the number of combinations of \( n \) distinct things taken \( m \) at a time, given by

\[
C(n, m) = \binom{n}{m} = \frac{n!}{m!(n-m)!}.
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3.3 GRAPHS

3.3.1 NOTATION

Notation for graphs

- \( E \) edge set
- \( V \) vertex set
- \( G \) graph
- \( \phi \) incidence mapping

Invariants

- \(|G|\) order
- \(\text{girth}(G)\) girth
- \(\text{Aut}(G)\) automorphism group
- \(\text{rad}(G)\) radius
- \(c(G)\) circumference
- \(P_G(x)\) chromatic polynomial
- \(d(u, v)\) distance between two vertices
- \(Z(G)\) center
- \(\text{deg } x\) degree of a vertex
- \(\alpha(G)\) independence number
- \(\text{diam}(G)\) diameter
- \(\delta(G)\) minimum degree
- \(e(G)\) size
- \(\Delta(G)\) maximum degree
- \(\text{ecc}(x)\) eccentricity
- \(\gamma(G)\) genus

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### Examples

<table>
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<td>$K_{1,n-1}$</td>
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<td>$K^{(m)}_n$</td>
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### 3.3.2 Basic Definitions

There are two standard definitions of graphs, a general definition and a more common simplification. Except where otherwise indicated, this book uses the simplified definition, according to which a graph $G$ is an ordered pair $(V, E)$ consisting of an arbitrary set $V$ and a set $E$ of 2-element subsets of $V$. Each element of $V$ is called a vertex (plural vertices). Each element of $E$ is called an edge.

According to the general definition, a graph $G$ is an ordered triple $G = (V, E, \phi)$ consisting of arbitrary sets $V$ and $E$ and an incidence mapping $\phi$ that assigns to each element $e \in E$ a nonempty set $\phi(e) \subseteq V$ of cardinality at most two. Again, the elements of $V$ are called vertices and the elements of $E$ are called edges. A loop is an edge $e$ for which $|\phi(e)| = 1$. A graph has multiple edges if edges $e \neq e'$ exist for which $\phi(e) = \phi(e')$.

A (general) graph is called simple if it has neither loops nor multiple edges. Because each edge in a simple graph can be identified with the two-element set $\phi(e) \subseteq V$, the simplified definition of graph given above is just an alternative definition of a simple graph.

The word multigraph is used to discuss general graphs with multiple edges but no loops. Occasionally the word pseudograph is used to emphasize that the graphs under discussion may have both loops and multiple edges. Every graph $G = (V, E)$ considered here is finite, i.e., both $V$ and $E$ are finite sets.

Specialized graph terms include the following:

- **acyclic**: A graph is acyclic if it has no cycles.
- **adjacency**: Two distinct vertices $v$ and $w$ in a graph are adjacent if the pair $\{v, w\}$ is an edge. Two distinct edges are adjacent if their intersection is nonempty, i.e., if there is a vertex incident with both of them.
adjacency matrix: For an ordering \( v_1, v_2, \ldots, v_n \) of the vertices of a graph \( G = (V, E) \) of order \(|G| = n\), there is a corresponding \( n \times n \) adjacency matrix \( A = (a_{ij}) \) defined as follows:

\[
a_{ij} = \begin{cases} 
1 & \text{if } \{v_i, v_j\} \in E; \\
0 & \text{otherwise.} 
\end{cases}
\]

arboricity: The arboricity \( \Upsilon(G) \) of a graph \( G \) is the minimum number of edge-disjoint spanning forests into which \( G \) can be partitioned.

automorphism: An automorphism of a graph is a permutation of its vertices that is an isomorphism.

automorphism group: The composition of two automorphisms is again an automorphism; with this binary operation, the automorphisms of a graph \( G \) form a group \( \text{Aut}(G) \) called the automorphism group of \( G \).

ball: The ball of radius \( k \) about a vertex \( u \) in a graph is the set \( B(u, k) = \{v \in V \mid d(u, v) \leq k\} \).

See also sphere and neighborhood.

block: A block is a graph with no cut vertex. A block of a graph is a maximal subgraph that is a block.

boundary operator: The boundary operator for a graph is the linear mapping from 1-chains (elements of the edge space) to 0-chains (elements of the vertex space) that sends each edge to the indicator mapping the set of two vertices incident with it. See also vertex space and edge space.

bridge: A bridge is an edge in a connected graph whose removal would disconnect the graph.

cactus: A cactus is a connected graph, each of whose blocks is a cycle.

cage: An \((r, n)\)-cage is a graph of minimal order among \( r \)-regular graphs with girth \( n \). A \((3, n)\)-cage is also called an \( n \)-cage.

center: The center \( Z(G) \) of a graph \( G = (V, E) \) consists of all vertices whose eccentricity equals the radius of \( G \):

\[
Z(G) = \{v \in V(G) \mid \text{ecc}(v) = \text{rad}(G)\}.
\]

Each vertex in the center of \( G \) is called a central vertex.

characteristic polynomial: All adjacency matrices of a graph \( G \) have the same characteristic polynomial, which is called the characteristic polynomial of \( G \).

chromatic index: The chromatic index \( \chi'(G) \) is the least \( k \) for which there exists a proper \( k \)-coloring of the edges of \( G \); in other words, it is the least number of matchings into which the edge set can be decomposed.

chromatic number: The chromatic number \( \chi(G) \) of a graph \( G \) is the least \( k \) for which there exists a proper \( k \)-coloring of the vertices of \( G \); in other words, it is the least \( k \) for which \( G \) is \( k \)-partite. See also multipartite.

chromatic polynomial: For a graph \( G \) of order \(|G| = n\) with exactly \( k \) connected components, the chromatic polynomial of \( G \) is the unique polynomial \( P_G(x) \) for which \( P_G(m) \) is the number of proper colorings of \( G \) with \( m \) colors for each positive integer \( m \).
circuit: A circuit in a graph is a trail whose first and last vertices are identical.
circulant graph: A graph $G$ is a circulant graph if its adjacency matrix is a circulant matrix; that is, the rows are circular shifts of one another.
circumference: The circumference of a graph is the length of its longest cycle.
clique: A clique is a set $S$ of vertices for which the induced subgraph $G[S]$ is complete.
clique number: The clique number $\omega(G)$ of a graph $G$ is the largest cardinality of a clique in $G$.
coboundary operator: The coboundary operator for a graph is the linear mapping from 0-chains (elements of the vertex space) to 1-chains (elements of the edge space) that sends each vertex to the indicator mapping of the set of edges incident with it.
cocycle vector: A cut vector is sometimes called a cocycle vector.
coloring: A partition of the vertex set of a graph is called a coloring, and the blocks of the partition are called color classes. A coloring with $k$ color classes is called a $k$-coloring. A coloring is proper if no two adjacent vertices belong to the same color class. See also chromatic number and chromatic polynomial.
complement: A graph $G$ is complete if every pair of distinct vertices is an edge; $K_n$ denotes a complete graph with $n$ vertices.
component: A component of a graph is a maximal connected subgraph.
connectedness: A graph is said to be connected if each pair of vertices is joined by a walk; otherwise, the graph is disconnected. A graph is $k$-connected if it has order at least $k + 1$ and each pair of vertices is joined by $k$ pairwise internally disjoint paths.
connectivity: The connectivity $\kappa(G)$ of $G$ is the largest $k$ for which $G$ is $k$-connected.
contraction: To contract an edge $\{v, w\}$ of a graph $G$ is to construct a new graph $G'$ from $G$ by removing the edge $\{v, w\}$ and identifying the vertices $v$ and $w$. A graph $G$ is contractible to a graph $H$ if $H$ can be obtained from $G$ via the contraction of one or more edges of $G$.
cover: A set $S \subseteq V$ is a vertex cover if every edge of $G$ is incident with some vertex in $S$. A set $T \subseteq E$ is an edge cover of a graph $G = (V, E)$ if each vertex of $G$ is incident to at least one edge in $T$.
crosscap number: The crosscap number $\hat{\gamma}(G)$ of a graph $G$ is the least $g$ for which $G$ has an embedding in a nonorientable surface obtained from the sphere by adding $g$ crosscaps. See also genus.
crossing: A crossing is a point lying in images of two edges of a drawing of a graph on a surface.
crossing number: The crossing number $\nu(G)$ of a graph $G$ is the minimum number of crossings among all drawings of $G$ in the plane. The rectilinear crossing number $\nu_r(G)$ of a graph $G$ is the minimum number of crossings among all drawings of $G$ in the plane for which the image of each edge is a straight line segment.
cubic: A graph is a cubic graph if it is regular of degree 3.
cut: For each partition \( V = V_1 \uplus V_2 \) of the vertex set of a graph \( G = (V, E) \) into two disjoint blocks, the set of all edges joining a vertex in \( V_1 \) to a vertex in \( V_2 \) is called a cut.

cut space: The cut space of a graph \( G \) is the subspace of the edge space of \( G \) spanned by the cut vectors.

cut vector: The cut vector corresponding to a cut \( C \) of a graph \( G = (V, E) \) is the mapping \( v: E \rightarrow GF(2) \) in the edge space of \( G \)

\[
v(e) = \begin{cases} 
1 & e \in C, \\
0 & \text{otherwise}.
\end{cases}
\]

cut vertex: A cut vertex of a connected graph is a vertex whose removal, along with all edges incident with it, leaves a disconnected graph.

cycle: A cycle is a circuit, each pair of whose vertices other than the first and the last are distinct.

cycle space: The cycle space of a graph \( G \) is the subspace of the edge space of \( G \) consisting of all 1-chains with boundary 0. An indicator mapping of a set of edges with which each vertex is incident an even number of times is called a cycle vector. The cycle space is the span of the cycle vectors.

degree: The degree \( \deg x \) of a vertex \( x \) in a graph is the number of vertices adjacent to it. The maximum and minimum degrees of vertices in a graph \( G \) are denoted \( \Delta(G) \) and \( \delta(G) \), respectively.

degree sequence: A sequence \( (d_1, \ldots, d_n) \) is a degree sequence of a graph if there is some ordering \( v_1, \ldots, v_n \) of the vertices for which \( d_i \) is the degree of \( v_i \) for each \( i \).

diameter: The diameter of \( G \) is the maximum distance between two vertices of \( G \); thus it is also the maximum eccentricity of a vertex in \( G \).

distance: The distance \( d(u, v) \) between vertices \( u \) and \( v \) in a graph \( G \) is the minimum among the lengths of \( u, v \)-paths in \( G \), or \( \infty \) if there is no \( u, v \)-path.

drawing: A drawing of a graph \( G \) in a surface \( S \) consists of a one-to-one mapping from the vertices of \( G \) to points of \( S \) and a one-to-one mapping from the edges of \( G \) to open arcs in \( X \) so that (i) no image of an edge contains an image of some vertex, (ii) the image of each edge \( \{v, w\} \) joins the images of \( v \) and \( w \), (iii) the images of adjacent edges are disjoint, (iv) the images of two distinct edges never have more than one point in common, and (v) no point of the surface lies on the images of more than two edges.

eccentricity: The eccentricity \( \text{ecc}(x) \) of a vertex \( x \) in a graph \( G \) is the maximum distance from \( x \) to a vertex of \( G \).

edge connectivity: The edge connectivity of \( G \), denoted \( \lambda(G) \), is the minimum number of edges whose removal results in a disconnected graph.

edge space: The edge space of a graph \( G = (V, E) \), is the vector space of all mappings from \( E \) to the two-element field \( GF(2) \). Elements of the edge space are called 1-chains.

embedding: An embedding of a graph \( G \) in a topological space \( X \) consists of an assignment of the vertices of \( G \) to distinct points of \( X \) and an assignment of the edges of \( G \) to disjoint open arcs in \( X \) so that no arc representing an edge contains some point representing a vertex and so that each arc representing an
edge joins the points representing the vertices incident with the edge. See also drawing.

end vertex: A vertex of degree 1 in a graph is called an end vertex.

Eulerian circuits and trails: A trail or circuit that includes every edge of a graph is said to be Eulerian, and a graph is Eulerian if it has an Eulerian circuit.

even: A graph is even if the degree of every vertex is even.

factor: A factor of a graph $G$ is a spanning subgraph of $G$. A factor in which every vertex has the same degree $k$ is called a $k$-factor. If $G_1, G_2, \ldots, G_k$ ($k \geq 2$) are edge-disjoint factors of the graph $G$, and if $\bigcup_{i=1}^{k} E(G_i) = E(G)$, then $G$ is said to be factored into $G_1, G_2, \ldots, G_k$ and we write $G = G_1 \oplus G_2 \oplus \cdots \oplus G_k$.

forest: A forest is an acyclic simple graph; see also tree.

genus: The genus $\gamma(G)$ (plural form genera) of a graph $G$ is the least $g$ for which $G$ has an embedding in an orientable surface of genus $g$. See also crosscap number.

girth: The girth $\text{gir}(G)$ of a graph $G$ is the minimum length of a cycle in $G$, or $\infty$ if $G$ is acyclic.

Hamiltonian cycles and paths: A path or cycle through all the vertices of a graph is said to be Hamiltonian. A graph is Hamiltonian if it has a Hamiltonian cycle.

homeomorphic graphs: Two graphs are homeomorphic to one another if there is a third graph of which each is a subdivision.

identification of vertices: To identify vertices $v$ and $w$ of a graph $G$ is to construct a new graph $G'$ from $G$ by removing the vertices $v$ and $w$ and all the edges of $G$ incident with them and introducing a new vertex $u$ and new edges joining $u$ to each vertex that was adjacent to $v$ or to $w$ in $G$. See also contraction.

incidence: A vertex $v$ and an edge $e$ are incident with one another if $v \in e$.

incidence matrix: For an ordering $v_1, v_2, \ldots, v_n$ of the vertices and an ordering $e_1, e_2, \ldots, e_m$ of the edges of a graph $G = (V, E)$ with order $|V| = n$ and size $e(G) = m$, there is a corresponding $n \times m$ incidence matrix $B = (b_{ij})$ defined as follows:

$$b_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } e_j \text{ are incident}, \\ 0, & \text{otherwise}. \end{cases}$$

independence number: The independence number $\alpha(G)$ of a graph $G = (V, E)$ is the largest cardinality of an independent subset of $V$.

independent set: A set $S \subseteq V$ is said to be independent if the induced subgraph $G[S]$ is empty. See also matching.

internally disjoint paths: Two paths in a graph with the same initial vertex $v$ and terminal vertex $w$ are internally disjoint if they have no internal vertex in common.

isolated vertex: A vertex is isolated if it is adjacent to no other vertex.

isomorphism: An isomorphism between graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ is a bijective mapping $\psi: V_G \to V_H$ for which $\{x, y\} \in E_G$ if and only if $\{\psi(x), \psi(y)\} \in E_H$. If there is an isomorphism between $G$ and $H$, then $G$ and $H$ are said to be isomorphic to one another; this is denoted as $G \cong H$. Figure 3.3.1 contains three graphs that are isomorphic.
**labeled graph:** Graph theorists sometimes speak loosely of *labeled graphs* of order $n$ and *unlabeled graphs* of order $n$ to distinguish between graphs with a fixed vertex set of cardinality $n$ and the family of isomorphism classes of such graphs. Thus, one may refer to *labeled graphs* to indicate an intention to distinguish between any two graphs that are *distinct* (i.e., have different vertex sets and/or different edge sets). One may refer to *unlabeled graphs* to indicate the intention to view any two distinct but isomorphic graphs as the ‘same’ graph, and to distinguish only between nonisomorphic graphs.

**matching:** A *matching* in a graph is a set of edges, no two having a vertex in common. A *maximal matching* is a matching that is not a proper subset of any other matching. A *maximum matching* is a matching of greatest cardinality. For a matching $M$, an $M$-*alternating* path is a path whose every other edge belongs to $M$, and an $M$-*augmenting* path is an $M$-alternating path whose first and last edges do not belong to $M$. A matching *saturates* a vertex if the vertex belongs to some edge of the matching.

**monotone graph property:** A property $\mathcal{P}$ that a graph may or may not enjoy is said to be *monotone* if, whenever $H$ is a graph enjoying $\mathcal{P}$, every supergraph $G$ of $H$ with $|G| = |H|$ also enjoys $\mathcal{P}$.

**multipartite graph:** A graph is $k$-*partite* if its vertex set can be partitioned into $k$ disjoint sets called *color classes* in such a way that every edge joins vertices in two different color classes (see also *coloring*). A two-partite graph is called *bipartite*.

**neighbor:** Adjacent vertices $v$ and $w$ in a graph are said to be *neighbors* of one another.

**neighborhood:** The sphere $S(x, 1)$ is called the *neighborhood* of $x$, and the ball $B(x, 1)$ is called the *closed neighborhood* of $x$.

**order:** The *order* $|G|$ of a graph $G = (V, E)$ is the number of vertices in $G$; in other words, $|G| = |V|$.

**path:** A path is a walk whose vertices are distinct.

**perfect graph:** A graph is *perfect* if $\chi(H) = \omega(H)$ for all induced subgraphs $H$ of $G$.

**planarity:** A graph is *planar* if it has a proper embedding in the plane.

**radius:** The *radius* $\text{rad}(G)$ of a graph $G$ is the minimum vertex eccentricity in $G$.

**regularity:** A graph is $k$-*regular* if each of its vertices has degree $k$. A graph is *strongly regular* with parameters $(k, \lambda, \mu)$ if (i) it is $k$-regular, (ii) every pair
of adjacent vertices has exactly \( \lambda \) common neighbors, and (iii) every pair of nonadjacent vertices has exactly \( \mu \) common neighbors. A graph \( G = (V, E) \) of order \( |G| \geq 3 \) is called highly regular if there exists an \( n \times n \) matrix \( C = (c_{ij}) \), where \( 2 \leq n < |G| \), called a collapsed adjacency matrix, so that, for each vertex \( v \) of \( G \), there is a partition of \( V \) into \( n \) subsets \( V_1 = \{v\}, V_2, \ldots, V_n \), so that every vertex \( y \in V_j \) is adjacent to exactly \( c_{ij} \) vertices in \( V_i \). Every highly regular graph is also regular.

**rooted graph:** A rooted graph is an ordered pair \( (G, v) \) consisting of a graph \( G \) and a distinguished vertex \( v \) of \( G \) called the root.

**self-complementary:** A graph is self-complementary if it is isomorphic to its complement.

**similarity:** Two vertices \( u \) and \( v \) of a graph \( G \) are similar (in symbols \( u \sim v \)) if there is an automorphism \( \alpha \) of \( G \) for which \( \alpha(u) = v \). Similarly, two edges \( (u, v) \) and \( (a, b) \) in the graph \( G \) are similar if an automorphism \( \alpha \) of \( G \) exists for which \( \{\alpha(u), \alpha(v)\} = \{a, b\} \).

**size:** The size \( e(G) \) of a graph \( G = (V, E) \) is the number of edges of \( G \), that is, \( e(G) = |E| \).

**spectrum:** The spectrum of a graph \( G \) is the spectrum of its characteristic polynomial, i.e., the nondecreasing sequence of \( |G| \) eigenvalues of the characteristic polynomial of \( G \).

**sphere:** The sphere of radius \( k \) about a vertex \( u \) is the set
\[
S(u, k) = \{v \in V \mid d(u, v) = k\}.
\]

See also ball and neighborhood.

**subdivision:** To subdivide an edge \( \{v, w\} \) of a graph \( G \) is to construct a new graph \( G' \) from \( G \) by removing the edge \( \{v, w\} \) and introducing new vertices \( x_i \) and new edges \( \{v, x_1\}, \{x_k, w\} \) and \( \{x_i, x_{i+1}\} \) for \( 1 \leq i < k \). A subdivision of a graph is a graph obtained by subdividing one or more edges of the graph.

**subgraph:** A graph \( H = (V_H, E_H) \) is a subgraph of a graph \( G = (V_G, E_G) \) (in symbols, \( H \subseteq G \)), if \( V_H \subseteq V_G \) and \( E_H \subseteq E_G \). In that case, \( G \) is a supergraph of \( H \) (in symbols, \( G \supseteq H \)). If \( V_H = V_G \), then \( H \) is called a spanning subgraph of \( G \). For each set \( S \subseteq V_G \), the subgraph \( G[S] \) of \( G \) induced by \( S \) is the unique subgraph of \( G \) with vertex set \( S \) for which every edge of \( G \) incident with two vertices in \( S \) is also an edge of \( G[S] \).

**symmetry:** A graph is vertex symmetric if every pair of vertices is similar. A graph is edge symmetric if every pair of edges is similar. A graph is symmetric if it is both vertex and edge symmetric.

**2-switch:** For vertices \( v, w, x, y \) in a graph \( G \) for which \( \{v, w\} \) and \( \{x, y\} \) are edges, but \( \{v, y\} \) and \( \{x, w\} \) are not edges, the construction of a new graph \( G' \) from \( G \) via the removal of edges \( \{v, w\} \) and \( \{x, y\} \) together with the insertion of the edges \( \{v, y\} \) and \( \{x, w\} \) is called a 2-switch.

**thickness:** The thickness \( \theta(G) \) of a graph \( G \) is the least \( k \) for which \( G \) is a union of \( k \) planar graphs.

**trail:** A trail in a graph is a walk whose edges are distinct.

**tree:** A tree is a connected forest, i.e., a connected acyclic graph. A spanning subgraph of a graph \( G \) that is a tree is called a spanning tree of \( G \).

**triangle:** A 3-cycle is called a triangle.
**trivial graph:** A *trivial* graph is a graph with exactly one vertex and no edges.

**unicyclic graph:** A unicyclic graph is a connected graph that contains exactly one cycle.

**vertex space:** The *vertex space* of a graph $G$ is the vector space of all mappings from $V$ to the two-element field $GF(2)$. The elements of the vertex space are called *0-chains*.

**walk:** A *walk* in a graph is an alternating sequence $v_0, e_1, v_1, \ldots, e_k, v_k$ of vertices $v_i$ and edges $e_i$ for which $e_i$ is incident with $v_{i-1}$ and with $v_i$ for each $i$. Such a walk is said to have length $k$ and to join $v_0$ and $v_k$. The vertices $v_0$ and $v_k$ are called the *initial vertex* and *terminal vertex* of the walk; the remaining vertices are called *internal vertices* of the walk.

### 3.3.3 CONSTRUCTIONS

**Operations on graphs**

For graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, there are several binary operations that yield a new graph from $G_1$ and $G_2$. The following table gives the names of those operations and the orders and sizes of the resulting graphs.

| Operation producing $G$ | Order $|G|$ | Size $e(G)$ |
|-------------------------|-----------|-------------|
| Composition $G_1[G_2]$ | $|G_1| \cdot |G_2|$ | $|G_1|e(G_2) + |G_2|^2e(G_1)$ |
| Conjunction $G_1 \land G_2$ | $|G_1| \cdot |G_2|$ | $e(G_1) + e(G_2)$ |
| Edge sum$^a$ $G_1 \oplus G_2$ | $|G_1| = |G_2|$ | $e(G_1) + e(G_2)$ |
| Join $G_1 + G_2$ | $|G_1| + |G_2|$ | $e(G_1) + e(G_2) + |G_1| \cdot |G_2|$ |
| Product $G_1 \times G_2$ | $|G_1| \cdot |G_2|$ | $|G_1|e(G_2) + |G_2|e(G_1)$ |
| Union $G_1 \cup G_2$ | $|G_1| + |G_2|$ | $e(G_1) + e(G_2)$ |

$^a$When applicable.

**composition:** For graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *composition* $G = G_1[G_2]$ is the graph with vertex set $V_1 \times V_2$ whose edges are the pairs \{(u, v), (u, w)\} with $u \in V_1$ and $v, w \in E_2$ and the pairs \{(t, u), (v, w)\} for which $\{t, v\} \in E_1$ and $\{u, w\} \in E_2$.

**conjunction:** The conjunction $G_1 \land G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_3 = (V_3, E_3)$ for which $V_3 = V_1 \times V_2$ and for which vertices $e_1 = (u_1, u_2)$ and $e_2 = (v_1, v_2)$ in $V_3$ are adjacent in $G_3$ if, and only if, $u_1$ is adjacent to $v_1$ in $G_1$ and $u_2$ is adjacent to $v_2$ in $G_2$.

**edge difference:** For graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with the same vertex set $V$, the *edge difference* $G_1 - G_2$ is the graph with vertex set $V$ and edge set $E_1 \setminus E_2$.

**edge sum:** For graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with the same vertex set $V$, the *edge sum* of $G_1$ and $G_2$ is the graph $G_1 \oplus G_2$ with vertex set $V$ and edge set $E_1 \cup E_2$. Sometimes the edge sum is denoted $G_1 \cup G_2$.

**join:** For graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $V_1 \cap V_2 = \emptyset$, the *join* $G_1 + G_2 = G_2 + G_1$ is the graph obtained from the union of $G_1$ and $G_2$ by adding edges joining each vertex in $V_1$ to each vertex in $V_2$.

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power: For a graph \( G = (V, E) \), the \( k \text{th power} \ G^k \) is the graph with the same vertex set \( V \) whose the edges are the pairs \( \{u, v\} \) for which \( d(u, v) \leq k \) in \( G \).

The square of \( G \) is \( G^2 \).

product: For graphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \), the product \( G_1 \times G_2 \) has vertex set \( V_1 \times V_2 \); its edges are all of the pairs \( \{(u, v), (u, w)\} \) for which \( u \in V_1 \) and \( (v, w) \in E_2 \) and all of the pairs \( \{(t, v), (u, v)\} \) for which \( \{t, u\} \in E_1 \) and \( v \in V_2 \).

union: For graphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) with \( V_1 \cap V_2 = \emptyset \), the union of \( G_1 \) and \( G_2 \) is the graph \( G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2) \). The union is sometimes called the disjoint union to distinguish it from the edge sum.

Graphs described by one parameter

complete graph, \( K_n \): A complete graph of order \( n \) is a graph isomorphic to the graph \( K_n \) with vertex set \( \{1, 2, \ldots, n\} \) whose every pair of vertices is an edge. The graph \( K_n \) has size \( \binom{n}{2} \) and is Hamiltonian. If \( G \) is a graph \( g \) of order \( n \), then \( K_n = G \overset{n}{\times} G \).

cube, \( Q_n \): An \( n \text{-cube} \) is a graph isomorphic to the graph \( Q_n \) whose vertices are the \( 2^n \) binary \( n \)-vectors and whose edges are the pairs of vectors that differ in exactly one place. It is an \( n \)-regular bipartite graph of order \( 2^n \) and size \( n2^{n-1} \). An equivalent recursive definition, \( Q_1 = K_2 \) and \( Q_n = Q_{n-1} \times K_2 \).

cycle, \( C_n \): A cycle of order \( n \) is a graph isomorphic to the graph \( C_n \) with vertex set \( \{0, 1, \ldots, n-1\} \) whose edges are the pairs \( \{v_i, v_{i+1}\} \) with \( 0 \leq i < n \) and arithmetic modulo \( n \). The cycle \( C_n \) has size \( n \) and is Hamiltonian.

The graph \( C_n \) is a special case of a circulant graph. The graph \( C_3 \) is called a triangle; the graph \( C_4 \) is called a square.

empty graph: A graph is empty if it has no edges; \( \overline{K}_n \) denotes an empty graph of order \( n \).

Kneser graphs, \( K^{(m)}_n \): For \( m \geq 2n \), the Kneser graph \( K^{(m)}_n \) is the complement of the intersection graph of the \( m \)-subsets of a \( n \)-set. The odd graph \( O_n \) is the Kneser graph \( K^{(m)}_{2m+1} \). The Petersen graph is the odd graph \( O_2 = K^{(2)}_5 \).

ladder: A ladder is a graph of the form \( P_n \times P_2 \). The Möbius ladder \( M_n \) is the graph obtained from the ladder \( P_n \times P_2 \) by joining the opposite end vertices of the two copies of \( P_n \).

path, \( P_n \): A path of order \( n \) is a graph isomorphic to the graph \( P_n \) whose vertex set is \( \{1, \ldots, n\} \) and whose edges are the pairs \( \{v_i, v_{i+1}\} \) with \( 1 \leq i < n \). A path of order \( n \) has size \( n - 1 \) and is a tree.

star, \( S_n \): A star of order \( n \) is a graph isomorphic to the graph \( S_n = K_{1,n-1} \). It has a vertex cover consisting of a single vertex, its size is \( n - 1 \), and it is a complete bipartite graph and a tree.

wheel, \( W_n \): The wheel \( W_n \) of order \( n \geq 4 \) consists of a cycle of order \( n - 1 \) and an additional vertex adjacent to every vertex in the cycle. Equivalently, \( W_n = C_{n-1} + K_1 \). This graph has size \( 2(n - 1) \).

Graphs described by two parameters

complete bipartite graph, \( K_{n,m} \): The complete bipartite graph \( K_{n,m} \) is the graph \( \overline{K}_n + \overline{K}_m \)
\(K_m\). Its vertex set can be partitioned into two color classes of cardinalities \(n\) and \(m\), respectively, so that each vertex in one color class is adjacent to every vertex in the other color class. The graph \(K_{n,m}\) has order \(n + m\) and size \(nm\).

**planar mesh:** A graph of the form \(P_n \times P_m\) is called a planar mesh.

**prism:** A graph of the form \(C_m \times P_n\) is called a prism.

**Toeplitz graph, \(TN(w, s)\):** The Toeplitz graph \(TN(w, s)\) is defined in terms of its adjacency matrix \(A = (a_{ij})\), for which

\[
a_{ij} = \begin{cases} 
1, & \text{if } |i - j| = 1 \pmod{w}, \\
0, & \text{otherwise.}
\end{cases}
\]

The Toeplitz graph is of order \(ws + 2\), size \((s + 1)(w - s + 2)/2\), and girth 3 or 4; it is \((s + 1)\)-regular and Hamiltonian. Moreover, \(TN(1, s) = K_{s+2}\) and \(TN(w, 1) = C_{w+2}\).

**toroidal mesh:** A graph of the form \(C_m \times C_n\) with \(m \geq 2\) and \(n \geq 2\) is called a toroidal mesh.

**Turán graph, \(T_{n,k}\):** The Turán graph \(T_{n,k}\) is the complete \(k\)-partite graph in which the cardinalities of any two color classes differ by, at most, one. It has \(n - k\lfloor n/k \rfloor\) color classes of cardinality \(\lfloor n/k \rfloor + 1\) and \(k - n + k \lfloor n/k \rfloor\) color classes of cardinality \(\lfloor n/k \rfloor\). Note that \(\omega(T_{n,k}) = k\).

**Graphs described by three or more parameters**

**Cayley graph:** For a group \(\Gamma\) and a set \(X\) of generators of \(\Gamma\), the Cayley graph of the pair \((\Gamma, X)\) is the graph with vertex set \(\Gamma\) in which \([\alpha, \beta]\) is an edge if either \(\alpha^{-1}\beta \in X\) or \(\beta^{-1}\alpha \in X\).

**complete multipartite graph, \(K_{n_1,n_2,...,n_k}\):** The complete \(k\)-partite graph \(K_{n_1,n_2,...,n_k}\) is the graph \(\overline{K_{n_1}} + \cdots + \overline{K_{n_k}}\). It is a \(k\)-partite graph with color classes \(V_i\) of cardinalities \(|V_i| = n_i\) for which every pair of vertices in two distinct color classes is an edge. The graph \(K_{n_1,n_2,...,n_k}\) has order \(\sum_{i=1}^{k} n_i\) and size \(\prod_{1 \leq i < j \leq k} n_i n_j\).

**double loop graph, \(DLG(n;a,b)\):** The double loop graph \(DLG(n;a,b)\) (with \(a\) and \(b\) between 1 and \((n-1)/2\)), consists of \(n\) vertices with every vertex \(i\) connected by an edge to the vertices \(i \pm a\) and \(i \pm b\) (modulo \(n\)). The name comes from the following fact: If \(\gcd(a, b, n) = 1\), then \(DLG(n; a, b)\) is Hamiltonian and, additionally, \(DLG(n; a, b)\) can be decomposed into two Hamiltonian cycles. These graphs are also known as circulant graphs.

**intersection graph:** For a family \(F = \{S_1, \ldots, S_n\}\) of subsets of a set \(S\), the intersection graph of \(F\) is the graph with vertex set \(F\) in which \(\{S_i, S_j\}\) is an edge if and only if \(S_i \cap S_j \neq \emptyset\). Each graph \(G\) is an intersection graph of some family of subsets of a set of cardinality at most \(\lceil |G|^2 / 4 \rceil\).

**interval graph:** An interval graph is an intersection graph of a family of intervals on the real line.

**Small examples**

The small graphs can be described in terms of the operations on page 190. Let \(\mathcal{G}_{n,m}\) denote the family of isomorphism classes of graphs of order \(n\) and size \(m\). Then
FIGURE 3.3.2
Examples of graphs with 6 vertices.

\[ \bar{K}_6 = \]

\[ P_6 = \]

\[ S_6 = \]

\[ W_6 = \]

\[ C_6 = \]

\[ \bar{C}_6 = \]

\[ K_{3,3} = \]

\[ G_1,0 = \{ K_1 \}, \]
\[ G_2,0 = \{ \bar{K}_2 \}, \]
\[ G_2,1 = \{ K_2 \}, \]
\[ G_3,0 = \{ \bar{K}_3 \}, \]
\[ G_{3,1} = \{ K_2 \cup K_1 \}, \]
\[ G_{3,2} = \{ P_3 \}, \]
\[ G_{3,2} = \{ K_3 \}, \]
\[ G_{4,0} = \{ \bar{K}_4 \}, \]
\[ G_{4,1} = \{ P_2 \cup \bar{K}_2 \}, \]
\[ G_{4,2} = \{ P_3 \cup \bar{K}_1, P_2 \cup P_2 \}, \]
\[ G_{4,3} = \{ P_4, K_3 \cup \bar{K}_1, K_{1,3} \}, \]
\[ G_{4,4} = \{ C_4, (K_2 \cup K_1)+K_1 \}, \]
\[ G_{4,5} = \{ K_4 - e \}, \text{ and} \]
\[ G_{4,6} = \{ K_4 \}. \]

3.3.4 FUNDAMENTAL RESULTS

Walks and connectivity
1. Every \( x, y \) walk includes all the edges of some \( x, y \) path.
2. Some path in \( G \) has length \( \delta(G) \).
3. Connectivity is a monotone graph property. If more edges are added to a
connected graph, the new graph is itself connected.

4. A graph is disconnected if, and only if, it is the sum of two graphs.

5. The sets \( S \) for which \( G[S] \) is a component partition of the vertex set \( V \).

6. Every vertex of a graph lies in at least one block.

7. For every graph \( G \), \( 0 \leq \kappa(G) \leq |G| - 1 \).

8. For all integers \( a, b, c \) with \( 0 < a \leq b \leq c \), a graph \( G \) exists with \( \kappa(G) = a \), \( \lambda(G) = b \), and \( \delta(G) = c \).

9. For any graph \( G \), \( \kappa(G) \leq \lambda(G) \leq \delta(G) \).

10. **Menger’s theorem**: Suppose that \( G \) is a connected graph of order greater than \( k \). Then \( G \) is \( k \)-connected if, and only if, it is impossible to disconnect \( G \) by removing fewer than \( k \) vertices, and \( G \) is \( k \)-edge connected if, and only if, it is impossible to disconnect \( G \) by removing fewer than \( k \) edges.

11. If \( G \) is a connected graph with a bridge, then \( \lambda(G) = 1 \). If \( G \) has order \( n \) and is \( r \)-regular with \( r \geq n/2 \), then \( \lambda(G) = r \).

**Trees**

1. A graph is a tree if, and only if, it is acyclic and has size \( |G| - 1 \).

2. A graph is a tree if, and only if, it is connected and has size \( |G| - 1 \).

3. A graph is a tree if, and only if, each of its edges is a bridge.

4. A graph is a tree if, and only if, each vertex of degree greater than 1 is a cut vertex.

5. A graph is a tree if, and only if, each pair of its vertices is joined by exactly one path.

6. Every tree of order greater than 1 has at least two end vertices.

7. The center of a tree consists of one vertex or two adjacent vertices.

8. For each graph \( G \), every tree with at most \( \delta(G) \) edges is a subgraph of \( G \).

9. Every connected graph has a spanning tree.

10. **Kirchhoff matrix-tree theorem**: Let \( G \) be a connected graph and let \( A \) be an adjacency matrix for \( G \). Obtain a matrix \( M \) from \( -A \) by replacing each term \( a_{ii} \) on the main diagonal with \( \deg v_i \). Then all cofactors of \( M \) have the same value, which is the number of spanning trees of \( G \).

11. **Nash–Williams arboricity theorem**: For a graph \( G \) and for each \( n \leq |G| \), define \( e_n(G) = \max\{e(H) : H \leq G, \text{ and } |H| = n \} \). Then

\[
\Upsilon(G) = \max_n \left[ \frac{e_n}{n - 1} \right].
\]

**Circuits and cycles**

1. **Euler’s theorem**: A multigraph is Eulerian if and only if it is connected and even.
2. If $G$ is Hamiltonian, and if $G'$ is obtained from $G$ by removing a nonempty set $S$ of vertices, then the number of components of $G'$ is at most $|S|$.

3. Ore’s theorem: If $G$ is a graph for which $\deg v + \deg w \geq |G|$ whenever $v$ and $w$ are nonadjacent vertices, then $G$ is Hamiltonian.

4. Dirac’s theorem: If $G$ is a graph of order $|G| \geq 3$ and $\deg v \geq |G|/2$ for each vertex $v$, then $G$ is Hamiltonian.

5. (Erdős–Chvátal) If $\alpha(G) \leq \kappa(G)$, then $G$ is Hamiltonian.

6. Every 4-connected planar graph is Hamiltonian.

Clique and independent sets

1. A set $S \subseteq V$ is a vertex cover if, and only if, $V \setminus S$ is an independent set.

2. Turán’s theorem: If $|G| = n$ and $\omega(G) \leq k$, then $e(G) \leq e(T_{n,k})$.

3. Ramsey’s theorem: For all positive integers $k$ and $l$, there is a least integer $R(k,l)$ for which every graph of order at least $R(k,l)$ has either a clique of cardinality $k$ or an independent set of cardinality $l$. For $k \geq 2$ and $l \geq 2$, $R(k,l) \leq R(k,l-1) + R(k-1,l)$. The following table gives the values of $R(k,l)$ for $k \leq 3$ and $l \leq 7$.

<table>
<thead>
<tr>
<th>$l$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k = 1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>9</td>
<td>14</td>
<td>18</td>
<td>23</td>
</tr>
</tbody>
</table>

Colorings and partitions

1. Every graph $G$ is $k$-partite for some $k$; in particular, $G$ is $|G|$-partite.

2. Every graph $G$ has a bipartite subgraph $H$ for which $e(H) \geq e(G)/2$.

3. $P_G(x) = P_{G-e}(x) - P_{G\cup e}(x)$.

4. Brooks’s theorem: If $G$ is a connected graph that is neither a complete graph nor a cycle of odd length, then $\chi(G) \leq \Delta(G)$.

5. For all positive integers $g$ and $c$, a graph $G$ exists with $\chi(G) \geq c$ and $\text{gir}(G) \geq g$.

6. Nordhaus–Gaddum bounds: For every graph $G$,

$$2\sqrt{|G|} \leq \chi(G) + \chi(G) \leq |G| + 1,$$

and

$$|G| \leq \chi(G)\chi(G) \leq \left(\frac{|G| + 1}{2}\right)^2.$$

7. Szekeres–Wilf theorem: For every graph $G = (V,E)$,

$$\chi(G) \leq 1 + \max_{S \subseteq V} \delta(G[S]).$$
8. (König) If $G$ is bipartite, then $\chi'(G) = \Delta(G)$.

9. Vizing’s theorem: For every graph $G$, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

10. The following table gives the chromatic numbers and chromatic polynomials of various graphs:

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\chi(G)$</th>
<th>$P_G(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_n$</td>
<td>$n$</td>
<td>$x(x-1) \cdots (x-n+1)$</td>
</tr>
<tr>
<td>$\overline{K}_n$</td>
<td>1</td>
<td>$x^n$</td>
</tr>
<tr>
<td>$T_n$</td>
<td>2</td>
<td>$x(x-1)^{n-1}$</td>
</tr>
<tr>
<td>$P_n$</td>
<td>2</td>
<td>$x(x-1)^{n-1}$</td>
</tr>
<tr>
<td>$C_4$</td>
<td>2</td>
<td>$x(x-1)(x^2 - 3x + 3)$</td>
</tr>
</tbody>
</table>

11. (Appel-Haken) Four-color theorem: $\chi(G) \leq 4$ for every planar graph $G$.

12. For each graph $G$ of order $|G| = n$ and size $e(G) = m$ with exactly $k$ components, the chromatic polynomial is of the form

$$P_G(x) = \sum_{i=0}^{n-k} (-1)^i a_i x^{n-i},$$

with $a_0 = 1$, $a_1 = m$ and every $a_i$ positive.

13. Not every polynomial is a chromatic polynomial. For example $P(x) = x^4 - 4x^3 + 3x^2$ is not a chromatic polynomial.

14. Sometimes a class of chromatic polynomials can only come from a specific class of graphs. For example:
   
   (a) If $P_G(x) = x^n$, then $G = \overline{K}_n$.
   
   (b) If $P_G(x) = (x)_n$, then $G = K_n$.
   
   (c) If $P_G(x) = x(x-1) \cdots (x-r+2)(x-r+1)^2 (x-r)^{n-r-1}$ for a graph of order $n \geq r + 1$, then $G$ can be obtained from a $r$-tree $T$ of order $n$ by deleting an edge contained in exactly $r - 1$ triangles of $T$.

**Distance**

1. A metric space $(X, d)$ is the metric space associated with a connected graph with vertex set $X$ if, and only if, it satisfies two conditions: (i) $d(u,v)$ is a nonnegative integer for all $u,v \in X$, and (ii) whenever $d(u,v) \geq 2$, some element of $X$ lies between $u$ and $v$. The edges of the graph are the pairs $\{u, v\} \subseteq X$ for which $d(u,v) = 1$. (In an arbitrary metric space $(X, d)$, a point $v \in X$ is said to lie between distinct points $u \in X$ and $w \in X$ if it satisfies the triangle equality $d(u,w) = d(u,v) + d(v,w)$.)

2. If $G = (V, E)$ is connected, then distance is always finite, and $d$ is a metric on $V$. Note that $\deg(x) = |S(x, 1)|$.

3. Moore bound: For every connected graph $G$,

$$|G| \leq 1 + \Delta(G) \sum_{i=1}^{\text{diam}(G)} (\Delta(G) - 1)^i.$$

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A graph for which the Moore bound holds exactly is called a *Moore graph* with parameters \((|G|, \Delta(G), \text{diam}(G))\). Every Moore graph is regular. If \(G\) is a Moore graph with parameters \((n, r, d)\), then \((n, r, d) = (n, n - 1, 1)\) (in which case \(G\) is complete) or \((n, r, d) = (2m + 1, 2, m)\) (in which case \(G\) is a \((2m + 1)\)-cycle), \((n, r, d) \in \{(10, 3, 2), (50, 7, 2), (3250, 57, 2)\}\).

### Drawings, embeddings, planarity, and thickness

1. Every graph has an embedding in \(\mathbb{R}^3\) for which the arcs representing edges are all straight line segments. Such an embedding can be constructed by using distinct points on the curve \(\{(t, t^2, t^3) : 0 \leq t \leq 1\}\) as representatives for the vertices.
2. For \(n \geq 2\),
   \(\gamma(Q_n) = (n - 4)2^{n-3} + 1\), and
   \(\tilde{\gamma}(Q_n) = (n - 4)2^{n-2} + 2\).
3. For \(r, s \geq 2\),
   \(\gamma(K_{r,s}) = \left\lceil \frac{(r - 2)(s - 2)}{4} \right\rceil\), and
   \(\tilde{\gamma}(K_{r,s}) = \left\lceil \frac{(r - 2)(s - 2)}{2} \right\rceil\).
4. For \(n \geq 3\),
   \(\gamma(K_n) = \left\lceil \frac{(n - 3)(n - 4)}{12} \right\rceil\), and
   \(\tilde{\gamma}(K_n) = \left\lceil \frac{(n - 3)(n - 4)}{6} \right\rceil\).
5. *Heawood map coloring theorem*: The greatest chromatic number among graphs of genus \(n\) is
   \[
   \max\{\chi(G) \mid \gamma(G) = n\} = \left\lceil \frac{7 + \sqrt{1 + 48n}}{2} \right\rceil.
   \]
6. *Kuratowski’s theorem*: A graph is planar if and only if it has no subgraph homeomorphic to \(K_5\) or \(K_{3,3}\).
7. A graph is planar if and only if it does not have a subgraph contractible to \(K_5\) or \(K_{3,3}\).
8. The graph \(K_n\) is nonplanar if and only if \(n \geq 5\).
9. Every planar graph can be embedded in the plane so that every edge is a straight line segment; this is a Fary embedding.
10. The four-color theorem states that any planar graph is four colorable.
11. For every graph \(G\) of order \(|G| \geq 3\), \(\theta(G) \geq \left\lceil \frac{e(G)}{3|G| - 6} \right\rceil\).

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12. The complete graphs $K_9$ and $K_{10}$ have thickness 3; for $n \notin \{9, 10\}$,

$$\theta(K_n) = \left\lfloor \frac{n + 7}{6} \right\rfloor.$$ 

13. The $n$-cube has thickness $\theta(Q_n) = \lceil n/4 \rceil + 1$.

14. For every planar graph $G$, $\nu(G) = \overline{\nu}(G)$. That equality does not hold for all graphs: $\nu(K_8) = 18$, and $\overline{\nu}(K_8) = 19$.

**Vertex degrees**

1. **Handshaking lemma**: For every graph $G$, $\sum_{v \in V} \deg v = 2e(G)$.

2. Every 2-switch preserves the degree sequence.

3. If $G$ and $H$ have the same degree sequence, then $H$ can be obtained from $G$ via a sequence of 2-switches.

4. (Havel–Hakimi) A sequence $\{d_1, d_2, \ldots, d_n\}$ of nonnegative integers with $d_1 \geq d_2 \geq \cdots \geq d_n$ (with $n \geq 2$ and $d_1 \geq 1$) is a degree sequence if and only if the sequence $\{d_2 - 1, d_3 - 1, \ldots, d_{d_1 + 1} - 1, d_{d_1 + 2}, \ldots, d_n\}$ is a degree sequence.

**Algebraic methods**

1. The bipartite graphs $K_{n,n}$ are circulant graphs.

2. For a graph $G$ with exactly $k$ connected components, the cycle space has dimension $e(G) - |G| + k$, and the cut space has dimension $|G| - k$.

3. In the $k$th power $A^k = (a_{ij}^k)$ of the adjacency matrix, each entry $a_{ij}^k$ is the number of $v_i, v_j$ walks of length $k$.

4. The incidence matrix of a graph $G$ is totally unimodular if, and only if, $G$ is bipartite.

5. Every odd graph is vertex-transitive.

6. The smallest graph that is vertex symmetric, but is not edge symmetric, is the prism $K_3 \times K_2$. The smallest graph that is edge symmetric, but is not vertex symmetric, is $S_2 = P_3 = K_{1,2}$.

7. The spectrum of a disconnected graph is the union of the spectra of its components.

8. The sum of the eigenvalues in the spectrum of a graph is zero.

9. The number of distinct eigenvalues in the spectrum of a graph is greater than the diameter of the graph.

10. The largest eigenvalue in the spectrum of a graph $G$ is, at most, $\Delta(G)$, with equality if, and only if, $G$ is regular.

11. (Wilf) If $G$ is a connected graph and its largest eigenvalue is $\lambda$, then $\chi(G) \leq 1 + \lambda$. Moreover, equality holds if, and only if, $G$ is a complete graph or a cycle of odd length.
12. (Hoffman) If \( G \) is a connected graph of order \( n \) with spectrum \( \lambda_1 \geq \cdots \geq \lambda_n \), then \( \chi(G) \geq 1 - \frac{\lambda_1}{\lambda_n} \).

13. **Integrality condition:** If \( G \) is a strongly regular graph with parameters \( (k, \lambda, \mu) \), then the quantities

\[
\frac{1}{2} \left( |G| - 1 \pm \frac{(|G| - 1)(\mu - \lambda) - 2k}{\sqrt{(\mu - \lambda)^2 + 4(k - \mu)}} \right)
\]

are nonnegative integers.

14. The following table gives the automorphism groups of various graphs:

<table>
<thead>
<tr>
<th>( G )</th>
<th>( \text{Aut}(G) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_n )</td>
<td>( D_n )</td>
</tr>
<tr>
<td>( K_n )</td>
<td>( S_n )</td>
</tr>
<tr>
<td>( K_n )</td>
<td>( S_n )</td>
</tr>
<tr>
<td>( K_{1,n} )</td>
<td>( E_1 + S_n )</td>
</tr>
<tr>
<td>( P_n )</td>
<td>( C_n )</td>
</tr>
</tbody>
</table>

15. A graph and its complement have the same group; \( \text{Aut}(G) = \text{Aut}(\overline{G}) \).

16. **Frucht’s theorem:** Every finite group is the automorphism group of some graph.

17. If \( G \) and \( G' \) are edge isomorphic, then \( G \) and \( G' \) are not required to be isomorphic. For example, the graphs \( C_3 \) and \( S_3 \) are edge isomorphic, but not isomorphic.

18. If the graph \( G \) has order \( n \), then the order of its automorphism group \( |\text{Aut}(G)| \) is a divisor of \( n! \). The order of the automorphism group equals \( n! \) if and only if \( G \simeq K_n \) or \( G \simeq K_n \).

**Matchings**

1. A matching \( M \) is a maximum matching if, and only if, there is no \( M \)-augmenting path.

2. **Hall’s theorem:** A bipartite graph \( G \) with bipartition \( (B_1, B_2) \) has a matching that saturates every vertex in \( B_1 \) if, and only if, \( |S(A, 1)| \geq |A| \) for every \( A \subseteq B_1 \).

3. **König’s theorem:** In a bipartite graph, the cardinality of a maximum matching equals the cardinality of a minimum vertex cover.

**Enumeration**

1. The number of labeled graphs of order \( n \) is \( 2^{\binom{n}{2}} \).

2. The number of labeled graphs of order \( n \) and size \( m \) is \( \binom{\binom{n}{2}}{m} \).

3. The number of different ways in which a graph \( G \) of order \( n \) can be labeled is \( n!/|\text{Aut}(G)| \).

4. **Cayley’s formula:** The number of labeled trees of order \( n \) is \( n^{n-2} \).

5. The number of labeled trees of order \( n \) with exactly \( t \) end vertices is \( \frac{n!}{t!} \binom{n-2}{n-t} \) for \( 2 \leq t \leq n - 1 \).
Define the generating functions $T(x) = \sum_{n=0}^{\infty} T_n x^n$ and $t(x) = \sum_{n=0}^{\infty} t_n x^n$. Then $T(x) = x \exp \left( \sum_{r=1}^{\infty} \frac{1}{r} T(x^r) \right)$, and $t(x) = T(x) - \frac{1}{2} \left[ T^2(x) - T(x^2) \right]$.

The following table lists the number of isomorphism classes of graphs of order $n$ and size $m$.
The following table gives the number of labeled graphs of order \( n \) having various properties:

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>All</td>
<td>1</td>
<td>2</td>
<td>8</td>
<td>64</td>
<td>1 024</td>
<td>( 2^{15} )</td>
<td>( 2^{21} )</td>
<td>( 2^{28} )</td>
</tr>
<tr>
<td>Connected</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>38</td>
<td>728</td>
<td>26 704</td>
<td>1 866 256</td>
<td>251 548 592</td>
</tr>
<tr>
<td>Even</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>8</td>
<td>64</td>
<td>( 2^{15} )</td>
<td>( 2^{21} )</td>
<td>( 2^{28} )</td>
</tr>
<tr>
<td>Trees</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>16</td>
<td>125</td>
<td>1 296</td>
<td>16 807</td>
<td>262 144</td>
</tr>
</tbody>
</table>

The following table gives the numbers of isomorphism classes of graphs of order \( n \) exhibiting various properties:

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>All</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>11</td>
<td>34</td>
<td>156</td>
<td>1 044</td>
<td>12 346</td>
</tr>
<tr>
<td>Connected</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>21</td>
<td>112</td>
<td>853</td>
<td>11 117</td>
</tr>
<tr>
<td>Even</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>7</td>
<td>16</td>
<td>54</td>
<td>243</td>
</tr>
<tr>
<td>Eulerian</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>8</td>
<td>37</td>
<td>184</td>
</tr>
<tr>
<td>Blocks</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>10</td>
<td>56</td>
<td>468</td>
<td>7 123</td>
</tr>
<tr>
<td>Trees</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td>11</td>
<td>23</td>
</tr>
<tr>
<td>Rooted trees</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>9</td>
<td>20</td>
<td>48</td>
<td>115</td>
</tr>
</tbody>
</table>

### 3.3.5 TREE DIAGRAMS

Let \( T_{n,m} \) denote the \( m \)th isomorphism class of trees of order \( n \). Figure 3.3.3 depicts trees of order at most 7. Figure 3.3.4 depicts trees of order 8.

### 3.4 PARTIALLY ORDERED SETS

Consider a set \( S \) and a relation on it. Given any two elements \( x \) and \( y \) in \( S \), we can determine whether or not \( x \) is “related” to \( y \); if it is, \( x \leq y \). The relation \( \leq \) will be a **partial order** on \( S \) if it satisfies the following three conditions:

- **Reflexive**: \( s \leq s \) for every \( s \in S \),
- **Antisymmetric**: \( s \leq t \) and \( t \leq s \) imply \( s = t \), and
- **Transitive**: \( s \leq t \) and \( t \leq u \) imply \( s \leq u \).

If \( \leq \) is a partial order on \( S \), then the pair \( (S, \leq) \) is called a **partially ordered set** or a **poset**. Given the partial order \( \leq \) on the set \( S \), define the relation \( < \) by

\[
x < y \quad \text{if and only if} \quad x \leq y \text{ and } x \neq y.
\]

We say that the element \( t \) **covers** the element \( s \) if \( s < t \) and there is no element \( u \) with \( s < u < t \). A Hasse diagram of the poset \( (S, \leq) \) is a figure consisting of the elements of \( S \) with a line segment directed generally upward from \( s \) to \( t \) whenever \( t \) covers \( s \).
FIGURE 3.3.3
Trees with 7 or fewer vertices.

\[ T_{1,1} \quad T_{7,1} \]
\[ T_{2,1} \quad T_{7,2} \]
\[ T_{3,1} \quad T_{7,3} \]
\[ T_{4,1} \quad T_{7,4} \]
\[ T_{4,2} \quad T_{7,5} \]
\[ T_{5,1} \quad T_{7,6} \]
\[ T_{5,2} \quad T_{7,7} \]
\[ T_{5,3} \quad T_{7,8} \]
\[ T_{6,1} \quad T_{7,9} \]
\[ T_{6,2} \quad T_{7,10} \]
\[ T_{6,3} \quad T_{7,11} \]
\[ T_{6,4} \]
\[ T_{6,5} \]
\[ T_{6,6} \]
FIGURE 3.3.4
Trees with 8 vertices.

\[ T_{8,1}, T_{8,2}, T_{8,3}, T_{8,4}, T_{8,5}, T_{8,6}, T_{8,7}, T_{8,8}, T_{8,9}, T_{8,10}, T_{8,11}, T_{8,12}, T_{8,13}, T_{8,14}, T_{8,15}, T_{8,16}, T_{8,17}, T_{8,18}, T_{8,19}, T_{8,20}, T_{8,21}, T_{8,22}, T_{8,23} \]
FIGURE 3.4.5
Left: Hasse diagram for integers up to 12 with \( x \leq y \) meaning “the number \( x \) divides the number \( y \).” Right: Hasse diagram for \( \{a, b, c\} \) with \( x \leq y \) meaning “the set \( x \) is a subset of the set \( y \).”

Two elements \( x \) and \( y \) in a poset \((S, \preceq)\) are said to be comparable if either \( x \preceq y \) or \( y \preceq x \). If every pair of elements in a poset is comparable, then \((S, \preceq)\) is a chain. An antichain is a poset in which no two elements are comparable (i.e., it is not true that \( x \preceq y \) or \( y \preceq x \) for all \( x \) and \( y \) in the antichain). A maximal chain is a chain that is not properly contained in another chain (and similarly for a maximal antichain).

For example:

1. Let \( S \) be the set of natural numbers up to 12 and let “\( x \leq y \)” mean “the number \( x \) divides the number \( y \).” Then \((S, \preceq)\) is a poset with the Hasse diagram shown in Figure 3.4.5 (left). Observe that the elements 2 and 4 are comparable, but elements 2 and 5 are not comparable.

2. Let \( S \) be the set of all subsets of the set \( \{a, b, c\} \) and let “\( x \leq y \)” mean “the set \( x \) is contained in the set \( y \).” Then \((S, \preceq)\) is a poset with the Hasse diagram shown in Figure 3.4.5 (right).

3.5 COMBINATORIAL DESIGN THEORY

Combinatorial design theory is the study of families of subsets with various prescribed regularity properties. An incidence structure is an ordered pair \((X, B)\):

- \( X = x_1, \ldots, x_r \) is a set of points.
- \( B = B_1, \ldots, B_b \) is a set of blocks or lines; each \( B_j \subseteq X \).
- The replication number \( r_i \) of \( x_i \) is the number of blocks that contain \( x_i \).
- The size of \( B_j \) is \( k_j \).
Counting the number of pairs \((x, B)\) with \(x \in B\) yields \(\sum_{i=1}^{v} r_i = \sum_{j=1}^{b} k_j\). The **incidence matrix** of an incidence structure is the \(v \times b\) matrix \(A = (a_{ij})\) with \(a_{ij} = 1\) if \(x_i \in B_j\) and 0 otherwise.

### 3.5.1 \(t\)-DESIGNS

The incidence structure \((X, B)\) is called a \(t\)-(\(v, k, \lambda\)) design if

1. For all \(j\), \(k_j = k\) and \(1 < k < v\), and
2. Any subset of \(t\) points lies on exactly \(\lambda\) blocks.

A 1-design is equivalent to a \(v \times b\) 0-1 matrix with constant row and column sums. Every \(t\)-(\(v, k, \lambda\)) design is also a \(\ell\)-(\(v, k, \lambda\) \(\ell\)) design \((1 \leq \ell \leq t\)), where

\[
\lambda_{\ell} = \lambda \left( \frac{v - \ell}{t - \ell} \right) \left/ \frac{\binom{k}{t} \binom{v - t}{\ell}}{\binom{v - t}{t - \ell}} \right..
\]  

A necessary condition for the existence of a \(t\)-(\(v, k, \lambda\)) design is that \(\lambda_{\ell}\) must be an integer for all \(\ell\), \(1 \leq \ell \leq t\). Another necessary condition is the generalized Fisher’s inequality when \(t = 2s\):

\[
b \geq \binom{v}{s}.
\]

**Related designs**

A \(t\)-(\(v, k, \lambda\)) design also implies the existence of the following designs:

**Complementary design**

Let \(B_C = \{X \setminus B \mid B \in B\}\). Then the incidence structure \((X, B_C)\) is a \(t\)-(\(v, v - k, \lambda(v-t) / \binom{v-t}{k-t}\)) design (provided \(v \geq k + t\)).

**Derived design**

Fix \(x \in X\) and let \(B_D = \{B \setminus \{x\} \mid B \in B\ with x \in B\}\). Then the incidence structure \((X \setminus \{x\}, B_D)\) is a \((t - 1)\)-(\(v - 1, k - 1, \lambda\)) design.

**Residual design**

Fix \(x \in X\) and let \(B_R = \{B \mid B \in B\ with x \notin B\}\). Then the incidence structure \((X \setminus \{x\}, B_R)\) is a \((t - 1)\)-(\(v - 1, k - 1, \lambda(v-t) / \binom{v-t}{k-t}\)) design.

**The Mathieu 5-design**

The following are the 132 blocks of a 5-(12,6,1) design. The blocks are the supports of the weight-6 codewords in the ternary Golay code (page 220). Similarly, the supports of the 759 weight-8 codewords in the binary Golay code form the blocks of a 5-(24,8,1) design.

```
0 1 2 3 4 11 0 2 5 7 8 10 1 3 4 7 8 11
0 1 2 3 5 10 0 2 5 7 9 11 1 3 5 6 7 10
0 1 2 3 6 8 0 2 6 7 8 9 1 3 5 6 8 11
0 1 2 3 7 9 0 2 6 8 10 11 1 3 5 8 9 10
```

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3.5.2 BALANCED INCOMPLETE BLOCK DESIGNS (BIBDS)

Balanced incomplete block designs (BIBDs) are \( t \)-designs with \( t = 2 \), so that every pair of points is on the same number of blocks. The relevant parameters are \( v, b, r, k \), and \( \lambda \) with

\[
vr = bk \quad \text{and} \quad v(v-1)\lambda = bk(k-1). \tag{3.5.2}
\]
If \( A \) is the \( v \times b \) incidence matrix, then \( A A^T = (r - \lambda)I_v + \lambda J_v \), where \( I_v \) is the \( n \times n \) identity matrix and \( J_n \) is the \( n \times n \) matrix of all ones.

**Symmetric designs**

Fisher’s inequality states that \( b \geq v \). If \( b = v \) (equivalently, \( r = k \)), then the BIBD is called a symmetric design, denoted as a \((v, k, \lambda)\)-design. The incidence matrix for a symmetric design satisfies

\[
J_v A = k J_v = A J_v \quad \text{and} \quad A^T A = (k - \lambda)I_v + \lambda J_v, \tag{3.5.3}
\]

that is, any two blocks intersect in \( \lambda \) points. The dualness of symmetric designs can be summarized by the following:

- \( v \) points \( \leftrightarrow \) \( v \) blocks,
- \( k \) blocks on a point \( \leftrightarrow \) \( k \) points in a block, and
- Any two points on \( \lambda \) blocks \( \leftrightarrow \) Any two blocks share \( \lambda \) points.

Some necessary conditions for symmetric designs are

1. If \( v \) is even, then \( k - \lambda \) is a square integer.
2. **Bruck–Ryser–Chowla theorem**: If \( v \) is odd, then the following equation has integer solutions (not all zero):

\[
\lambda^2 = (k - \lambda)y^2 + (-1)^{(v-1)/2}\lambda z^2.
\]

**Existence table for BIBDs**

Some of the most fruitful construction methods for BIBD are dealt with in separate sections, difference sets (page 208), finite geometry (page 209), Steiner triple systems (page 211), and Hadamard matrices (page 211). The table below gives all parameters for which BIBDs exists with \( k \leq v/2 \) and \( b \leq 30 \).

<table>
<thead>
<tr>
<th>( v )</th>
<th>( b )</th>
<th>( r )</th>
<th>( k )</th>
<th>( \lambda )</th>
<th>( v )</th>
<th>( b )</th>
<th>( r )</th>
<th>( k )</th>
<th>( \lambda )</th>
<th>( v )</th>
<th>( b )</th>
<th>( r )</th>
<th>( k )</th>
<th>( \lambda )</th>
</tr>
</thead>
<tbody>
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<td>27</td>
<td>13</td>
<td>13</td>
<td>6</td>
</tr>
</tbody>
</table>

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3.5.3 DIFFERENCE SETS

Let $G$ be a finite group of order $v$ (see page 139). A subset $D$ of size $k$ is a $(v, k, \lambda)$-difference set in $G$ if every nonidentity element of $G$ can be written $\lambda$ times as a “difference” $d_1d_2^{-1}$ with $d_1$ and $d_2$ in $D$. If $G$ is the cyclic group $\mathbb{Z}_v$, then the difference set is a cyclic difference set. The order of a difference set is $n = k - \lambda$.

For example, $\{1, 2, 4\}$ is a $(7, 3, 1)$ cyclic difference set of order 2. A $(v, k, \lambda)$-difference set implies the existence of a $(v, k, \lambda)$-design. The points are the elements of $G$ and the blocks are the translates of $D$: all sets $Dg = \{dg : d \in D\}$ for $g \in G$. Note that each translate $Dg$ is itself a difference set. Here are the 7 blocks for a $(7, 3, 1)$-design based on $D = \{1, 2, 4\}$:

\[
1 \ 2 \ 4 \ 2 \ 3 \ 5 \ 3 \ 4 \ 6 \ 4 \ 5 \ 0 \ 5 \ 6 \ 1 \ 6 \ 0 \ 2 \ 0 \ 1 \ 3
\]

Example: A $(16, 6, 2)$-difference set in $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is

\[
\begin{array}{cccccc}
0000 & 0001 & 0010 & 0100 & 1000 & 1111
\end{array}
\]

Example: A $(21, 5, 1)$-difference set in $G = \{a, b : a^3 = b^2 = 1, a^{-1}ba = a^3\}$ is $\{a, a^2, b, b^2, b^4\}$.

Some families of cyclic difference sets

Paley: Let $v$ be a prime congruent to 3 modulo 4. Then the nonzero squares in $\mathbb{Z}_v$ form a $(v, (v-1)/2, (v-3)/4)$-difference set. Example: $(v, k, \lambda) = (11, 5, 2)$.

Stanton–Sprott: Let $v = p(p+2)$, where $p$ and $p+2$ are both primes. Then there is a $(v, (v-1)/2, (v-3)/4)$-difference set. Example: $(v, k, \lambda) = (35, 17, 8)$.

Biquadratic Residues (I): If $v = 4a^2 + 1$ is a prime with $a$ odd, then the nonzero fourth powers modulo $v$ form a $(v, (v-1)/4, (v-5)/16)$-difference set. Example: $(v, k, \lambda) = (37, 9, 2)$.

Biquadratic Residues (II): If $v = 4a^2 + 9$ is a prime with $a$ odd, then zero and the fourth powers modulo $v$ form a $(v, (v+3)/4, (v+3)/16)$-difference set. Example: $(v, k, \lambda) = (13, 4, 1)$.

Singer: If $q$ is a prime power, then there exists a $((q^m-1)/(q-1), (q^{m-1}-1)/(q-1), (q^{m-2}-1)/(q-1))$-difference set for all $m \geq 3$.

Existence table of cyclic difference sets

This table gives all cyclic difference sets for $k \leq v/2$ and $v \leq 50$ up to equivalence by translation and multiplication by a number relatively prime to $v$.

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### 3.5.4 FINITE GEOMETRY

#### Affine planes

A finite **affine plane** is a finite set of points that satisfy the axioms:

- Any two points are on exactly one line.
- *(Parallel postulate)* Given a point \( P \) and a line \( L \) not containing \( P \), there is exactly one line through \( P \) that does not intersect \( L \).
- There are four points, no three of which are collinear.

These axioms are sufficient to show that a finite affine plane is a BIBD (see page 206) with

\[
v = n^2 \quad b = n^2 + n \quad r = n + 1 \quad k = n \quad \lambda = 1
\]

\( n \) is the *order* of the plane. The lines of an affine plane can be divided into \( n + 1 \) parallel classes each containing \( n \) lines. A sufficient condition for affine planes to exist is for \( n \) to be a prime power.

Below are two views of the affine plane of order 2 showing the parallel classes.
Below is the affine plane of order 3 showing the parallel classes.

Projective planes
A finite projective plane is a finite set of points that satisfy the axioms:

1. Any two points are on exactly one line.
2. Any two lines intersect in exactly one point.
3. There are four points, no three of which are collinear.

These axioms are sufficient to show that a finite affine plane is a symmetric design (see page 207) with

\[ v = n^2 + n + 1 \quad k = n + 1 \quad \lambda = 1 \]

(\(n\) is the order of the plane). A sufficient condition for affine planes to exist is for \(n\) to be a prime power.

A projective plane of order \(n\) can be constructed from an affine plane of order \(n\) by adding a line at infinity. A line of \(n + 1\) new points is added to the affine plane. For each parallel class, one distinct new point is added to each line. The construction works in reverse: removing any one line from a projective plane of order \(n\) and its points leaves an affine plane of order \(n\). Below is the projective plane of order 2. The center circle functions as a line at infinity; removing it produces the affine plane of order 2.

3.5.5 Steiner Triple Systems
A Steiner triple system (STS) is a 2-(\(v,3,1\)) design. In particular, STSs are BIBDs (see page 206). STSs exist if, and only if, \(v \equiv 1\) or 3 (mod 6). The number of blocks in an STS is \(b = v(v - 1)/6\).
Some families of Steiner triple systems

\( v = 2^m - 1 \): Take as points all nonzero vectors over \( \mathbb{Z}_2 \) of length \( m \). A block consists of any set of three distinct vectors \( \{x, y, z\} \) such that \( x + y + z = 0 \).

\( v = 3^m \): Take as points all vectors over \( \mathbb{Z}_3 \) of length \( m \). A block consists of any set of three distinct vectors \( \{x, y, z\} \) such that \( x + y + z = 0 \).

Resolvable Steiner triple systems

An STS is resolvable if the blocks can be divided into parallel classes such that each point occurs in exactly one block per class. A resolvable STS exists if and only if \( v \equiv 3 \pmod{6} \). For example, the affine plane of order 3 is a resolvable STS with \( v = 9 \) (see page 209).

A resolvable STS with \( v = 15 \) (\( b = 35 \)) is known as the Kirkman schoolgirl problem and dates from 1850. Here is an example. Each column of 5 triples is a parallel class:

\[
\begin{array}{cccccccccccc}
\text{abi} & \text{acj} & \text{adk} & \text{ael} & \text{afm} & \text{agn} & \text{aho} \\
\text{cdf} & \text{deg} & \text{efh} & \text{fgb} & \text{ghc} & \text{hbd} & \text{bce} \\
\text{gjo} & \text{hki} & \text{blj} & \text{cmk} & \text{dnl} & \text{eom} & \text{fin} \\
\text{ekn} & \text{flo} & \text{gmi} & \text{hnj} & \text{bok} & \text{cil} & \text{djm} \\
\text{hlm} & \text{bmn} & \text{cno} & \text{doi} & \text{eij} & \text{fjk} & \text{gkl}.
\end{array}
\]

3.5.6 HADAMARD MATRICES

A Hadamard matrix of order \( n \) is an \( n \times n \) matrix \( H \) with entries \( \pm 1 \) such that \( HH^T = nI_n \). In order for a Hadamard matrix to exist, \( n \) must be 1, 2, or a multiple of 4. It is conjectured that these values are also sufficient. If \( H_1 \) and \( H_2 \) are Hadamard matrices, then so is the Kronecker product \( H_1 \otimes H_2 \).

Some Hadamard matrices

We use “−” to denote \(-1\).

\[
\begin{align*}
\text{\( n = 2 \)} & & \text{\( n = 4 \)} & & \text{\( n = 8 \)} \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{align*}
\]

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Designs and Hadamard matrices

Without loss of generality, a Hadamard matrix can be assumed to have a first row and column consisting of all $+1$s.

**BIBDs:** Delete the first row and column. The points of the design are the remaining column indices. Each row produces a block of the design, namely those indices where the entry is $+1$. The resulting design is an $(n-1, (n-1)/2, (n-5)/4)$ symmetric design (see page 211).

**3-Designs:** The points are the indices of the columns. Each row, except the first row, yields two blocks, one block for those indices where the entries are $+1$ and one block for those indices where the entries are $-1$. The resulting design is a 3-$(n, n/2, (n-5)/4)$ design.
3.5.7 LATIN SQUARES

A Latin square of size $n$ is an $n \times n$ array $S = [s_{ij}]$ of $n$ symbols such that every symbol appears exactly once in each row and column. Two Latin squares $S$ and $T$ are orthogonal if every pair of symbols occurs exactly once as a pair $(s_{ij}, t_{ij})$. Let $M(n)$ be the maximum size of a set of mutually orthogonal Latin squares (MOLS).

- $M(n) \leq n - 1$.
- $M(n) = n - 1$ if $n$ is a prime power.
- $M(n_1n_2) \geq \min(M(n_1), M(n_2))$.
- $M(6) = 1$ (i.e., there are no two MOLS of size 6).
- $M(n) \geq 2$ for all $n \geq 3$ except $n = 6$.

The existence of $n - 1$ MOLS of size $n$ is equivalent to the existence of an affine plane of order $n$ (see page 209).

Examples of mutually orthogonal Latin squares

These are complete sets of MOLS for $n = 3$, 4, and 5.

\[
\begin{array}{ccc}
0 & 1 & 2 \\
1 & 2 & 0 \\
2 & 0 & 1 \\
0 & 1 & 2 \\
1 & 2 & 0 \\
2 & 0 & 1 \\
0 & 1 & 2 \\
1 & 2 & 0 \\
2 & 0 & 1 \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & 1 & 2 \\
1 & 0 & 3 \\
2 & 3 & 0 \\
3 & 2 & 1 \\
0 & 1 & 2 \\
1 & 0 & 3 \\
2 & 3 & 0 \\
3 & 2 & 1 \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & 1 & 2 \\
1 & 0 & 3 \\
2 & 3 & 0 \\
3 & 2 & 1 \\
0 & 1 & 2 \\
1 & 0 & 3 \\
2 & 3 & 0 \\
3 & 2 & 1 \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & 1 & 2 \\
1 & 2 & 3 \\
2 & 3 & 4 \\
3 & 4 & 0 \\
4 & 0 & 1 \\
0 & 1 & 2 \\
1 & 2 & 3 \\
2 & 3 & 4 \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & 1 & 2 \\
1 & 2 & 3 \\
2 & 3 & 4 \\
3 & 4 & 0 \\
4 & 0 & 1 \\
0 & 1 & 2 \\
1 & 2 & 3 \\
2 & 3 & 4 \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & 1 & 2 \\
1 & 0 & 3 \\
2 & 3 & 0 \\
3 & 2 & 1 \\
0 & 1 & 2 \\
1 & 0 & 3 \\
2 & 3 & 0 \\
3 & 2 & 1 \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & 1 & 2 \\
1 & 0 & 3 \\
2 & 3 & 0 \\
3 & 2 & 1 \\
0 & 1 & 2 \\
1 & 0 & 3 \\
2 & 3 & 0 \\
3 & 2 & 1 \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & 1 & 2 \\
1 & 0 & 3 \\
2 & 3 & 0 \\
3 & 2 & 1 \\
0 & 1 & 2 \\
1 & 0 & 3 \\
2 & 3 & 0 \\
3 & 2 & 1 \\
\end{array}
\]

These are two superimposed MOLS for $n = 7$, 8, 9, and 10.

\[
\begin{array}{cccccccc}
00 & 11 & 22 & 33 & 44 & 55 & 66 \\
16 & 20 & 31 & 42 & 53 & 64 & 05 \\
25 & 36 & 40 & 51 & 62 & 03 & 14 \\
34 & 45 & 56 & 60 & 01 & 12 & 23 \\
43 & 54 & 65 & 06 & 10 & 21 & 32 \\
52 & 63 & 04 & 15 & 26 & 30 & 41 \\
61 & 02 & 13 & 24 & 35 & 46 & 50 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
00 & 11 & 22 & 33 & 44 & 55 & 66 & 77 \\
12 & 03 & 30 & 21 & 56 & 47 & 74 & 65 \\
24 & 35 & 06 & 17 & 60 & 71 & 42 & 53 \\
33 & 22 & 11 & 00 & 77 & 66 & 55 & 44 \\
46 & 57 & 64 & 75 & 02 & 13 & 20 & 31 \\
57 & 46 & 75 & 64 & 13 & 02 & 31 & 20 \\
65 & 74 & 47 & 56 & 21 & 30 & 03 & 12 \\
71 & 60 & 53 & 42 & 35 & 24 & 17 & 06 \\
\end{array}
\]

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3.5.8 ROOM SQUARES

A Room square of side n is an $n \times n$ array with entries either empty or consisting of an unordered pair of symbols from a symbol set of size $n + 1$ with the requirements:

1. Each symbol appears exactly once in each row and column.
2. Every unordered pair occurs exactly once in the array.

Room squares exist if and only if $n$ is odd and $n \geq 7$.

A Room square yields a construction of a round-robin tournament between $n + 1$ opponents:

1. Rows of the square represent rounds in the tournament.
2. Columns in the square represent locales.
3. Each pair represents one competition.

Then each team plays exactly once in each round, at each locale, and against each opponent.

For example, this is a Room square of side 7:

\[
\begin{array}{cccccccccc}
00 & 11 & 22 & 33 & 44 & 55 & 66 & 77 & 88 \\
12 & 20 & 01 & 45 & 53 & 34 & 78 & 86 & 67 \\
21 & 02 & 10 & 54 & 35 & 43 & 87 & 68 & 76 \\
36 & 47 & 58 & 60 & 71 & 82 & 03 & 14 & 25 \\
48 & 56 & 37 & 72 & 80 & 61 & 15 & 23 & 04 \\
57 & 38 & 46 & 81 & 62 & 70 & 24 & 05 & 13 \\
63 & 74 & 85 & 06 & 17 & 28 & 30 & 41 & 52 \\
75 & 83 & 64 & 18 & 26 & 07 & 42 & 50 & 31 \\
84 & 65 & 73 & 27 & 08 & 16 & 51 & 32 & 40 \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
00 & 67 & 58 & 49 & 91 & 83 & 75 & 12 & 24 & 36 \\
76 & 11 & 07 & 68 & 59 & 92 & 84 & 23 & 35 & 40 \\
85 & 70 & 22 & 17 & 08 & 69 & 93 & 34 & 46 & 51 \\
94 & 86 & 71 & 33 & 27 & 18 & 09 & 45 & 59 & 62 \\
19 & 95 & 80 & 72 & 44 & 37 & 28 & 56 & 61 & 03 \\
38 & 29 & 96 & 81 & 73 & 55 & 47 & 60 & 02 & 14 \\
57 & 48 & 39 & 90 & 82 & 74 & 66 & 01 & 13 & 25 \\
21 & 32 & 43 & 54 & 65 & 06 & 10 & 77 & 88 & 99 \\
42 & 53 & 64 & 05 & 16 & 20 & 31 & 89 & 97 & 78 \\
63 & 04 & 15 & 26 & 30 & 41 & 52 & 98 & 79 & 87 \\
\end{array}
\]
3.6 INCLUSION/EXCLUSION

Let \( \{a_1, a_2, \ldots, a_r\} \) be properties that the elements of a set may or may not have. If the set has \( N \) objects, then the number of objects having exactly \( m \) properties (with \( m \leq r \)), \( e_m \), is given by

\[
e_m = s_m - \left( \begin{array}{c} m+1 \\ 1 \end{array} \right) s_{m+1} + \left( \begin{array}{c} m+2 \\ 2 \end{array} \right) s_{m+2} - \left( \begin{array}{c} m+3 \\ 3 \end{array} \right) s_{m+3} + \cdots + (-1)^p \left( \begin{array}{c} m+p \\ p \end{array} \right) s_{m+p} + \cdots + (-1)^{r-m} \left( \begin{array}{c} m+(r-m) \\ (r-m) \end{array} \right) s_r.
\]

Here \( s_t = \sum N(a_{i_1}a_{i_2}\ldots a_{i_t}) \). When \( m = 0 \), this is the usual inclusion/exclusion rule:

\[
e_0 = s_0 - s_1 + s_2 - \cdots + (-1)^r s_r,
\]

\[
= N - \sum_i N(a_i) + \sum_{i,j} N(a_ia_j) - \sum_{i,j,k} N(a_ia_ja_k) + \cdots + (-1)^r N(a_1a_2\ldots a_r).
\]

3.7 COMMUNICATION THEORY

3.7.1 INFORMATION THEORY

Definitions

Let \( p_x \) be a probability distribution on the discrete random variable \( X \) with \( \text{Prob}(X = x) = p_x \). The *entropy* of the distribution is

\[
H(p_X) = -\sum_x p_x \log_2 p_x. \tag{3.7.1}
\]

When \( X \) takes only two values,

\[
H(p_X) = H(p) = -p \log_2 p - (1-p) \log_2(1-p). \tag{3.7.2}
\]

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The units for entropy are bits. The range of \( H(p) \) is from 0 to 1 with a maximum at \( p = 0.5 \). The maximum of \( H(p_X) \) is \( \log_2 n \) and is obtained when \( X \) is a uniform random variable taking \( n \) values. Entropy measures how much information is gained from learning the value of \( X \). Below is a plot of \( p \) versus \( H(p) \).

Given two discrete random variables \( X \) and \( Y \), \( p_{X,Y} \) is the joint distribution of \( X \) and \( Y \). The mutual information of \( X \) and \( Y \) is defined by

\[
I(X, Y) = H(p_X) + H(p_Y) - H(p_{X,Y}).
\] (3.7.3)

Note that \( I(X, Y) \geq 0 \) and that \( I(X, Y) = 0 \) if, and only if, \( X \) and \( Y \) are independent. Mutual information gives the amount of information that learning a value of \( X \) says about the value of \( Y \) (and vice versa).

**Channel capacity**

The transition probabilities are defined by \( t_{x,y} = \text{Prob}(Y = y \mid X = x) \). The distribution \( p_X \) determines \( p_Y \) by \( p_y = \sum t_{x,y} p_x \). The matrix \( T = (t_{x,y}) \) is the transition matrix. The matrix \( T \) defines a channel given by a transition diagram (input is \( X \), output is \( Y \)). For example (here \( X \) and \( Y \) only take two values),

The capacity of the channel is defined as

\[
C = \max_{p_X} I(X, Y).
\] (3.7.4)
A channel is symmetric if each row is a permutation of the first row, and the transition matrix is a symmetric matrix. The capacity of a symmetric channel is \( C = \log_2 n - H(p) \), where \( p \) is the first row. The capacity of a symmetric channel is achieved with equally likely inputs. The channel below on the left is symmetric; both channels achieve capacity with equally likely inputs.

\[
\begin{array}{c|c}
\text{Binary Symmetric Channel} & \text{Binary Erasure Channel} \\
\hline
0 & 0 \\
1 & 1 \\
\end{array}
\]

\[
\begin{array}{c|c}
1 - p & 0 \\
p & p \\
1 - p & 1 \\
p & 1 - p \\
\end{array}
\]

\[
C = 1 - H(p).
\]

Shannon's theorem

Let both \( X \) and \( Y \) be discrete random variables with values in an alphabet \( A \). A code is a set of \( n \)-tuples (codewords) with entries from \( A \) that is in one-to-one correspondence with \( M \) messages. The rate \( R \) of the code is defined as \( \frac{1}{n} \log_2 M \). Assume that the codeword is sent via a channel with transition matrix \( T \) by sending each vector element independently. Define

\[
e = \max_{\text{all codewords}} \text{Prob(codeword incorrectly decoded)}.
\]

Then Shannon's coding theorem states:

- If \( R < C \), then there is a sequence of codes with \( n \to \infty \) such that \( e \to 0 \).
- If \( R \geq C \), then \( e \) is always bounded away from 0.

3.7.2 BLOCK CODING

Definitions

A code \( C \) over an alphabet \( A \) is a set of vectors of a fixed length \( n \) with entries from \( A \). Let \( A \) be the finite field \( GF(q) \) (see Section 3.7.3). If \( C \) is a vector space over \( A \), then \( C \) is a linear code; the dimension \( k \) of a linear code is its dimension as a vector space.

The Hamming distance \( d_H(\mathbf{u}, \mathbf{v}) \) between two vectors, \( \mathbf{u} \) and \( \mathbf{v} \), is the number of places in which they differ. For a vector \( \mathbf{u} \) over \( GF(q) \), define the weight, \( wt(\mathbf{u}) \), as the number of nonzero components. Then \( d_H(\mathbf{u}, \mathbf{v}) = wt(\mathbf{u} - \mathbf{v}) \). The minimum Hamming distance between two distinct vectors in a code \( C \) is called the minimum distance \( d \). A code can detect \( e \) errors if \( e < d \). A code can correct \( t \) errors if \( 2t + 1 < d \).
Coding diagram for linear codes

1. A message \( x \) consists of \( k \) information symbols.
2. The message is encoded as \( xG \in C \), where \( G \) is a \( k \times n \) matrix called the generating matrix.
3. After transmission over a channel, a (possibly corrupted) vector \( y \) is received.
4. There exists a parity check matrix \( H \) such that \( c \in C \) if and only if \( cH = 0 \). Thus the syndrome \( z = yH \) can be used to try to decode \( y \).
5. If \( G \) has the form \( [I \ A] \), where \( I \) is the \( k \times k \) identity matrix, then \( H = \begin{bmatrix} -A \\ I \end{bmatrix} \).

Cyclic codes

A linear code \( C \) of length \( n \) is cyclic if \( (a_0, a_1, \ldots, a_{n-1}) \in C \) implies \( (a_{n-1}, a_0, \ldots, a_{n-2}) \in C \). To each codeword \( (a_0, a_1, \ldots, a_{n-1}) \in C \) is associated the polynomial \( a(x) = \sum_{i=0}^{n-1} a_i x^i \). Every cyclic code has a generating polynomial \( g(x) \) such that \( a(x) \) corresponds to a codeword if, and only if, \( a(x) \equiv d(x)g(x) \pmod{x^n - 1} \) for some \( d(x) \). The roots of a cyclic code are roots of \( g(x) \) under some extension field \( \text{GF}(q') \) with primitive element \( \alpha \).

**BCH Bound:** If a cyclic code \( C \) has roots \( \alpha^i, \alpha^{i+1}, \ldots, \alpha^{i+d-2} \), then the minimum distance of \( C \) is at least \( d \).

**Binary BCH codes** (BCH stands for Bose, Ray-Chaudhuri, and Hocquenghem): Fix \( m \), define \( n = 2^m - 1 \), and let \( \alpha \) be a primitive element in \( \text{GF}(2^m) \). Define \( f_i(x) \) as the minimum binary polynomial of \( \alpha^i \). Then

\[
g(x) = \text{LCM}(f_1(x) \cdots f_{2e}(x)) \tag{3.7.6}
\]

defines a generating polynomial for a binary BCH code of length \( n \) and minimum distance at least \( \delta = 2e + 1 \) (\( \delta \) is called the designed distance). The code dimension is at least \( n - me \).
### Table of binary BCH codes

<table>
<thead>
<tr>
<th>$n$</th>
<th>$k$</th>
<th>$d$</th>
<th>$n$</th>
<th>$k$</th>
<th>$d$</th>
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<td>4</td>
<td>3</td>
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<td>92</td>
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<td>99</td>
<td>9</td>
<td></td>
<td>8</td>
<td>63</td>
<td></td>
</tr>
</tbody>
</table>

**Dual code:** Given a code $C$, the dual code is $C^\perp = \{a \mid a \cdot x = 0 \text{ for all } x \in C\}$. The code $C^\perp$ is an $(n, n-k)$ linear code over $F_q$. A code is self-doubt if $C = C^\perp$.

**MDS codes:** A linear code that meets the Singleton bound, $n + 1 = k + d$, is called MDS (for maximum distance separable). The generating matrix of an MDS code has any $k$ columns linearly independent.

**Reed–Solomon codes:** Let $\alpha$ be a primitive element for GF($q$) and $n = q - 1$. The generating polynomial $g(x) = (x - \alpha)(x - \alpha^2) \cdots (x - \alpha^{d-1})$ defines a cyclic MDS code with distance $d$ and dimension $k = n - d + 1$.

**Hexacode:** The hexacode is a $(6, 3, 4)$ self-dual MDS code over GF(4). Let the finite field of four elements be $\{0, 1, a, b\}$ with $b = a^2 = a + 1$. The code is generated by the vectors $(1, 0, 0, 1, a, b)$, $(0, 1, 0, 1, a, b)$, and $(0, 0, 1, 1, 1, 1)$. The 64 codewords are

<table>
<thead>
<tr>
<th>Codeword</th>
<th>Codeword</th>
<th>Codeword</th>
<th>Codeword</th>
<th>Codeword</th>
<th>Codeword</th>
<th>Codeword</th>
<th>Codeword</th>
</tr>
</thead>
<tbody>
<tr>
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<td>10110a</td>
<td>a00ab0</td>
<td>b00ba0</td>
<td>0101ab</td>
<td>11001a</td>
<td>a10ba0</td>
<td>b10a0a</td>
</tr>
<tr>
<td>0a0ab0</td>
<td>1a00ab</td>
<td>ba00aa</td>
<td>0ba0b1</td>
<td>1b0aa0</td>
<td>ab010b</td>
<td>bb00bb</td>
<td>ba1a0b</td>
</tr>
<tr>
<td>001111</td>
<td>1010ab</td>
<td>a010b0a</td>
<td>b010ab0</td>
<td>0110ba</td>
<td>11110a</td>
<td>a11a1a</td>
<td>b11b1b</td>
</tr>
<tr>
<td>0alba0</td>
<td>1alala0</td>
<td>a11bba0</td>
<td>b1a0b0</td>
<td>1ba1b0</td>
<td>ab1010</td>
<td>bb11aa</td>
<td>ba1a0b</td>
</tr>
<tr>
<td>00aaa0</td>
<td>10a0b1</td>
<td>a0a0b1</td>
<td>b0a10b</td>
<td>010ba1</td>
<td>11a0b0a</td>
<td>ab0000a</td>
<td>bbabab</td>
</tr>
<tr>
<td>0aa01b</td>
<td>1aa01a</td>
<td>a0aa00a</td>
<td>baabba</td>
<td>0ba00b</td>
<td>ab01a0</td>
<td>bb0000b</td>
<td>aabbba</td>
</tr>
<tr>
<td>0babbb</td>
<td>10ba01a</td>
<td>a0b01a</td>
<td>b0b01a</td>
<td>0ba10b</td>
<td>11bbba</td>
<td>aib00b</td>
<td>b1b1b1</td>
</tr>
<tr>
<td>0ab10a</td>
<td>1a0b0b0</td>
<td>a0b1ba0</td>
<td>bababa</td>
<td>0bb0a1</td>
<td>1b1b1b</td>
<td>abbaba</td>
<td>bbb0000</td>
</tr>
</tbody>
</table>

**Perfect codes:** A linear code is perfect if it satisfies the Hamming bound, $q^{n-k} = \sum_{i=0}^{d-1} \binom{n}{i}(q-1)^i$. The binary Hamming codes and Golay codes are perfect.

**Binary Hamming codes:** Have the parameters $[n = 2^m - 1, k = 2^m - 1 - m, d = 3]$. The parity check matrix is the $2^m - 1 \times m$ matrix whose rows are all of the binary $m$-tuples in a fixed order. The generating and parity check matrices for the...
(7, 4) Hamming code are

\[
G = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

(3.7.7)

**Binary Golay code:** This has the parameters \( \{n = 24, k = 12, d = 8\} \). The generating matrix is

\[
G = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

(3.7.8)

**Ternary Golay code:** This has the parameters \( \{n = 12, k = 6, d = 6\} \). The generating matrix is

\[
G = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

(3.7.9)

**Bounds**

Bounds for block codes investigate the trade-offs between the length \( n \), the number of codewords \( M \), the minimum distance \( d \), and the alphabet size \( q \). The number of errors that can be corrected is \( e \) with \( 2^e + 1 \leq d \). If the code is linear, then the bounds concern the dimension \( k \) with \( M = q^k \).

1. Hamming or sphere-packing bound: \( M \leq q^n / \sum_{i=0}^{d-1} \binom{n}{i} (q-1)^i \).
2. Plotkin bound: Suppose that \( d > n(q-1)/q \). Then \( M \leq \frac{qd}{qd-n(q-1)} \).
3. Singleton bound: For any code, \( M \leq q^{n-d+1} \); if the code is linear, then \( k + d \leq n + 1 \).
4. Varsharmov–Gilbert bound: There is a block code with minimum distance at least \( d \) and \( M \geq q^n / \sum_{i=0}^{d-1} \binom{n}{i} (q-1)^i \).

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Table of best binary codes

$A(n, d)$ is the number of codewords in the largest binary code of length $n$ and minimum distance $d$. Note that $A(n−1, d−1) = A(n, d)$ if $d$ is odd and $A(n, 2) = 2^{n−1}$ (given, e.g., by even weight words).

<table>
<thead>
<tr>
<th>$n$</th>
<th>$d = 4$</th>
<th>$d = 6$</th>
<th>$d = 8$</th>
<th>$d = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>16</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>20</td>
<td>4</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>40</td>
<td>6</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>11</td>
<td>72–79</td>
<td>12</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>12</td>
<td>144–158</td>
<td>24</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>13</td>
<td>256</td>
<td>32</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>14</td>
<td>512</td>
<td>64</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>15</td>
<td>1024</td>
<td>128</td>
<td>16</td>
<td>4</td>
</tr>
<tr>
<td>16</td>
<td>2048</td>
<td>256</td>
<td>32</td>
<td>4</td>
</tr>
<tr>
<td>17</td>
<td>2720–3276</td>
<td>256–340</td>
<td>36–37</td>
<td>6</td>
</tr>
<tr>
<td>18</td>
<td>5248–6552</td>
<td>512–680</td>
<td>64–74</td>
<td>10</td>
</tr>
<tr>
<td>19</td>
<td>10496–13104</td>
<td>1024–1288</td>
<td>128–144</td>
<td>20</td>
</tr>
<tr>
<td>20</td>
<td>20480–26208</td>
<td>2048–2372</td>
<td>256–279</td>
<td>40</td>
</tr>
<tr>
<td>21</td>
<td>36864–43690</td>
<td>2560–4096</td>
<td>512</td>
<td>42–48</td>
</tr>
<tr>
<td>22</td>
<td>73728–87380</td>
<td>4096–6942</td>
<td>1024</td>
<td>68–88</td>
</tr>
<tr>
<td>23</td>
<td>147456–173784</td>
<td>8192–13774</td>
<td>2048</td>
<td>64–150</td>
</tr>
<tr>
<td>24</td>
<td>294912–344636</td>
<td>16384–24106</td>
<td>4096</td>
<td>128–280</td>
</tr>
</tbody>
</table>

3.7.3 FINITE FIELDS

Pertinent definitions for finite fields can be found in Section 2.6.5.

Irreducible and primitive polynomials

Let $N_q(n)$ be the number of monic irreducible polynomials over $GF(q)$. Then

$$q^n = \sum_{d|n} d N_q(d) \quad \text{and} \quad N_q(n) = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) q^d,$$

(3.7.10)

where $\mu(\cdot)$ is the number theoretic Möbius function (see Section 2.3.8).

Table of binary irreducible polynomials

The table lists the nonzero coefficients of the polynomial, e.g., 2 1 0 corresponds to $x^2 + x^1 + x^0 = x^2 + x + 1$. The exponent of an irreducible polynomial is the smallest

---

such that \( f(x) \) divides \( x^L - 1 \). A \( P \) after the exponent indicates that the polynomial is primitive.

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>Exponent</th>
<th>( f(x) )</th>
<th>Exponent</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 1 0</td>
<td>3 P</td>
<td>7 6 5 3 2 1 0</td>
<td>127 P</td>
</tr>
<tr>
<td>3 1 0</td>
<td>7 P</td>
<td>7 6 5 4 0</td>
<td>127 P</td>
</tr>
<tr>
<td>3 2 0</td>
<td>7 P</td>
<td>7 6 5 4 2 1 0</td>
<td>127 P</td>
</tr>
<tr>
<td>4 1 0</td>
<td>15 P</td>
<td>7 6 5 4 3 2 0</td>
<td>127 P</td>
</tr>
<tr>
<td>4 2 0</td>
<td>15 P</td>
<td>8 4 3 1 0</td>
<td>51</td>
</tr>
<tr>
<td>4 3 2 1 0</td>
<td>5</td>
<td>8 4 3 2 0</td>
<td>255 P</td>
</tr>
<tr>
<td>5 2 0</td>
<td>31 P</td>
<td>8 5 3 1 0</td>
<td>255 P</td>
</tr>
<tr>
<td>5 3 0</td>
<td>31 P</td>
<td>8 5 3 2 0</td>
<td>255 P</td>
</tr>
<tr>
<td>5 3 2 1 0</td>
<td>31 P</td>
<td>8 5 4 3 0</td>
<td>17</td>
</tr>
<tr>
<td>5 4 2 1 0</td>
<td>31 P</td>
<td>8 5 4 3 2 1 0</td>
<td>85</td>
</tr>
<tr>
<td>5 4 3 1 0</td>
<td>31 P</td>
<td>8 6 3 2 0</td>
<td>255 P</td>
</tr>
<tr>
<td>5 4 3 2 0</td>
<td>31 P</td>
<td>8 6 4 3 2 1 0</td>
<td>255 P</td>
</tr>
<tr>
<td>6 1 0</td>
<td>63 P</td>
<td>8 6 5 1 0</td>
<td>255 P</td>
</tr>
<tr>
<td>6 3 0</td>
<td>9</td>
<td>8 6 5 2 0</td>
<td>255 P</td>
</tr>
<tr>
<td>6 4 2 1 0</td>
<td>21</td>
<td>8 6 5 3 0</td>
<td>255 P</td>
</tr>
<tr>
<td>6 4 3 1 0</td>
<td>63 P</td>
<td>8 6 5 4 0</td>
<td>255 P</td>
</tr>
<tr>
<td>6 5 0</td>
<td>63 P</td>
<td>8 6 5 4 2 1 0</td>
<td>85</td>
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<tr>
<td>6 5 2 1 0</td>
<td>63 P</td>
<td>8 6 5 4 3 1 0</td>
<td>85</td>
</tr>
<tr>
<td>6 5 3 2 0</td>
<td>63 P</td>
<td>8 7 2 1 0</td>
<td>255 P</td>
</tr>
<tr>
<td>6 5 4 1 0</td>
<td>63</td>
<td>8 7 3 1 0</td>
<td>85</td>
</tr>
<tr>
<td>6 5 4 2 0</td>
<td>21 P</td>
<td>8 7 3 2 0</td>
<td>255 P</td>
</tr>
<tr>
<td>7 1 0</td>
<td>127 P</td>
<td>8 7 4 3 2 1 0</td>
<td>51</td>
</tr>
<tr>
<td>7 3 0</td>
<td>127 P</td>
<td>8 7 5 1 0</td>
<td>85</td>
</tr>
<tr>
<td>7 3 2 1 0</td>
<td>127 P</td>
<td>8 7 5 3 0</td>
<td>255 P</td>
</tr>
<tr>
<td>7 4 0</td>
<td>127 P</td>
<td>8 7 5 4 0</td>
<td>51</td>
</tr>
<tr>
<td>7 4 3 2 0</td>
<td>127 P</td>
<td>8 7 5 4 3 2 0</td>
<td>85</td>
</tr>
<tr>
<td>7 5 2 1 0</td>
<td>127 P</td>
<td>8 7 6 1 0</td>
<td>255 P</td>
</tr>
<tr>
<td>7 5 3 1 0</td>
<td>127 P</td>
<td>8 7 6 3 2 1 0</td>
<td>255 P</td>
</tr>
<tr>
<td>7 5 4 3 0</td>
<td>127 P</td>
<td>8 7 6 4 2 1 0</td>
<td>17</td>
</tr>
<tr>
<td>7 5 4 3 2 1 0</td>
<td>127 P</td>
<td>8 7 6 4 3 2 0</td>
<td>85</td>
</tr>
<tr>
<td>7 6 0</td>
<td>127 P</td>
<td>8 7 6 5 2 1 0</td>
<td>255 P</td>
</tr>
<tr>
<td>7 6 3 1 0</td>
<td>127 P</td>
<td>8 7 6 5 4 1 0</td>
<td>51</td>
</tr>
<tr>
<td>7 6 4 1 0</td>
<td>127 P</td>
<td>8 7 6 5 4 2 0</td>
<td>255 P</td>
</tr>
<tr>
<td>7 6 4 2 0</td>
<td>127 P</td>
<td>8 7 6 5 4 3 0</td>
<td>85</td>
</tr>
<tr>
<td>7 6 5 2 0</td>
<td>127 P</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table of binary primitive polynomials

Listed below are primitive polynomials, with the least number of nonzero terms, of degree from 1 to 64. Only the nonzero terms are listed, e.g., 2 1 0 corresponds to \( x^2 + x + 1 \).


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3.7.4 BINARY SEQUENCES

Barker sequences

A *Barker sequence* is a sequence \( s = (s_1, \ldots, s_N) \) with \( s_j = \pm 1 \) such that \( \sum_{j=1}^{N-i} s_js_{j+i} = \pm 1 \) or 0, for \( i = 1, \ldots, N-1 \). The following table lists all known Barker sequences (up to reversal, multiplication by \(-1\), and multiplying alternate values by \(-1\)).

<table>
<thead>
<tr>
<th>Length</th>
<th>Barker sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>(+1 +1)</td>
</tr>
<tr>
<td>3</td>
<td>(+1 +1 -1)</td>
</tr>
<tr>
<td>4</td>
<td>(+1 +1 -1 +1)</td>
</tr>
<tr>
<td>5</td>
<td>(+1 +1 +1 -1 +1)</td>
</tr>
<tr>
<td>7</td>
<td>(+1 +1 +1 +1 -1 -1 +1)</td>
</tr>
<tr>
<td>11</td>
<td>(+1 +1 +1 +1 +1 -1 -1 +1 -1 +1)</td>
</tr>
</tbody>
</table>

Periodic sequences

Let \( s = (s_0, s_1, \ldots, s_{N-1}) \) be a periodic sequence with period \( N \). A (left) shift of \( s \) is the sequence \( (s_1, \ldots, s_{N-1}, s_0) \). For \( \tau \) relatively prime to \( N \), the *decimation* of \( s \) is the sequence \( (s_0, s_\tau, s_{2\tau}, \ldots) \), which also has period \( N \). The *periodic autocorrelation* is defined as the vector \((a_0, \ldots, a_{N-1})\), with

\[
a_i = \sum_{j=0}^{N-1} s_j s_{j+i}, \quad \text{subscripts taken modulo } N. \tag{3.7.11}
\]

An autocorrelation is *two-valued* if all values are equal except possibly for the 0th term.
The \( m \)-sequences

A binary \( m \)-sequence of length \( N = 2^r - 1 \) is the sequence of period \( N \)
\[ (s_0, s_1, \ldots, s_{N-1}), \quad s_i = \text{Tr}(\alpha^i), \]
where \( \alpha \) is a primitive element of \( \text{GF}(2^r) \) and \( \text{Tr} \) is the trace function from \( \text{GF}(2^r) \) to \( \text{GF}(2) \).

- All \( m \)-sequences of a given length are equivalent under decimation.
- Binary \( m \)-sequences have a two-valued autocorrelation (with the identification that \( 0 \leftrightarrow +1 \) and \( 1 \leftrightarrow -1 \)).
- All \( m \)-sequences possess the span property: all binary \( r \)-tuples occur in the sequence except the all-zeros \( r \)-tuple.

A binary sequence of length \( 2^n - 1 \) with a two-valued autocorrelation is equivalent to the existence of a cyclic difference set with parameters \( (2^n - 1, 2^{n-1} - 1, 2^{n-2} - 1) \).

Shift registers

Below are examples of the two types of shift registers used to generate binary \( m \)-sequences. The generating polynomial in each case is \( x^4 + x + 1 \), the initial register loading is 1000, and the generated sequence is \{0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, \ldots\}.

Additive shift register

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
\downarrow & + & & \\
0 & 0 & 0 & 0
\end{array}
\]

Multiplicative shift register

\[
\begin{array}{cccc}
1 & + & 0 & 0 \\
\downarrow & & & \\
0 & 0 & 0 & 0
\end{array}
\]

Table of binary sequences with two-valued autocorrelation

The following table lists all binary sequences with two-valued periodic autocorrelation of length \( 2^n - 1 \) for \( n = 3 \) to 8 (up to shifts, decimations, and complementation). The table indicates the positions of 0; the remaining values are 1. An \( S \) indicates that that sequence has the span property.
3.7.5 MORSE CODE

International version of Morse code is

A ——
B —····
C ———
D ——
E ·
F ·——
G ——
H ···
I ···
J ——
K ——
L ···
M ——
N —
O ——
P ···
Q ·—
R ·
S ···
T —
U ···
V ···
W ———
X ——
Y —·
Z ——

3.7.6 SOURCE CODING FOR ENGLISH TEXT

English text has, on average, 4.08 bits/character.
### 3.7.7 GRAY CODE

A Gray code is a sequence ordering such that a small change in the sequence number results in a small change in the sequence. For example, the 16 4-bit strings {0000, 0001, ..., 1111} can be ordered so that adjacent bit strings differ in only 1 bit:

<table>
<thead>
<tr>
<th>Sequence number</th>
<th>Bit string</th>
<th>Sequence number</th>
<th>Bit string</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0000</td>
<td>8</td>
<td>1100</td>
</tr>
<tr>
<td>1</td>
<td>0001</td>
<td>9</td>
<td>1101</td>
</tr>
<tr>
<td>2</td>
<td>0011</td>
<td>10</td>
<td>1111</td>
</tr>
<tr>
<td>3</td>
<td>0010</td>
<td>11</td>
<td>1110</td>
</tr>
<tr>
<td>4</td>
<td>0110</td>
<td>12</td>
<td>1010</td>
</tr>
<tr>
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<td>0111</td>
<td>13</td>
<td>1011</td>
</tr>
<tr>
<td>6</td>
<td>0101</td>
<td>14</td>
<td>1001</td>
</tr>
<tr>
<td>7</td>
<td>0100</td>
<td>15</td>
<td>1000</td>
</tr>
</tbody>
</table>
As another example, the subsets of \( \{a, b, c\} \) can be ordered so that adjacent subsets differ by only the insertion or deletion of a single element:

\[ \phi, \ {a}, \ {a, b}, \ {b}, \ {b, c}, \ {a, b, c}, \ {a, c}, \ {c}. \]

### 3.8 COSTAS ARRAYS

An \( n \times n \) Costas array is an array of zeros and ones whose autocorrelation function is \( n \) at the origin and no more than 1 anywhere else. There are \( B_n \) basic Costas arrays; there are \( C_n \) arrays when rotations and flips are allowed. Each array can be interpreted as a permutation.

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B_n )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>17</td>
<td>17</td>
<td>30</td>
<td>60</td>
<td>555</td>
<td>990</td>
<td></td>
</tr>
<tr>
<td>( C_n )</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>12</td>
<td>40</td>
<td>116</td>
<td>200</td>
<td>444</td>
<td>760</td>
<td>2160</td>
<td>4368</td>
<td>7852</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( n )</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B_n )</td>
<td>1616</td>
<td>2168</td>
<td>2467</td>
<td>2648</td>
<td>2294</td>
<td>1892</td>
<td>1283</td>
<td>810</td>
</tr>
<tr>
<td>( C_n )</td>
<td>12828</td>
<td>17252</td>
<td>19612</td>
<td>21104</td>
<td>18276</td>
<td>15096</td>
<td>10240</td>
<td>6464</td>
</tr>
</tbody>
</table>

\( B_n \): 

\( C_4 \): 

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3.9 DIFFERENCE EQUATIONS

3.9.1 THE CALCULUS OF FINITE DIFFERENCES

1. \[ \Delta f(x) = f(x + h) - f(x) \] (forward difference).
2. \[ \Delta^2 f(x) = f(x + 2h) - 2f(x + h) + f(x) \]
3. \[ \Delta^n f(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x + (n - k)h) \]
4. \[ \Delta(cf(x)) = c \Delta f(x) \]
5. \[ \Delta(f(x) + g(x)) = \Delta f(x) + \Delta g(x) \]
6. \[ \Delta(f(x)g(x)) = g(x) \Delta f(x) + f(x) \Delta g(x) \]
7. \[ \Delta \left( \frac{f(x)}{g(x)} \right) = \frac{g(x) \Delta f(x) - f(x) \Delta g(x)}{g(x)(x + h)} \]
   provided that \( g(x)g(x + h) \neq 0 \)
8. \[ \Delta^n (x^n) = n! h^n, \quad n = 0, 1, \ldots \]

3.9.2 EXISTENCE AND UNIQUENESS

A difference equation of order \( k \) has the form
\[
 x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+(k-1)}, n)
\]
where \( f \) is a given function and \( k \) is a positive integer. A solution to Equation (3.9.1) is a sequence of numbers \( \{x_n\}_{n=0}^{\infty} \) which satisfies the equation. Any constant solution of Equation (3.9.1) is called an equilibrium solution.

A linear difference equation of order \( k \) has the form
\[
 a_n^{(k)} x_{n+k} + a_n^{(k-1)} x_{n+(k-1)} + \cdots + a_n^{(1)} x_{n-1} + a_n^{(0)} x_n = g_n,
\]
where \( k \) is a positive integer and the coefficients \( a_n^{(0)}, \ldots, a_n^{(k)} \) along with \( g_n \) are given sequences. If the sequence \( g_n \) is identically zero, then Equation (3.9.2) is called homogeneous; otherwise, it is called nonhomogeneous. If the coefficients \( a_n^{(0)}, \ldots, a_n^{(k)} \) are constants, Equation (3.9.2) is a difference equation with constant coefficients; otherwise it is a difference equation with variable coefficients.

**THEOREM 3.9.1** (Existence and uniqueness)

Consider the initial value problem (IVP)
\[
 x_{n+k} + b_n^{(k-1)} x_{n+(k-1)} + \cdots + b_n^{(1)} x_{n+1} + b_n^{(0)} x_n = f_n, \quad n = 0, 1, \ldots, k-1
\]
for \( n = 0, 1, \ldots \), where \( b^{(i)}_n \) and \( f_n \) are given sequences with \( b^{(0)}_n \neq 0 \) for all \( n \) and the \( \{a_i\} \) are given initial conditions. Then the above equations have exactly one solution.

### 3.9.3 LINEAR INDEPENDENCE: GENERAL SOLUTION

The sequences \( x^{(1)}_n, x^{(2)}_n, \ldots, x^{(k)}_n \) are linearly dependent if constants \( c_1, c_2, \ldots, c_k \) (not all of them zero) exist such that

\[
\sum_{i=1}^{k} c_i x^{(i)}_n = 0 \quad \text{for } n = 0, 1, \ldots
\]

Otherwise the sequences \( x^{(1)}_n, x^{(2)}_n, \ldots, x^{(k)}_n \) are linearly independent.

The Casoratian of the \( k \) sequences \( x^{(1)}_n, x^{(2)}_n, \ldots, x^{(k)}_n \) is the \( k \times k \) determinant

\[
C \left( x^{(1)}_n, x^{(2)}_n, \ldots, x^{(k)}_n \right) = \begin{vmatrix}
    x^{(1)}_n & x^{(2)}_n & \cdots & x^{(k)}_n \\
    x^{(1)}_{n+1} & x^{(2)}_{n+1} & \cdots & x^{(k)}_{n+1} \\
    \vdots & \vdots & \ddots & \vdots \\
    x^{(1)}_{n+k-1} & x^{(2)}_{n+k-1} & \cdots & x^{(k)}_{n+k-1}
\end{vmatrix}
\]

### THEOREM 3.9.2

The solutions \( x^{(1)}_n, x^{(2)}_n, \ldots, x^{(k)}_n \) of the linear homogeneous difference equation,

\[
x_{n+k} + b^{(k-1)}_n x_{n+(k-1)} + \cdots + b^{(1)}_n x_{n+1} + b^{(0)}_n x_n = 0, \quad n = 0, 1, \ldots
\]

are linearly independent if, and only if, their Casoratian is different from zero for \( n = 0 \).

The set \( \{x^{(1)}_n, x^{(2)}_n, \ldots, x^{(k)}_n\} \) is a fundamental system of solutions for Equation (3.9.7) if, and only if, the sequences \( x^{(1)}_n, x^{(2)}_n, \ldots, x^{(k)}_n \) are linearly independent solutions of the homogeneous difference equation (3.9.7).

### THEOREM 3.9.3

Consider the nonhomogeneous linear difference equation

\[
x_{n+k} + b^{(k-1)}_n x_{n+(k-1)} + \cdots + b^{(1)}_n x_{n+1} + b^{(0)}_n x_n = d_n, \quad n = 0, 1, \ldots
\]

where \( b^{(i)}_n \) and \( d_n \) are given sequences. Let \( x^{(h)}_n \) be the general solution of the corresponding homogeneous equation

\[
x_{n+k} + b^{(k-1)}_n x_{n+(k-1)} + \cdots + b^{(1)}_n x_{n+1} + b^{(0)}_n x_n = 0, \quad n = 0, 1, \ldots
\]

and let \( x^{(p)}_n \) be a particular solution of Equation (3.9.8). Then \( x^{(p)}_n + x^{(h)}_n \) is the general solution of Equation (3.9.8).

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THEOREM 3.9.4 (Superposition principle)

Let \( x^{(1)}_n \) and \( x^{(2)}_n \) be solutions of the nonhomogeneous linear difference equations

\[
x_{n+k} + b_n^{(k-1)}x_{n+(k-1)} + \cdots + b_n^{(1)}x_{n+1} + b_n^{(0)}x_n = \alpha_n, \quad n = 0, 1, \ldots,
\]

and

\[
x_{n+k} + b_n^{(k-1)}x_{n+(k-1)} + \cdots + b_n^{(1)}x_{n+1} + b_n^{(0)}x_n = \beta_n, \quad n = 0, 1, \ldots,
\]

respectively, where \( b_n^{(i)} \) and \( \{\alpha_n\} \) and \( \{\beta_n\} \) are given sequences. Then \( x^{(1)}_n + x^{(2)}_n \) is a solution of the equation

\[
x_{n+k} + b_n^{(k-1)}x_{n+(k-1)} + \cdots + b_n^{(1)}x_{n+1} + b_n^{(0)}x_n = \alpha_n + \beta_n, \quad n = 0, 1, \ldots.
\]

3.9.4 HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

The results given below for second-order linear difference equations extend naturally to higher order equations.

Consider the second-order linear homogeneous difference equation,

\[
\alpha_2 x_{n+2} + \alpha_1 x_{n+1} + \alpha_0 x_n = 0,
\]

(3.9.9)

where the \( \{\alpha_i\} \) are real constant coefficients with \( \alpha_2\alpha_0 \neq 0 \). The characteristic equation corresponding to Equation (3.9.9) is defined as the quadratic equation

\[
\alpha_2 \lambda^2 + \alpha_1 \lambda + \alpha_0 = 0.
\]

(3.9.10)

The solutions \( \lambda_1, \lambda_2 \) of the characteristic equation are the eigenvalues or the characteristic roots of Equation (3.9.9).

THEOREM 3.9.5

Let \( \lambda_1 \) and \( \lambda_2 \) be the eigenvalues of Equation (3.9.9). Then the general solution of Equation (3.9.9) is given as described below with arbitrary constants \( c_1 \) and \( c_2 \).

Case 1: \( \lambda_1 \neq \lambda_2 \) with \( \lambda_1, \lambda_2 \in \mathbb{R} \) (real and distinct roots).

The general solution is given by \( x_n = c_1 \lambda_1^n + c_2 \lambda_2^n \).

Case 2: \( \lambda_1 = \lambda_2 \in \mathbb{R} \) (real and equal roots).

The general solution is given by \( x_n = c_1 \lambda^n + c_2 n \lambda^n \).

Case 3: \( \lambda_1 = \overline{\lambda_2} \) (complex conjugate roots).

Suppose that \( \lambda_1 = re^{i\phi} \). The general solution is given by

\[
x_n = c_1 r^n \cos(n\phi) + c_2 r^n \sin(n\phi).
\]
Example: The unique solution of the initial value problem

\[ F_{n+2} = F_{n+1} + F_n, \quad n = 0, 1, \ldots, \]
\[ F_0 = 0, \quad F_1 = 1, \]

is the Fibonacci sequence. Using Theorem 3.9.5 one can show that

\[ F_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right), \quad n = 0, 1, \ldots. \]

### 3.9.5 Nonhomogeneous Equations with Constant Coefficients

**Theorem 3.9.6 (Variation of parameters)**

Consider the difference equation,

\[ x_{n+2} + 2x_{n+1} + 3x_n = \gamma_n, \]

where \( \alpha_n, \beta_n, \) and \( \gamma_n \) are given sequences with \( \beta_n \neq 0. \) Let \( x_n^{(1)} \) and \( x_n^{(2)} \) be two linearly independent solutions of the homogeneous equation corresponding to this equation. A particular solution is given by

\[ x_n^{(p)} = x_n^{(1)} v_n^{(1)} + x_n^{(2)} v_n^{(2)}, \]

where the sequences \( v_n^{(1)} \) and \( v_n^{(2)} \) satisfy the following system of equations:

\[ x_n^{(1)} \left( v_{n+1}^{(1)} - v_n^{(1)} \right) + x_n^{(2)} \left( v_{n+1}^{(2)} - v_n^{(2)} \right) = 0, \quad \text{and} \]
\[ x_n^{(1)} \left( v_{n+2}^{(1)} - v_n^{(1)} \right) + x_n^{(2)} \left( v_{n+2}^{(2)} - v_n^{(2)} \right) = \gamma_n. \]

### 3.9.6 Generating Functions and Z Transforms

Generating functions can be used to solve initial value problems of difference equations in the same way that Laplace transforms are used to solve initial value problems of differential equations.

The generating function of the sequence \( \{x_n\} \), denoted by \( G[x_n] \), is defined by the infinite series

\[ G[x_n] = \sum_{n=0}^{\infty} x_n s^n \]

provided that the series converges for \( |s| < r \), for some positive number \( r \). The following are useful properties of the generating function:

1. **Linearity:** \( G \{c_1 x_n + c_2 y_n\} = c_1 G \{x_n\} + c_2 G \{y_n\} \).
2. **Translation invariance:** \( G \{x_{n+k}\} = \frac{1}{s^k} \left( G \{x_n\} - \sum_{n=0}^{k-1} x_n s^n \right) \).
3. **Uniqueness:** \( G \{x_n\} = G \{y_n\} \iff x_n = y_n \) for \( n = 0, 1, \ldots \).
The \( Z \)-transform of a sequence \( \{ x_n \} \) is denoted by \( Z[x_n] \) and is defined by the infinite series,

\[
Z[x_n] = \sum_{n=0}^{\infty} \frac{x_n}{z^n},
\]

(3.9.13)

provided that the series converges for \( |z| > r \), for some positive number \( r \).
Comparing the definitions for the generating function and the Z-transform one can see that they are connected because Equation (3.9.13) can be obtained from Equation (3.9.12) by setting $s = z^{-1}$.

### 3.9.7 CLOSED FORM SOLUTIONS FOR SPECIAL EQUATIONS

In general, it is difficult to find a closed form solution for a difference equation which is not linear of order one or linear of any order with constant coefficients. A few special difference equations which possess closed form solutions are presented below.

**THEOREM 3.9.7**

The general solution of the first-order linear difference equation with variable coefficients,

\[ x_{n+1} - \alpha_n x_n = \beta_n, \quad n = 0, 1, \ldots \]  

(3.9.14)

is given by

\[ x_n = \left( \prod_{k=0}^{n-1} \alpha_k \right) x_0 + \sum_{m=0}^{n-2} \left( \prod_{k=m+1}^{n-1} \alpha_k \right) \beta_m + \beta_{n-1}, \quad n = 0, 1, \ldots \]  

(3.9.15)

where $x_0$ is an arbitrary constant.

**Riccati equation**

Consider the nonlinear first order equation,

\[ x_{n+1} = \frac{\alpha_n x_n + \beta_n}{\gamma_n x_n + \delta_n}, \quad n = 0, 1, \ldots \]  

(3.9.16)

where $\alpha_n$, $\beta_n$, $\gamma_n$, $\delta_n$ are given sequences of real numbers with

\[ \gamma_n \neq 0 \quad \text{and} \quad \begin{vmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{vmatrix} \neq 0, \quad n = 0, 1, \ldots \]  

(3.9.17)

The following statements are true:

1. The change of variables,

\[ \frac{u_{n+1}}{u_n} = \frac{\alpha_n x_n + \beta_n}{\gamma_n x_n + \delta_n}, \quad n = 0, 1, \ldots \]  

(3.9.18)

reduces Equation (3.9.16) to the linear second order equation,

\[ u_{n+2} = A_n u_n + B_n, \quad n = 0, 1, \ldots \]  

(3.9.19)

where $A_n = \delta_{n+1} + \alpha_n \frac{\gamma_{n+1}}{\gamma_n}$, and $B_n = (\beta_n \gamma_n - \alpha_n \delta_n) \frac{\gamma_{n+1}}{\gamma_n}$.
2. Let \( \left\{ x_n^{(p)} \right\} \) be a particular solution of Equation (3.9.16). The change of variables,

\[
v_n = \frac{1}{x_n - x_n^{(p)}}, \quad n = 0, 1, \ldots , \quad (3.9.20)
\]

reduces Equation (3.9.16) to the linear first-order equation,

\[
v_{n+1} + C_n v_n + D_n = 0, \quad n = 0, 1, \ldots , \quad (3.9.21)
\]

where \( C_n = \left( \gamma_n x_n^{(p)} + \delta_n \right)^2 \beta_n \gamma_n - \alpha_n \delta_n \), and \( D_n = \gamma_n \left( \gamma_n x_n^{(p)} + \delta_n \right) \beta_n \gamma_n - \alpha_n \delta_n \).

3. Let \( x_n^{(1)} \) and \( x_n^{(2)} \) be two particular solutions of Equation (3.9.16) with \( x_n^{(1)} \neq x_n^{(2)} \) for \( n = 0, 1, \ldots \). Then the change of variables,

\[
w_n = \frac{1}{x_n - x_n^{(1)}} + \frac{1}{x_n^{(1)} - x_n^{(2)}}, \quad n = 0, 1, \ldots , \quad (3.9.22)
\]

reduces Equation (3.9.16) to the linear homogeneous first-order equation,

\[
w_{n+1} + E_n w_n = 0, \quad n = 0, 1, \ldots , \quad (3.9.23)
\]

where \( E_n = \left( \gamma_n x_n^{(1)} + \delta_n \right)^2 \beta_n \gamma_n - \alpha_n \delta_n \).

**Logistic equation**

Consider the initial value problem

\[
x_{n+1} = r x_n \left( 1 - \frac{x_n}{k} \right), \quad n = 0, 1, \ldots , \quad (3.9.24)
\]

\[
x_0 = \alpha, \quad \text{with} \quad \alpha \in [0, k],
\]

where \( r \) and \( k \) are positive numbers with \( r \leq 4 \). The following are true:

1. When \( r = k = 2 \), Equation (3.9.24) reduces to

\[
x_{n+1} = 2 x_n - x_n^2. \quad (3.9.25)
\]

By setting \( \alpha = 4 \sin^2(\theta) \) with \( \theta \in [0, \frac{\pi}{2}] \), then Equation (3.9.25) has the closed form solution

\[
x_{n+1} = 4 \sin^2(2^{n+1}\theta), \quad n = 0, 1, \ldots , \quad (3.9.26)
\]

\[
x_0 = 4 \sin^2(\theta), \quad \text{with} \quad \theta \in \left[ 0, \frac{\pi}{2} \right].
\]

2. When \( r = 4 \) and \( k = 1 \), Equation (3.9.24) reduces to

\[
x_{n+1} = 4 x_n - 4 x_n^2. \quad (3.9.27)
\]
By setting \( \alpha = \sin^2(\theta) \) with \( \theta \in [0, \frac{\pi}{2}] \), then Equation (3.9.27) has the closed form solution

\[
x_{n+1} = \sin^2(2^n \theta), \quad n = 0, 1, \ldots,
\]

\[
x_0 = \sin^2(\theta), \quad \text{with } \theta \in \left[0, \frac{\pi}{2}\right].
\]

### 3.10 DISCRETE DYNAMICAL SYSTEMS AND CHAOS

Let the distance between successive bifurcations of a process be \( d_k \). The limiting ratio, \( \delta = \lim_{k \to \infty} d_k/d_{k+1} \), Feigenbaum’s constant, is constant in many situations; \( \delta \approx 4.6692016091029 \).

#### 3.10.1 CHAOTIC ONE-DIMENSIONAL MAPS

1. Logistic map: \( x_{n+1} = 4x_n(1 - x_n) \).
   Solution is \( x_n = \frac{1}{2} - \frac{1}{2} \cos[2^n \cos^{-1}(1 - 2x_0)] \).

2. Tent map: \( x_{n+1} = 1 - 2|x_n - \frac{1}{2}| \).
   Solution is \( x_n = \frac{1}{\pi} \cos^{-1}[\cos(2^n \pi x_0)] \).

3. Baker transformation: \( x_{n+1} = 2x_n \) (mod 1).
   Solution is \( x_n = \frac{1}{\pi} \cot^{-1}[\cot(2^n \pi x_0)] \).

#### 3.10.2 LOGISTIC MAP

Consider \( u_{n+1} = f(u_n) = au_n(1 - u_n) \). The fixed points satisfy \( u = f(u) = au(1 - u) \); they are \( u = 0 \) and \( u = (a - 1)/a \).

- If \( a = 0 \) then \( u_n = 0 \).
- If \( 0 < a \leq 1 \) then \( u_n \to 0 \).
- If \( 1 < a < 3 \) then \( u_n \to (a - 1)/a \).
- If \( 3 < a < 3.449490 \ldots \) then \( u_n \) oscillates between the two roots of \( u = f(f(u)) \) which are not roots of \( u = f(u) \), that is, \( u_{\pm} = (a+1+\sqrt{a^2-2a-3})/2a \).
3.10.3 JULIA SETS AND THE MANDELBROT SET

For the function \( f(x) = x^2 + c \) consider the iterates of all complex points \( z \), \( z_{n+1} = f(z_n) \) with \( z_0 = z \). For each \( z \), either the iterates remain bounded (\( z \) is in the prisoner set) or they escape to infinity (\( z \) is in the escape set). The Julia set \( J_c \) is the common boundary between these two sets. Using lighter colors to indicate a “faster” escape to infinity, Figure 3.10.6 shows two Julia sets. One of these Julia sets is connected, the other is disconnected. The Mandelbrot set, \( M \), is the set of those complex values \( c \) for which \( J_c \) is a connected set (see Figure 3.10.7). Alternately, the Mandelbrot set consists of all points \( c \) for which the discrete dynamical system, \( z_{n+1} = z_n^2 + c \) with \( z_0 = 0 \), converges.

The boundary of the Mandelbrot set is a fractal. There is no universally agreed upon definition of “fractal”. One definition is that it is a set whose fractal dimension differs from its topological dimension.

3.11 OPERATIONS RESEARCH

3.11.1 LINEAR PROGRAMMING

Linear programming (LP) is a technique for modeling problems with linear objective functions and linear constraints. The standard form for an LP model with \( n \) decision
FIGURE 3.10.6
Connected Julia set for $c = -0.5i$ (left). Disconnected Julia set for $c = -\frac{3}{4}(1+i)$ (right).

FIGURE 3.10.7
The Mandelbrot set. The leftmost point has the coordinates $(-2, 0)$.

variables and $m$ resource constraints is

\[
\begin{align*}
\text{Minimize} & \quad \sum_{j=1}^{n} c_j x_j \quad \text{(objective function)}, \\
\text{Subject to} & \quad x_j \geq 0 \text{ for } j = 1, \cdots, n, \\
& \quad \sum_{j=1}^{n} a_{ij} x_j = b_i \text{ for } i = 1, \cdots, m \quad \text{(constraint functions)},
\end{align*}
\]

(3.11.1)

where $x_j$ is the amount of decision variable $j$ used, $c_j$ is decision $j$'s per unit contribution to the objective, $a_{ij}$ is decision $j$'s per unit usage of resource $i$, and $b_i$ is the total amount of resource $i$ available. Let $\mathbf{x}$ represent the $(n \times 1)$ vector $(x_1, x_2, \cdots, x_n)$, $\mathbf{c}$ the $(1 \times n)$ vector $(c_1, c_2, \cdots, c_n)$, $\mathbf{b}$ the $(m \times 1)$ vector $(b_1, b_2, \cdots, b_m)$, $A$ the $(m \times n)$ matrix $(a_{ij})$, and $A_j$ the $(n \times 1)$ column of $A$ associated with $x_j$. 

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The standard model, written in matrix notation, is “minimize $cx$ subject to $Ax \geq b$ and $x \geq 0$.” A vector $x$ is called feasible if, and only if, $Ax = b$ and $x \geq 0$.

**Modeling in LP**

LP is an appropriate modeling technique if the following four assumptions are satisfied by the situation:

1. All data coefficients are known with certainty.
2. There is a single objective.
3. The problem relationships are linear functions of the decisions.
4. The decisions can take on continuous values.

Branches of optimization such as stochastic programming, multiobjective programming, nonlinear programming, and integer programming have developed in operations research to allow a richer variety of models and solution techniques for situations where the assumptions required for LP are inappropriate.

• **Product mix problem:**

Consider a company that has three products to sell. Each product requires four operations and the per unit data are given in the following table:

<table>
<thead>
<tr>
<th>Product</th>
<th>Drilling</th>
<th>Assembly</th>
<th>Finishing</th>
<th>Packing</th>
<th>Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>45</td>
</tr>
<tr>
<td>B</td>
<td>3</td>
<td>6</td>
<td>2</td>
<td>4</td>
<td>90</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>55</td>
</tr>
<tr>
<td>Hours available</td>
<td>480</td>
<td>960</td>
<td>540</td>
<td>320</td>
<td></td>
</tr>
</tbody>
</table>

Let $x_A$, $x_B$, and $x_C$ represent the number of units of $A$, $B$, and $C$ manufactured daily. A model to maximize profit subject to the labor restrictions is

Maximize $45x_A + 90x_B + 55x_C$ (total profit),

Subject to:

\[
\begin{align*}
2x_A + 3x_B + 2x_C & \leq 480 \quad \text{(drilling hours),} \\
3x_A + 6x_B + 1x_C & \leq 960 \quad \text{(assembly hours),} \\
1x_A + 2x_B + 4x_C & \leq 540 \quad \text{(finishing hours),} \\
2x_A + 4x_B + 1x_C & \leq 320 \quad \text{(packing hours),} \\
x_A \geq 0, \quad x_B \geq 0, \quad x_C \geq 0.
\end{align*}
\]  

(3.11.2)

• **Maximum flow through a network:**

Consider the directed network in Figure 3.11.8. Node $S$ denotes the source node and node $T$ denotes the terminal node. On each arc shipping material up to the arc capacity $C_{ij}$ is permitted. Material is neither created nor destroyed at nodes other than $S$ and $T$. The goal is to maximize the amount of material that can be shipped through the network from $S$ to $T$. Letting $x_{ij}$ represent the amount of material shipped from node $i$ to node $j$, a model that determines the maximum flow is shown below.
Maximize \( x_{1T} + x_{3T} + x_{4T}, \)

Subject to

\[
\begin{align*}
  x_{S1} &= x_{13} + x_{14} + x_{1T} \quad \text{(node 1 conservation),} \\
  x_{S2} &= x_{23} + x_{24} \quad \text{(node 2 conservation),} \\
  x_{13} + x_{23} &= x_{3T} \quad \text{(node 3 conservation),} \\
  x_{S4} + x_{24} + x_{14} &= x_{4T} \quad \text{(node 4 conservation),} \\
  0 &\leq x_{ij} \leq C_{ij} \quad \text{for all } (i, j) \text{ pairs} \quad \text{(arc capacity).}
\end{align*}
\]

**Transformation to standard form**

Any LP model can be transformed to standard form using the operations in the following table:

<table>
<thead>
<tr>
<th>Original model</th>
<th>Standard form</th>
</tr>
</thead>
<tbody>
<tr>
<td>maximize objective</td>
<td>minimize objective</td>
</tr>
<tr>
<td>multiply ( c_j ) by (-1) (for example: ( c_j = -c_j ))</td>
<td></td>
</tr>
<tr>
<td>( \leq ) constraint</td>
<td>( = ) constraint</td>
</tr>
<tr>
<td>add slack variable(s) to ( \sum_{j=1}^{n} a_{ij}x_j \leq b_i )</td>
<td></td>
</tr>
<tr>
<td>(for example: ( \sum_{j=1}^{n} a_{ij}x_j + s_j = b_i ))</td>
<td></td>
</tr>
<tr>
<td>( \geq ) constraint</td>
<td>( = ) constraint</td>
</tr>
<tr>
<td>subtract surplus variable(s) from ( \sum_{j=1}^{n} a_{ij}x_j \geq b_i )</td>
<td></td>
</tr>
<tr>
<td>(for example: ( \sum_{j=1}^{n} a_{ij}x_j - u_i = b_i ))</td>
<td></td>
</tr>
<tr>
<td>( = ) constraint</td>
<td>( = ) constraint</td>
</tr>
<tr>
<td>no change required</td>
<td></td>
</tr>
</tbody>
</table>
LP requires that the slack and surplus variables be nonnegative, therefore, decisions are feasible to the new constraint if, and only if, they are feasible to the original constraint. All slack and surplus variables have $c_j = 0$.

Assume that there is at least one feasible $x$ vector, and that $A$ has rank $m$. Geometrically, because all constraints are linear, the set of feasible $x$ forms a convex polyhedral set (bounded or unbounded) that must have at least one extreme point. The motivation for the simplex method for solving LP models is the following:

For any LP model with a bounded optimal solution, an optimal solution exists at an extreme point of the feasible set.

Given a feasible solution $x$, let $x^B$ be the components of $x$ with $x_j > 0$ and $x^N$ be the components with $x_j = 0$. Associated with $x^B$, define $B$ as the columns of $A$ associated with each $x_j$ in $x^B$. For example, if $x_2$, $x_4$, and $S_1$ are positive in $x$, then $x^B = (x_2, x_4, S_1)$, and $B$ is the matrix with columns $A_2, A_4, A_{S_1}$. Define $N$ as the remaining columns of $A$, i.e., those associated with $x^N$. A basic feasible solution (BFS) is a feasible solution where the columns of $B$ are linearly independent. The following theorem relates a BFS with extreme points:

A feasible solution $x$ is at an extreme point of the feasible region if, and only if, $x$ is a BFS.

The following simplex method finds an optimal solution to the LP by finding the optimal partition of $x$ into $x^B$ and $x^N$:

1. Find an initial basic feasible solution. Define $x^B, x^N, B, N, c^B$, and $c^N$ as above.
2. Compute the vector $c' = (c^N - c^B B^{-1} N)$. If $c'_j \geq 0$ for all $j$, then stop; the solution $x^B = B^{-1} b$ is optimal with objective value $c^B B^{-1} b$. Otherwise, select the variable $x_j$ in $x^N$ with the most negative $c'_j$ value, and go to step 3.
3. Compute $A'_j = B^{-1} A_j$. If $A'_{ij} \leq 0$ for all $j$, then stop; the problem is unbounded and the objective can decrease to $-\infty$. Otherwise, compute $b' = B^{-1} b$ and find $\min_{i, a_{ij} > 0} \frac{b'_i}{a_{ij}}$. Assume the minimum ratio occurs in row $r$. Insert $x_j$ into the $r$th position of $x^B$, take the variable that was in this position, and move it to $x^N$. Update $B, N, c^B$, and $c^N$ accordingly. Return to step 2.

Ties in the selections in steps 2 and 3 can be broken arbitrarily. The unboundedness signal suggests that the model is missing constraints or that there has been a data entry or computational error, because, in real problems, the profit or cost cannot be unbounded. For maximization problems, only step 2 changes. The solution is optimal when $c'_j \leq 0$ for all $j$, then choose the variable with the maximum $c'_j$ value to move into $x^B$. Effective methods for updating $B^{-1}$ in each iteration of step 3 exist to ease the computational burden.

To find an initial basic feasible solution define a variable $A_i$ and add this variable to the left hand side of constraint $i$ transforming to $\sum_{j=1}^{n} a_{ij} + A_i = b_i$. The new
constraint is equivalent to the original constraint if, and only if, \( A_i = 0 \). Also, because \( A_i \) appears only in constraint \( i \) and there are \( m A_i \) variables, the columns corresponding to the \( A_i \) variables are a rank \( m \) linearly independent set. We now solve a “new” LP model with the adjusted constraints and the new objective “minimize \( \sum_{i=1}^{m} A_i \).” If the optimal solution to this new model is 0, the solution is a basic feasible solution to the original problem. Otherwise, no basic feasible solution exists for the original problem.

Solving LP models: interior point method

An alternative method to searching extreme points is to cut through the middle of the polyhedron and go directly towards the optimal solution. Extreme points, however, provide an efficient method of determining movement directions (step 1 of simplex method) and movement distances (step 2 of simplex method). There were no effective methods on the interior of the feasible set until Karmarkar’s method was developed in 1984. The method assumes that the model has the following form:

Minimize \( \sum_{j=1}^{n} c_j x_j \),

Subject to \[
\begin{align*}
\sum_{j=1}^{n} a_{ij} x_j &= 0 & \text{for } i = 1, \cdots, m, \\
\sum_{j=1}^{n} x_j &= 1, \\
x_j &\geq 0 & \text{for } j = 1, \cdots, n.
\end{align*}
\] (3.11.4)

Also, assume that the optimal objective value is 0 and that the \( x_j = 1/n \) for \( j = 1, \cdots, n \) is feasible. Any model can be transformed so that these assumptions hold.

The following centering transformation, relative to the \( k \)th estimate of solution vector \( x^k \), takes any feasible solution vector \( x \) and transforms it to \( y \) such that the \( x^k \) is transformed to the center of the feasible simplex: \( y_j = \frac{x_j / x^k_j}{\sum_{r=1}^{n} (x_r / x^k_r)} \). Let \( \text{Diag}(x^k) \) represent an \( n \times n \) matrix with off-diagonal entries equal to 0 and the diagonal entry in row \( j \) equal to \( x^k_j \). The formal algorithm is as follows:

1. Initialize \( x^0 = 1/n \) and set the iteration count \( k = 0 \).

2. If \( \sum_{j=1}^{n} x^k_j c_j \) is sufficiently close to 0, then stop. \( x^k \) is optimal. Otherwise go to step 3.

3. Move from the center of the transformed space in an improving direction using \[
x^{k+1} = \left[ \frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n} \right]^T - \frac{\theta(I - P^T (P P^T)^{-1} P) [\text{Diag}(x^k)] c^T}{\| C_p \| \sqrt{n(n - 1)}} \],
\] where \( \| C_p \| \) is the length of the vector \( (I - P^T (P P^T)^{-1} P) [\text{Diag}(x^k)] c^T \), \( P \) is an \((m + 1) \times n \) matrix whose first \( m \) rows are \( A [\text{Diag}(x^k)] \) and whose last row is a vector of 1’s, and \( \theta \) is a parameter that must be between 0 and 1. Go to step 4.

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4. Find the new point $x^{k+1}$ in the original space by applying the inverse transformation of the centering transformation to $y^{k+1}$. Set $k = k + 1$ and return to step 2.

The method is guaranteed to converge to the optimal solution when $\theta = \frac{1}{4}$ is used.

### 3.11.2 DUALITY AND COMPLEMENTARY SLACKNESS

Define $y_i$ as the dual variable (shadow price) representing the purchase price for a unit of resource $i$. The dual problem to the primal model (maximize objective, all $\leq$ constraints) is

Minimize $\sum_{i=1}^{m} b_i y_i,$

Subject to

\[
\begin{align*}
\sum_{i=1}^{m} a_{ij} y_i &\geq c_j \quad \text{for } j = 1, \ldots, n, \\
y_i &\geq 0 \quad \text{for } i = 1, \ldots, m.
\end{align*}
\]  

(3.11.5)

The objective minimizes the amount of money spent to obtain the resources. The constraints ensure that the marginal cost of the resources is greater than or equal to the marginal profit for each product.

The following results link the dual model (minimization) with its primal model (maximization).

- **Weak duality theorem**: Assume that $x$ and $y$ are feasible solutions to the respective primal and dual problems. Then

\[
\sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i.
\]

- **Strong duality theorem**: Assume that the primal has a finite optimal solution $x^*$. Then the dual has a finite optimal solution $y^*$, and

\[
\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*.
\]

- **Complementary slackness theorem**: Assume that $x$ and $y$ are feasible solutions to the respective primal and dual problems. Then, $x$ is optimal for the primal and $y$ is optimal for the dual if and only if:

\[
y_i \cdot \left(b_i - \sum_{j=1}^{n} a_{ij} x_j\right) = 0, \quad \text{for } i = 1, \ldots, m,
\]

and

\[
x_j \cdot \left(\sum_{i=1}^{m} a_{ij} y_i - c_j\right) = 0, \quad \text{for } j = 1, \ldots, n.
\]  

(3.11.6)
3.11.3 LINEAR INTEGER PROGRAMMING

Linear integer programming models result from restricting the decisions in linear programming models to be integer valued. The standard form is

\[
\begin{align*}
\text{Minimize} & \quad \sum_{j=1}^{n} c_j x_j \quad \text{(objective function)}, \\
\text{Subject to} & \quad x_j \geq 0, \quad \text{and integer for } j = 1, \ldots, n, \\
& \quad \sum_{j=1}^{n} a_{ij} x_j = b_i \quad \text{for } i = 1, \ldots, m \quad \text{(constraint functions)}.
\end{align*}
\]

(3.11.7)

As long as the variable values are bounded, then the general model can be transformed into a model where all variable values are restricted to either 0 or 1. Therefore, algorithms that can solve 0–1 integer programming models are sufficient for most applications.

3.11.4 BRANCH AND BOUND

Branch and bound implicitly enumerates all feasible integer solutions to find the optimal solution. The main idea is to break the feasible set into subsets (branching) and then evaluate the best solution in each subset or determine that the subset cannot contain the optimal solution (bounding). When a subset is evaluated, it is said to be “fathomed”. The following algorithm performs the branching by partitioning on variables with fractional values and uses a linear programming relaxation to generate a bound on the best solution in a subset:

1. Assume that a feasible integer solution, denoted “the incumbent”, is known whose objective function value is \(z\) (initially, \(z\) may be set to infinity if no feasible solution is known). Set \(p\), the subset counter equal to 1. Set the original model as the first problem in the subset list.

2. If \(p = 0\), then stop. The incumbent solution is the optimal solution. Otherwise go to step 3.

3. Solve the LP relaxation of problem \(p\) in the subset list (allow all integer valued variables to take on continuous values). Denote the LP objective value by \(v\). If \(v \geq z\) or the LP is infeasible, then set \(p = p - 1\) (fathom by bound or infeasibility), and return to step 2. If the LP solution is integer valued, then update the incumbent to the LP solution, set \(z = \min(z, v)\) and \(p = p - 1\), and return to step 2. Otherwise, go to step 4.

4. Take any variable \(x_j\) with fractional value in the LP solution. Replace problem \(p\) with two problems created by individually adding the constraints \(x_j \leq \lfloor x_j \rfloor\) and \(x_j \geq \lceil x_j \rceil\) to problem \(p\). Add these two problems to the bottom of the subset list, set \(p = p + 1\), and go to step 2.
3.11.5 NETWORK FLOW METHODS

A network consists of \( N \), the set of nodes, and \( A \), the set of arcs. Each arc \((i, j)\) defines a connection from node \( i \) to node \( j \). Depending on the application, arc \((i, j)\) may have an associated cost and upper and lower capacity on flow.

Decision problems on networks can often be modeled using linear programming models, and these models usually have the property that solutions from the simplex method are integer valued (the total unimodularity property). Because of the underlying graphical structure, more efficient algorithms are also available. We present the augmenting path algorithm for the maximum flow problem and the Hungarian method for the assignment problem.

Maximum flow

Let \( x_{ij} \) represent the flow on arc \((i, j)\), \( c_{ij} \) the flow capacity of \((i, j)\), \( S \) be the source node, and \( T \) be the terminal node. The maximum flow problem is to ship as much flow from \( S \) to \( T \) without violating the capacity on any arc, and all flow sent into node \( i \) must leave \( i \) (for \( i \neq S, T \)). The following algorithm solves the problem by continually adding flow-carrying paths until no path can be found:

1. Initialize \( x_{ij} = 0 \) for all \((i, j)\).
2. Find a flow-augmenting path from \( S \) to \( T \) using the following labeling method. Start by labeling \( S \) with \( a^* \). From any labeled node \( i \), label node \( j \) with the label \( i \) if \( j \) is unlabeled and \( x_{ij} < c_{ij} \) (“forward labeling arc”). From any labeled node \( i \), label node \( j \) with the label \( i \) if \( j \) is unlabeled and \( x_{ji} > 0 \) (“backward labeling arc”). Perform labeling until no additional nodes can be labeled. If \( T \) cannot be labeled, then stop. The current \( x_{ij} \) values are optimal. Otherwise, go to step 3.
3. There is a path from \( S \) to \( T \) where flow is increased on the forward labeling arcs, decreased on the backward labeling arcs, and gets more flow from \( S \) to \( T \). Let \( F \) be the minimum of \( c_{ij} - x_{ij} \) over all forward labeling arcs and of \( x_{ij} \) over all backward labeling arcs. Set \( x_{ij} = x_{ij} + F \) for the forward arcs and \( x_{ij} = x_{ij} - F \) for the backward arcs. Return to step 2.

The algorithm terminates with a set of arcs with \( x_{ij} = c_{ij} \) and if these are deleted, then \( S \) and \( T \) are in two disconnected pieces of the network. The algorithm finds the maximum flow by finding the minimum capacity set of arcs that disconnects \( S \) and \( T \) (minimum capacity cutset).

3.11.6 ASSIGNMENT PROBLEM

Consider a set \( J \) of jobs and a set \( I \) of employees. Each employee can do 1 job, and each job must be done by 1 employee. If job \( j \) is assigned to employee \( i \), then the cost to the company is \( c_{ij} \). The problem is to assign employees to jobs to minimize the overall cost.
This problem can be formulated as an optimization problem on a bipartite graph where the jobs are one part and the employees are the other. Let \( m \) be the cardinality of \( J \) and \( I \) (they must be equal cardinality sets or there is no feasible solution), and let \( C \) be the \( m \times m \) matrix of costs \( c_{ij} \). The following algorithm solves for the optimal assignment.

1. Find \( l_i = \min_j c_{ij} \) for each row \( i \). Let \( c_{ij} = c_{ij} - l_i \). Find \( n_j = \min_i c_{ij} \) for each column \( j \). Let \( c_{ij} = c_{ij} - n_j \).
2. Construct a graph with nodes for \( S \), \( T \), and each element of the sets \( J \) and \( I \). Construct an arc from \( S \) to each node in \( I \), and set its capacity to 1. Construct an arc from each node in \( J \) to \( T \), and set its capacity to 1. If \( c_{ij} = 0 \), then construct an arc from \( i \in I \) to \( j \in J \), and set its capacity to 2. Solve a maximum flow problem on the constructed graph. If \( m \) units of flow can go through the network, then stop. The maximum flow solution on the arcs between \( I \) and \( J \) represents the optimal assignment. Otherwise, go to step 3.
3. Update \( C \) using the following rules based on the labels in the solution to the maximum flow problem: Let \( L_I \) and \( L_J \) be the set of elements of \( I \) and \( J \) respectively with labels when the maximum flow algorithm terminates. Let \( \delta = \min_{i \in L_I, j \in (J - L_J)} c_{ij} \); note that \( \delta > 0 \). For \( i \in L_I \) and \( j \in (J - L_J) \), set \( c_{ij} = c_{ij} - \delta \). For \( i \in (I - L_I) \) and \( j \in L_J \), set \( c_{ij} = c_{ij} + \delta \). Leave all other \( c_{ij} \) values unchanged. Return to step 2.

In step 3, the algorithm creates new arcs, eliminates some unused arcs, and leaves unchanged arcs with \( x_{ij} = 1 \). When returning to step 2, you can solve the next maximum flow problem by adding and deleting the appropriate arcs and starting with the flows and labels of the preceding execution of the maximum flow algorithm.

### 3.11.7 Dynamic Programming

Dynamic programming is a technique for determining a sequence of optimal decisions for a system or process that operates over time and requires successive dependent decisions. The following five properties are required for using dynamic programming:

1. The system can be characterized by a set of parameters called state variables.
2. At each decision point or stage of the process, there is a choice of actions.
3. Given the current state and the decision, it is possible to specify how the state evolves before the next decision.
4. Only the current state matters, not the path by which the system arrived at the state (termed “time separability”).
5. An objective function depends on the state and the decisions made.

The separability requirement is necessary to formulate a functional form for the decision problem. Let \( f_r^*(i) \) denote the optimal objective value to take the system
from stage $t$ to the end of the process, given that the process is now in state $i$, $A_t(i)$ is the set of decisions possible at stage $t$ and state $i$, $a_j$ is a particular action in $A_t(i)$, $\Delta_{it}(a_j)$ is the change from state $i$ between stage $t$ and stage $t + 1$ based on the action taken, and $c_{it}(a_j)$ is the immediate impact on the objective of taking action $a_j$ at stage $t$ and state $i$.

The **principle of optimality** states as follows:

An optimal sequence of decisions has the property that, whatever the initial state and initial decision are, the remaining decisions must be an optimal policy based on the state resulting from the initial information.

Using the principle, Bellman’s equations are

$$f^*_t(i) = \min_{a_j \in A_t(i)} \{c_{it}(a_j) + f^*_{t+1}[\Delta_{it}(a_j)]\}, \text{ for all } i.$$  

### 3.11.8 SHORTEST PATH PROBLEM

Consider a network $(N, A)$ where $N$ is the set of nodes, $A$ the set of arcs, and $d_{ij}$ represents the “distance” of traveling on arc $(i, j)$ (if no arc exists between $i$ and $j$, $d_{ij} = \infty$). For any two nodes $R$ and $S$, the shortest path problem is to find the shortest distance route through the network from $R$ to $S$. Let the state space be $N$ and a stage representing travel along one arc. $f^*(i)$ is the optimal distance from node $i$ to $S$. The resultant recursive equations to solve are

$$f^*(i) = \min_{j \in N} \{d_{ij} + f^*(j)\}, \text{ for all } i.$$

Dijkstra’s algorithm can be used successively to approximate the solution to the equations when $d_{ij} > 0$ for all $(i, j)$.

1. Set $f^*(S) = 0$ and $f^*(i) = d_{is}$ for all $i \in N$. Let $P$ be the set of permanently labeled nodes. $P = \{S\}$. Let $T = N - P$ be the set of temporarily labeled nodes.
2. Find $i \in T$ with $f^*(i) = \min_{j \in T} f^*(j)$. Set $T = T - i$ and $P = P + i$. If $T = \emptyset$ (the empty set), then stop; $f^*(R)$ is the optimal path length. Otherwise, go to step 3.
3. Set $f^*(j) = \min[f^*(j), f^*(i) + d_{ij}]$ for all $j \in T$. Return to step 2.

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4.1 COORDINATE SYSTEMS IN THE PLANE

Convention
When we talk about “the point with coordinates \((x, y)\)” or “the curve with equation \(y = f(x)\)”, we always mean Cartesian coordinates. If a formula involves other coordinates, this fact will be stated explicitly.

4.1.1 SUBSTITUTIONS AND TRANSFORMATIONS

Formulas for changes in coordinate systems can lead to confusion because (for example) moving the coordinate axes \(\text{up}\) has the same effect on equations as moving objects \(\text{down}\) while the axes stay fixed. (To read the next paragraph, you can move your eyes down or slide the page up.)

To avoid confusion, we will carefully distinguish between transformations of the plane and substitutions, as explained below. Similar considerations will apply to transformations and substitutions in three dimensions (Section 4.9).

Substitutions
A substitution, or change of coordinates, relates the coordinates of a point in one coordinate system to those of the same point in a different coordinate system. Usually one coordinate system has the superscript \(\prime\) and the other does not, and we write

\[
\begin{align*}
x &= F_x(x', y'), \\
y &= F_y(x', y'),
\end{align*}
\]

or

\[(x, y) = F(x', y')\] (4.1.1)

(where subscripts are not derivatives). This means: given the equation of an object in the unprimed coordinate system, one obtains the equation of the same object in the primed coordinate system by substituting \(F_x(x', y')\) for \(x\) and \(F_y(x', y')\) for \(y\) in the equation. For instance, suppose the primed coordinate system is obtained from the unprimed system by moving the axes up a distance \(d\). Then \(x = x'\) and \(y = y' + d\). The circle with equation \(x^2 + y^2 = 1\) in the unprimed system has equation \(x'^2 + (y' + d)^2 = 1\) in the primed system. Thus, transforming an implicit equation in \((x, y)\) into one in \((x', y')\) is immediate.

The point \(P = (a, b)\) in the unprimed system, with equation \(x = a, y = b\), has equation \(F_x(x', y') = a, F_y(x', y') = b\) in the new system. To get the primed coordinates explicitly, one must solve for \(x'\) and \(y'\) (in the example just given we have

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\( x' = a, \ y' + d = b \), which yields \( x' = a, \ y' = b - d \). Therefore, if possible, we give the inverse equations

\[
\begin{align*}
    x' &= G_x(x, y), \\
    y' &= G_y(x, y)
\end{align*}
\]

or \((x', y') = G(x, y)\),

which are equivalent to Equation (4.1.1) if \( G(F(x, y)) = (x, y) \) and \( F(G(x, y)) = (x, y) \). Then to go from the unprimed to the unprimed system, one merely inserts the known values of \( x \) and \( y \) into these equations. This is also the best strategy when dealing with a curve expressed parametrically, that is, \( x = x(t), \ y = y(t) \).

**Transformations**

A transformation associates with each point \((x, y)\) a different point in the same coordinate system; we denote this by

\[
(x, y) \mapsto F(x, y),
\]

where \( F \) is a map from the plane to itself (a two-component function of two variables). For example, translating down by a distance \( d \) is accomplished by \( (x, y) \mapsto (x, y - d) \) (see Section 4.2). Thus, the action of the transformation on a point whose coordinates are known (or on a curve expressed parametrically) can be immediately computed.

If, on the other hand, we have an object (say a curve) defined implicitly by the equation \( C(x, y) = 0 \), finding the equation of the transformed object requires using the inverse transformation \( (x, y) \mapsto G(x, y) \) defined by \( G(F(x, y)) = (x, y) \) and \( F(G(x, y)) = (x, y) \). The equation of the transformed object is \( C(G(x, y)) = 0 \). For instance, if \( C \) is the circle with equation \( x^2 + y^2 = 1 \) and we are translating down by a distance \( d \), the inverse transformation is \( (x, y) \mapsto (x, y + d) \) (translating up), and the equation of the translated circle is \( x^2 + (y + d)^2 = 1 \). Compare the example following Equation (4.1.1).

**Using transformations to perform changes of coordinates**

Usually, we will not give formulas of the form (4.1.1) for changes between two coordinate systems of the same type, because they can be immediately derived from the corresponding formulas (4.1.2) for transformations, which are given in Section 4.2. We give two examples for clarity.

Let the two Cartesian coordinate systems \((x, y)\) and \((x', y')\) be related as follows:

They have the same origin, and the positive \( x' \)-axis is obtained from the positive \( x \)-axis by a (counterclockwise) rotation through an angle \( \theta \) (Figure 4.1.1). If a point has coordinates \((x, y)\) in the unprimed system, its coordinates \((x', y')\) in the primed system are the same as the coordinates in the unprimed system of a point that undergoes the inverse rotation, that is, a rotation by an angle \( \alpha = -\theta \). According to Equation (4.2.1) (page 257), this transformation acts as follows:

\[
(x, y) \mapsto \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix} (x, y) = (x \cos \theta + y \sin \theta, -x \sin \theta + y \cos \theta),
\]

\[(4.1.3)\]
Therefore the right-hand side of Equation (4.1.3) is \((x', y')\), and the desired substitution is

\[
\begin{align*}
  x' &= x \cos \theta + y \sin \theta, \\
  y' &= -x \sin \theta + y \cos \theta.
\end{align*}
\]

Switching the roles of the primed and unprimed systems we get the equivalent substitution

\[
\begin{align*}
  x &= x' \cos \theta - y' \sin \theta, \\
  y &= x' \sin \theta + y' \cos \theta
\end{align*}
\]

(because the \(x\)-axis is obtained from the \(x'\)-axis by a rotation through an angle \(-\theta\)).

Similarly, let the two Cartesian coordinate systems \((x, y)\) and \((x', y')\) differ by a translation: \(x\) is parallel to \(x'\) and \(y\) to \(y'\), and the origin of the second system coincides with the point \((x_0, y_0)\) of the first system. The coordinates \((x, y)\) and \((x', y')\) of a point are related by

\[
\begin{align*}
  x &= x' + x_0, & x' &= x - x_0, \\
  y &= y' + y_0, & y' &= y - y_0.
\end{align*}
\] (4.1.4)

### 4.1.2 Cartesian Coordinates in the Plane

In *Cartesian coordinates* (or *rectangular coordinates*), the “address” of a point \(P\) is given by two real numbers indicating the positions of the perpendicular projections from the point to two fixed, perpendicular, graduated lines, called the *axes*. If one coordinate is denoted \(x\) and the other \(y\), the axes are called the *x-axis* and the *y-axis*, and we write \(P = (x, y)\). Usually the *x*-axis is horizontal, with \(x\) increasing to the right, and the *y*-axis is vertical, with \(y\) increasing vertically up. The point \(x = 0, y = 0\), where the axes intersect, is the *origin*. See Figure 4.1.2.

### 4.1.3 Polar Coordinates in the Plane

In *polar coordinates* a point \(P\) is also characterized by two numbers: the distance \(r \geq 0\) to a fixed *pole* or *origin* \(O\), and the angle \(\theta\) that the ray \(OP\) makes with a fixed
FIGURE 4.1.2
In Cartesian coordinates, \( P_1 = (4, 3) \), \( P_2 = (-1.3, 2.5) \), \( P_3 = (-1.5, -1.5) \), \( P_4 = (3.5, -1) \), and \( P_5 = (4.5, 0) \). The axes divide the plane into four quadrants. \( P_1 \) is in the first quadrant, \( P_2 \) in the second, \( P_3 \) in the third, and \( P_4 \) in the fourth. \( P_5 \) is on the positive \( x \)-axis.

Among the possible sets of polar coordinates for \( P \) are \((10, 30^\circ)\), \((10, 390^\circ)\) and \((10, -330^\circ)\). Among the sets of polar coordinates for \( Q \) are \((2.5, 210^\circ)\) and \((-2.5, 30^\circ)\).

Relations between Cartesian and polar coordinates

Consider a system of polar coordinates and a system of Cartesian coordinates with the same origin. Assume that the initial ray of the polar coordinate system coincides with the positive \( x \)-axis, and that the ray \( \theta = 90^\circ \) coincides with the positive \( y \)-axis. Then the polar coordinates \((r, \theta)\) and the Cartesian coordinates \((x, y)\) of the same
point are related as follows:

\[
\begin{align*}
  x &= r \cos \theta, \\
  y &= r \sin \theta, \\
  r &= \sqrt{x^2 + y^2}, \\
  \theta &= \tan^{-1} \frac{y}{x}, \\
  \sin \theta &= \frac{y}{\sqrt{x^2 + y^2}}, \\
  \cos \theta &= \frac{x}{\sqrt{x^2 + y^2}}.
\end{align*}
\]

### 4.1.4 HOMOGENEOUS COORDINATES IN THE PLANE

A triple of real numbers \((x : y : t)\), with \(t \neq 0\), is a set of **homogeneous coordinates** for the point \(P\) with Cartesian coordinates \((x/t, y/t)\). Thus the same point has many sets of homogeneous coordinates: \((x : y : t)\) and \((x' : y' : t')\) represent the same point if and only if there is some real number \(\alpha\) such that \(x' = \alpha x, y' = \alpha y, z' = \alpha z\).

When we think of the same triple of numbers as the Cartesian coordinates of a point in three-dimensional space (page 296), we write it as \((x, y, t)\) instead of \((x : y : t)\). The connection between the point in space with Cartesian coordinates \((x, y, t)\) and the point in the plane with homogeneous coordinates \((x : y : t)\) becomes apparent when we consider the plane \(t = 1\) in space, with Cartesian coordinates given by the first two coordinates \(x, y\) of space (*Figure 4.1.4*). The point \((x, y, t)\) in space can be connected to the origin by a line \(L\) that intersects the plane \(t = 1\) in the point with Cartesian coordinates \((x/t, y/t)\) or homogeneous coordinates \((x : y : t)\).

**FIGURE 4.1.4**

*The point \(P\) with spatial coordinates \((x, y, t)\) projects to the point \(Q\) with spatial coordinates \((x/t, y/t, 1)\). The plane Cartesian coordinates of \(Q\) are \((x/t, y/t)\), and \((x : y : t)\) is one set of homogeneous coordinates for \(Q\). Any point on the line \(L\) (except for the origin \(O\)) would also project to \(P'\).*

Projective coordinates are useful for several reasons. One the most important is that they allow one to unify all symmetries of the plane (as well as other transformations) under a single umbrella. All of these transformations can be regarded as linear maps in the space of triples \((x : y : t)\), and so can be expressed in terms of matrix multiplications (see page 258).
If we consider triples \((x : y : t)\) such that at least one of \(x, y, t\) is nonzero, we can name not only the points in the plane but also points “at infinity”. Thus, \((x : y : 0)\) represents the point at infinity in the direction of the line whose slope is \(y/x\).

### 4.1.5 OBLIQUE COORDINATES IN THE PLANE

The following generalization of Cartesian coordinates is sometimes useful. Consider two axes (graduated lines), intersecting at the origin but not necessarily perpendicularly. Let the angle between them be \(\omega\). In this system of oblique coordinates, a point \(P\) is given by two real numbers indicating the positions of the projections from the point to each axis, in the direction of the other axis (see Figure 4.1.5). The first axis (\(x\)-axis) is generally drawn horizontally. The case \(\omega = 90^\circ\) yields a Cartesian coordinate system.

**FIGURE 4.1.5**

In oblique coordinates, \(P_1 = (4, 3), P_2 = (-1, 3, 5), P_3 = (-1.5, -1.5), P_4 = (3.5, -1),\) and \(P_5 = (4.5, 0)\). Compare to Figure 4.1.2.

Relations between two oblique coordinate systems

Let the two oblique coordinate systems \((x, y)\) and \((x', y')\), with angles \(\omega\) and \(\omega'\), share the same origin, and suppose the \(x'\)-axis makes an angle \(\theta\) with the \(x\)-axis. The coordinates \((x, y)\) and \((x', y')\) of a point in the two systems are related by

\[
\begin{align*}
x &= \frac{x' \sin(\omega - \theta) + y' \sin(\omega - \omega' - \theta)}{\sin \omega}, \\
y &= \frac{x' \sin \theta + y' \sin(\omega' + \theta)}{\sin \omega}.
\end{align*}
\]

This formula also covers passing from a Cartesian system to an oblique system and vice versa, by taking \(\omega = 90^\circ\) or \(\omega' = 90^\circ\).

The relation between two oblique coordinate systems that differ by a translation is the same as for Cartesian systems. See Equation (4.1.4).
4.2 PLANE SYMMETRIES OR ISOMETRIES

A transformation of the plane (invertible map of the plane to itself) that preserves distances is called an isometry of the plane. Every isometry of the plane is of one of the following types:

- The identity (which leaves every point fixed)
- A translation by a vector \( \mathbf{v} \)
- A rotation through an angle \( \alpha \) around a point \( P \)
- A reflection in a line \( L \)
- A glide-reflection in a line \( L \) with displacement \( d \)

Although the identity is a particular case of a translation and a rotation, and reflections are particular cases of glide-reflections, it is more intuitive to consider each case separately.

4.2.1 FORMULAS FOR SYMMETRIES: CARTESIAN COORDINATES

In the formulas below, a multiplication between a matrix and a pair of coordinates should be carried out regarding the pair as a column vector (or a matrix with two rows and one column). Thus \[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
(x, y) = (ax + by, cx + dy).
\]

Translation by \((x_0, y_0)\):

\[
(x, y) \mapsto (x + x_0, \ y + y_0).
\]

Rotation through \(\alpha\) (counterclockwise) around the origin:

\[
(x, y) \mapsto \begin{bmatrix}
  \cos \alpha & -\sin \alpha \\
  \sin \alpha & \cos \alpha
\end{bmatrix} (x, y).
\] (4.2.1)

Rotation through \(\alpha\) (counterclockwise) around an arbitrary point \((x_0, y_0)\):

\[
(x, y) \mapsto (x_0, y_0) + \begin{bmatrix}
  \cos \alpha & -\sin \alpha \\
  \sin \alpha & \cos \alpha
\end{bmatrix} (x - x_0, \ y - y_0).
\]

Reflection:

- in the \(x\)-axis: \((x, y) \mapsto (x, -y)\),
- in the \(y\)-axis: \((x, y) \mapsto (-x, y)\),
- in the diagonal \(x = y\): \((x, y) \mapsto (y, x)\).

Reflection in a line with equation \(ax + by + c = 0\):

\[
(x, y) \mapsto \frac{1}{a^2 + b^2} \begin{bmatrix}
  b^2 - a^2 & -2ab \\
  -2ab & a^2 - b^2
\end{bmatrix} (x, y) - (2ac, 2bc).
\]
Reflection in a line going through \((x_0, y_0)\) and making an angle \(\alpha\) with the \(x\)-axis:

\[(x, y) \mapsto (x_0, y_0) + \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix} (x - x_0, y - y_0).\]

Glide-reflection in a line \(L\) with displacement \(d\): Apply first a reflection in \(L\), then a translation by a vector of length \(d\) in the direction of \(L\), that is, by the vector

\[
\frac{1}{a^2 + b^2} (\pm ad, \mp bd)
\]

if \(L\) has equation \(ax + by + c = 0\).

### 4.2.2 FORMULAS FOR SYMMETRIES: HOMOGENEOUS COORDINATES

All isometries of the plane can be expressed in homogeneous coordinates in terms of multiplication by a matrix. This fact is useful in implementing these transformations on a computer. It also means that the successive application of transformations reduces to matrix multiplication. The corresponding matrices are as follows:

**Translation** by \((x_0, y_0)\):

\[
T(x_0, y_0) = \begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

**Rotation** through \(\alpha\) around the origin:

\[
R_\alpha = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

**Reflection** in a line going through the origin and making an angle \(\alpha\) with the \(x\)-axis:

\[
M_\alpha = \begin{bmatrix} \cos 2\alpha & \sin 2\alpha & 0 \\ \sin 2\alpha & -\cos 2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

From this one can deduce all other transformations. For example, to find the matrix for a rotation through \(\alpha\) around an arbitrary point \(P = (x_0, y_0)\), we apply a translation by \(- (x_0, y_0)\) to move \(P\) to the origin, a rotation through \(\alpha\) around the origin, and then a translation by \((x_0, y_0)\):

\[
T(x_0, y_0)R_\alpha T^{-1}(x_0, y_0) = \begin{bmatrix} \cos \alpha & -\sin \alpha & x_0 - x_0 \cos \alpha + y_0 \sin \alpha \\ \sin \alpha & \cos \alpha & y_0 - y_0 \cos \alpha - x_0 \sin \alpha \\ 0 & 0 & 1 \end{bmatrix}
\]

(notice the order of the multiplication).

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4.2.3 FORMULAS FOR SYMMETRIES: POLAR COORDINATES

Rotation around the origin through an angle $\alpha$:
$$(r, \theta) \mapsto (r, \theta + \alpha).$$

Reflection in a line through the origin and making an angle $\alpha$ with the positive x-axis:
$$(r, \theta) \mapsto (r, 2\alpha - \theta).$$

4.2.4 CRYSTALLOGRAPHIC GROUPS

A group of symmetries of the plane that is doubly infinite is a wallpaper group, or crystallographic group. There are 17 types of such groups, corresponding to 17 essentially distinct ways to tile the plane in a doubly periodical pattern. (There are also 230 three-dimensional crystallographic groups.)

The simplest crystallographic group involves translations only (page 261, top left). The others involve, in addition to translations, one or more of the other types of symmetries (rotations, reflections, glide-reflections). The Conway notation for crystallographic groups is based on the types of nontranslational symmetries occurring in the “simplest description” of the group: $\ast$ indicates a reflection (mirror symmetry), $\times$ a glide-reflection, and a number $n$ indicates a rotational symmetry of order $n$ (rotation by $360^\circ/n$). In addition, if a number $n$ comes after the $\ast$, the center of the corresponding rotation lies on mirror lines, so that the symmetry there is actually dihedral of order $2n$.

Thus the group $\ast\ast$ in the table below (page 261, middle left) has two inequivalent lines of mirror symmetry; the group 333 (page 263, top left) has three inequivalent centers of order-3 rotation; the group 22$^\ast$ (page 261, bottom right) has two inequivalent centers of order-2 rotation as well as mirror lines; and $\ast632$ (page 263, bottom) has points of dihedral symmetry of order 12($= 2 \times 6$), 6, and 4.

The following table gives the groups in the Conway notation and in the notation traditional in crystallography. It also gives the quotient space of the plane by the action of the group. The entry “4,4,2 turnover” means the surface of a triangular puff pastry with corner angles $45^\circ (= 180^\circ /4), 45^\circ$ and $90^\circ$. The entry “4,4,2 turnover slit along 2,4” means the same surface, slit along the edge joining a $45^\circ$ vertex to the $90^\circ$ vertex. Open edges are silvered (mirror lines); such edges occur exactly for those groups whose Conway notation includes a $\ast$.

The last column of the table gives the dimension of the space of inequivalent groups of the given type (equivalent groups are those that can be obtained from one another by proportional scaling or rigid motion). For instance, there is a group of type $\circ$ for every shape parallelogram, and there are two degrees of freedom for the choice of such a shape (say the ratio and angle between sides). Thus, the $\circ$ group of page 261 (top left) is based on a square fundamental domain, while for the $\circ$ group of page 263 (top left) a fundamental parallelogram would have the shape of two juxtaposed equilateral triangles. These two groups are inequivalent, although they are of the same type.

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The figures on pages 261–263 show wallpaper patterns based on each of the 17 types of crystallographic groups (two patterns are shown for the \(\circ\), or translations-only, type). Thin lines bound unit cells, or fundamental domains. When solid, they represent lines of mirror symmetry, and are fully determined. When dashed, they represent arbitrary boundaries, which can be shifted so as to give different fundamental domains. One can even make these lines into curves, provided the symmetry is respected. Dots at the intersections of thin lines represent centers of rotational symmetry.

Some of the relationships between the types are made obvious by the patterns. For instance, on the first row of page 261, we see that the group on the right, of type \(\times\times\), contains the one on the left, of type \(\circ\), with index two. However, there are more relationships than can be indicated in a single set of pictures. For instance, there is a group of type \(\times\times\) hiding in any group of type \(3^3\).
4.2.5 CLASSIFYING THE CRYSTALLOGRAPHIC GROUPS

To classify an image representing a crystallographic group, answer the following sequence of questions starting with: “What is the minimal rotational invariance?”.

- None
  Is there a reflection?
  - No.
  - Is there a glide-reflection?
    - No: p1 (page 261)
    - Yes: pg (page 261)
  - Yes.
    - Is there a glide-reflection in an axis that is not a reflection axis?
      - No: pm (page 261)
      - Yes: cm (page 261)

- 2-fold (180° rotation)
  Is there a reflection?
  - No.
    - Is there a glide-reflection?
      - No: p2 (page 262)
      - Yes: pgg (page 261)
  - Yes.
    - Are there reflections in two directions?
      - No: pmg (page 261)
      - Yes: Are all rotation centers on reflection axes?
        - No: cmm (page 262)
        - Yes: pmm (page 262)

- 3-fold (120° rotation)
  Is there a reflection?
  - No: p3 (page 263)
  - Yes.
    - Are all centers of threefold reflections on reflection axes?
      - No: p31m (page 263)
      - Yes: p3m1 (page 263)

- 4-fold (90° rotation)
  Is there a reflection?
  - No: p4 (page 262)
  - Yes.
    - Are there four reflection axes?
      - No: p4g (page 262)
      - Yes: p4m (page 262)

- 6-fold (60° rotation)
  Is there a reflection?
  - No: p6 (page 263)
  - Yes: p6m (page 263)
4.3 OTHER TRANSFORMATIONS OF THE PLANE

4.3.1 SIMILARITIES

A transformation of the plane that preserves shapes is called a similarity. Every similarity of the plane is obtained by composing a proportional scaling transformation (also known as a homothety) with an isometry. A proportional scaling transformation centered at the origin has the form

\[(x, y) \mapsto (ax, ay),\]

where \(a \neq 0\) is the scaling factor (a real number). The corresponding matrix in homogeneous coordinates is

\[
H_a = \begin{bmatrix}
    a & 0 & 0 \\
    0 & a & 0 \\
    0 & 0 & 1
\end{bmatrix}.
\]

In polar coordinates, the transformation is \((r, \theta) \mapsto (ar, \theta)\).

4.3.2 AFFINE TRANSFORMATIONS

A transformation that preserves lines and parallelism (maps parallel lines to parallel lines) is an affine transformation. There are two important particular cases of such transformations:

A nonproportional scaling transformation centered at the origin has the form \((x, y) \mapsto (ax, by)\), where \(a, b \neq 0\) are the scaling factors (real numbers). The corresponding matrix in homogeneous coordinates is

\[
H_{a,b} = \begin{bmatrix}
    a & 0 & 0 \\
    0 & b & 0 \\
    0 & 0 & 1
\end{bmatrix}.
\]

A shear preserving horizontal lines has the form \((x, y) \mapsto (x + ry, y)\), where \(r\) is the shearing factor (see Figure 4.3.6). The corresponding matrix in homogeneous coordinates is

\[
S_r = \begin{bmatrix}
    1 & r & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 1
\end{bmatrix}.
\]

Every affine transformation is obtained by composing a scaling transformation with an isometry, or a shear with a homothety and an isometry.
4.3.3 PROJECTIVE TRANSFORMATIONS

A transformation that maps lines to lines (but does not necessarily preserve parallelism) is a projective transformation. Any plane projective transformation can be expressed by an invertible $3 \times 3$ matrix in homogeneous coordinates; conversely, any invertible $3 \times 3$ matrix defines a projective transformation of the plane. Projective transformations (if not affine) are not defined on all of the plane but only on the complement of a line (the missing line is “mapped to infinity”).

A common example of a projective transformation is given by a perspective transformation (Figure 4.3.7). Strictly speaking this gives a transformation from one plane to another, but, if we identify the two planes by (for example) fixing a Cartesian system in each, we get a projective transformation from the plane to itself.

FIGURE 4.3.7
A perspective transformation with center $O$, mapping the plane $P$ to the plane $Q$. The transformation is not defined on the line $L$, where $P$ intersects the plane parallel to $Q$ and going through $O$. 
4.4 LINES

The (Cartesian) equation of a straight line is linear in the coordinates \(x\) and \(y\):

\[ ax + by + c = 0. \]  \hspace{1cm} (4.4.1)

The slope of this line is \(-a/b\), the intersection with the \(x\)-axis (or \(x\)-intercept) is \(x = -c/a\), and the intersection with the \(y\)-axis (or \(y\)-intercept) is \(y = -c/b\). If \(a = 0\), the line is parallel to the \(x\)-axis, and if \(b = 0\), then the line is parallel to the \(y\)-axis.

(In an oblique coordinate system, everything in the preceding paragraph remains true, except for the value of the slope.)

When \(a^2 + b^2 = 1\) and \(c \leq 0\) in the equation \(ax + by + c = 0\), the equation is said to be in normal form. In this case \(c\) is the distance of the line to the origin, and \(\omega = \sin^{-1} a = \cos^{-1} b\) is the angle that the perpendicular dropped to the line from the origin makes with the positive \(x\)-axis (Figure 4.4.8).

**FIGURE 4.4.8**

The normal form of the line \(L\) is \(x \cos \omega + y \sin \omega = p\).

To reduce an arbitrary equation \(ax + by + c = 0\) to normal form, divide by \(\pm \sqrt{a^2 + b^2}\), where the sign of the radical is chosen opposite the sign of \(c\) when \(c \neq 0\) and the same as the sign of \(b\) when \(c = 0\).

**Lines with prescribed properties**

- Line of slope \(m\) intersecting the \(x\)-axis at \(x = x_0\): \(y = m(x + x_0)\).
- Line of slope \(m\) intersecting the \(y\)-axis at \(y = y_0\): \(y = mx + y_0\).
- Line intersecting the \(x\)-axis at \(x = x_0\) and the \(y\)-axis at \(y = y_0\):

  \[
  \frac{x}{x_0} + \frac{y}{y_0} = 1.
  \]

  (This formula remains true in oblique coordinates.)
• Line of slope $m$ passing through $(x_0, y_0)$: 
  \[ y - y_0 = m(x - x_0). \]

• Line passing through points $(x_0, y_0)$ and $(x_1, y_1)$:
  \[
  \begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0. \tag{4.4.2}
  \]

  (These formulas remain true in oblique coordinates.)

• Slope of line going through points $(x_0, y_0)$ and $(x_1, y_1)$:
  \[
  \frac{y_1 - y_0}{x_1 - x_0}.
  \]

• Line passing through points with polar coordinates $(r_0, \theta_0)$ and $(r_1, \theta_1)$:
  \[
  r(r_0 \sin(\theta - \theta_0) - r_1 \sin(\theta - \theta_1)) = r_0r_1 \sin(\theta_1 - \theta_0). \tag{4.4.3}
  \]

### 4.4.1 DISTANCES

The distance between two points in the plane is the length of the line segment joining the two points. If the points have Cartesian coordinates $(x_0, y_0)$ and $(x_1, y_1)$, this distance is

\[
\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}. \tag{4.4.4}
\]

If the points have polar coordinates $(r_0, \theta_0)$ and $(r_1, \theta_1)$, this distance is

\[
\sqrt{r_0^2 + r_1^2 - 2r_0r_1 \cos(\theta_0 - \theta_1)}. \tag{4.4.5}
\]

If the points have oblique coordinates $(x_0, y_0)$ and $(x_1, y_1)$, this distance is

\[
\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + 2(x_1 - x_0)(y_1 - y_0) \cos \omega}, \tag{4.4.6}
\]

where $\omega$ is the angle between the axes. (Figure 4.1.5).

The point $k\%$ of the way from $P_0 = (x_0, y_0)$ to $P_1 = (x_1, y_1)$ is

\[
\left( \frac{kx_1 + (100 - k)x_2}{100}, \frac{ky_1 + (100 - k)y_2}{100} \right). \tag{4.4.7}
\]

(The same formula also works in oblique coordinates.) This point divides the segment $P_0P_1$ in the ratio $k : (100 - k)$. As a particular case, the midpoint of $P_0P_1$ is given by $\left( \frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2) \right)$.

The distance from the point $(x_0, y_0)$ to the line $ax + by + c = 0$ is

\[
\frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.
\]

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4.4.2 ANGLES

The angle between two lines \( a_0x + b_0y + c_0 = 0 \) and \( a_1x + b_1y + c_1 = 0 \) is

\[
\tan^{-1}\frac{b_1}{a_1} - \tan^{-1}\frac{b_0}{a_0} = \tan^{-1}\frac{a_0b_1 - a_1b_0}{a_0a_1 + b_0b_1}.
\]

(4.4.8)

In particular, the two lines are parallel when \( a_0b_1 = a_1b_0 \), and perpendicular when \( a_0b_1 = -a_1b_0 \).

The angle between two lines of slopes \( m_0 \) and \( m_1 \) is

\[
\tan^{-1}\left(\frac{m_1 - m_0}{1 + m_0m_1}\right).
\]

(4.4.8)

In particular, the two lines are parallel when \( m_0 = m_1 \) and perpendicular when \( m_0m_1 = -1 \).

4.4.3 CONCURRENCE AND COLLINEARITY

Three lines \( a_0x + b_0y + c_0 = 0 \), \( a_1x + b_1y + c_1 = 0 \), and \( a_2x + b_2y + c_2 = 0 \) are concurrent if and only if

\[
\begin{vmatrix}
  a_0 & b_0 & c_0 \\
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2
\end{vmatrix} = 0.
\]

(This remains true in oblique coordinates.)

Three points \((x_0, y_0), (x_1, y_1)\) and \((x_2, y_2)\) are collinear if and only if

\[
\begin{vmatrix}
  x_0 & y_0 & 1 \\
  x_1 & y_1 & 1 \\
  x_2 & y_2 & 1
\end{vmatrix} = 0.
\]

(This remains true in oblique coordinates.)

Three points with polar coordinates \((r_0, \theta_0), (r_1, \theta_1)\) and \((r_2, \theta_2)\) are collinear if and only if

\[
r_1r_2 \sin(\theta_2 - \theta_1) + r_0r_1 \sin(\theta_1 - \theta_0) + r_2r_0 \sin(\theta_0 - \theta_2) = 0.
\]

4.5 POLYGONS

Given \( k \geq 3 \) points \( A_1, \ldots, A_k \) in the plane, in a certain order, we obtain a \( k \)-sided polygon or \( k \)-gon by connecting each point to the next, and the last to the first, with a line segment. The points \( A_i \) are the vertices and the segments \( A_iA_{i+1} \) are the sides or edges of the polygon. When \( k = 3 \) we have a triangle, when \( k = 4 \) we have a quadrangle or quadrilateral, and so on (see page 276). Here we will assume that all polygons are simple: this means that no consecutive edges are on the same line and no two edges intersect (except that consecutive edges intersect at the common vertex) (see Figure 4.5.9).
When we refer to the angle at a vertex \( A_k \) we have in mind the interior angle (as marked in the leftmost polygon in Figure 4.5.9). We denote this angle by the same symbol as the vertex. The complement of \( A_k \) is the exterior angle at that vertex; geometrically, it is the angle between one side and the extension of the adjacent side. In any \( k \)-gon, the sum of the angles equals \( 2(k - 2) \) right angles, or \( 2(k - 2) \times 90^\circ \); for example, the sum of the angles of a triangle is \( 180^\circ \).

The area of a polygon whose vertices \( A_i \) have coordinates \((x_i, y_i)\), for \( 1 \leq i \leq k \), is the absolute value of

\[
\text{area} = \frac{1}{2} (x_1 y_2 - x_2 y_1) + \cdots + \frac{1}{2} (x_k y_1 - x_1 y_k),
\]

where in the summation we take \( x_{k+1} = x_1 \) and \( y_{k+1} = y_1 \). In particular, for a triangle we have

\[
\text{area} = \frac{1}{2} (x_1 y_2 - x_2 y_1 + x_2 y_3 - x_3 y_2 + x_3 y_1 - x_1 y_3) = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.
\]

In oblique coordinates with angle \( \omega \) between the axes, the area is as given above, multiplied by \( \sin \omega \).

If the vertices have polar coordinates \((r_i, \theta_i)\), for \( 1 \leq i \leq k \), the area is the absolute value of

\[
\text{area} = \frac{1}{2} \sum_{i=1}^{k} r_i r_{i+1} \sin(\theta_{i+1} - \theta_i),
\]

where we take \( r_{k+1} = r_1 \) and \( \theta_{k+1} = \theta_1 \).

Formulas for specific polygons in terms of side lengths, angles, etc., are given below.
4.5.1 TRIANGLES

Because the angles of a triangle add up to 180°, at least two of them must be acute (less than 90°). In an acute triangle all angles are acute. A right triangle has one right angle, and an obtuse triangle has one obtuse angle.

The altitude corresponding to a side is the perpendicular dropped to the line containing that side from the opposite vertex. The bisector of a vertex is the line that divides the angle at that vertex into two equal parts. The median is the segment joining a vertex to the midpoint of the opposite side. See Figure 4.5.10.

FIGURE 4.5.10
Notations for an arbitrary triangle of sides \(a, b, c\) and vertices \(A, B, C\). The altitude corresponding to \(C\) is \(h_c\), the median is \(m_c\), the bisector is \(t_c\). The radius of the circumscribed circle is \(R\), that of the inscribed circle is \(r\).

Every triangle also has an inscribed circle tangent to its sides and interior to the triangle (in other words, any three nonconcurrent lines determine a circle). The center of this circle is the point of intersection of the bisectors. We denote the radius of the inscribed circle by \(r\).

Every triangle has a circumscribed circle going through its vertices; in other words, any three noncollinear points determine a circle. The point of intersection of the medians is the center of mass of the triangle (considered as an area in the plane). We denote the radius of the circumscribed circle by \(R\).

Introduce the following notations for an arbitrary triangle of vertices \(A, B, C\) and sides \(a, b, c\) (see Figure 4.5.10). Let \(h_c, t_c,\) and \(m_c\) be the lengths of the altitude, bisector and median originating in vertex \(C\), let \(r\) and \(R\) be as usual the radii of the inscribed and circumscribed circles, and let \(s\) be the semi-perimeter: \(s = \frac{1}{2}(a+b+c)\).
Then
\[ c^2 = a^2 + b^2 - 2ab \cos C \quad \text{(law of cosines)}, \]
\[ a = b \cos C + c \cos B, \]
\[ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{(law of sines)}, \]
\[ \text{area} = \frac{1}{2}hc = \frac{1}{2}ab \sin C = \frac{c^2 \sin A \sin B}{2 \sin C} = rs = \frac{abc}{4R}, \]
\[ = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{(Heron)}, \]
\[ r = c \sin \left( \frac{1}{2} A \right) \sin \left( \frac{1}{2} B \right) \sec \left( \frac{1}{2} C \right) = \frac{ab \sin C}{2s} = (s-c) \tan \left( \frac{1}{2} C \right), \]
\[ = \left( \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} \right)^{-1}, \]
\[ R = \frac{abc}{2 \sin C} = \frac{4 \text{area}}{c}, \]
\[ h_c = a \sin B = b \sin A = \frac{2 \text{area}}{c}, \]
\[ t_c = \frac{2ab}{a+b} \cos \frac{1}{2}C = \sqrt{ab \left( 1 - \frac{c^2}{(a+b)^2} \right)}, \quad \text{and} \]
\[ m_c = \sqrt{\frac{1}{4}a^2 + \frac{1}{2}b^2 - \frac{1}{4}c^2}. \]

A triangle is **equilateral** if all of its sides have the same length, or, equivalently, if all of its angles are the same (and equal to 60°). It is **isosceles** if two sides are the same, or, equivalently, if two angles are the same. Otherwise it is **scalene**.

For an **equilateral triangle** of side \( a \) we have

\[ \text{area} = \frac{1}{4}a^2 \sqrt{3}, \quad r = \frac{1}{2}a \sqrt{3}, \quad R = \frac{1}{2}a \sqrt{3}, \quad h = \frac{1}{2}a \sqrt{3}, \]

where \( h \) is any altitude. The altitude, the bisector, and the median for each vertex coincide.

For an **isosceles triangle**, the altitude for the unequal side is also the corresponding bisector and median, but this is not true for the other two altitudes. Many formulas for an isosceles triangle of sides \( a, a, c \) can be immediately derived from those for a right triangle of legs \( a, \frac{1}{2}c \) (see Figure 4.5.11, left).

For a **right triangle**, the **hypotenuse** is the longest side opposite the right angle; the **legs** are the two shorter sides adjacent to the right angle. The altitude for each leg equals the other leg. In Figure 4.5.11 (right), \( h \) denotes the altitude for the hypotenuse, while \( m \) and \( n \) denote the segments into which this altitude divides the hypotenuse.

The following formulas apply for a right triangle:

\[ A + B = 90^\circ, \]
\[ c^2 = a^2 + b^2 \quad \text{(Pythagoras)}, \]
\[ r = \frac{ab}{a+b+c}, \quad R = \frac{1}{2}c, \]
\[ a = c \sin A = c \cos B, \quad b = c \sin B = c \cos A, \]
\[ mc = b^2, \quad nc = a^2, \]
\[ \text{area} = \frac{1}{2}ab, \quad hc = ab. \]
The hypotenuse is a diameter of the circumscribed circle. The median joining the midpoint of the hypotenuse (the center of the circumscribed circle) to the right angle makes angles $2A$ and $2B$ with the hypotenuse.

Additional facts about triangles:

1. In any triangle, the longest side is opposite the largest angle, and the shortest side is opposite the smallest angle. This follows from the law of sines.

2. **Ceva’s theorem** (see Figure 4.5.12, left): In a triangle $ABC$, let $D$, $E$, and $F$ be points on the lines $BC$, $CA$, and $AB$, respectively. Then the lines $AD$, $BE$, and $CF$ are concurrent if, and only if, the signed distances $BD$, $CE$, ... satisfy

   \[ BD \cdot CE \cdot AF = DC \cdot EA \cdot FB. \]

   This is so in three important particular cases: when the three lines are the medians, when they are the bisectors, and when they are the altitudes.

3. **Menelaus’s theorem** (see Figure 4.5.12, right): In a triangle $ABC$, let $D$, $E$, and $F$ be points on the lines $BC$, $CA$, and $AB$, respectively. Then $D$, $E$, and $F$ are collinear if, and only if, the signed distances $BD$, $CE$, ... satisfy

   \[ BD \cdot CE \cdot AF = -DC \cdot EA \cdot FB. \]
4. Each side of a triangle is less than the sum of the other two. For any three lengths such that each is less than the sum of the other two, there is a triangle with these side lengths.

### 4.5.2 QUADRILATERALS

The following formulas give the area of a general quadrilateral (see Figure 4.5.13, left, for the notation).

\[
\text{area} = \frac{1}{2} pq \sin \theta = \frac{1}{4} (b^2 + d^2 - a^2 - c^2) \tan \theta \\
= \frac{1}{4} \sqrt{4p^2 q^2 - (b^2 + d^2 - a^2 - c^2)^2} \\
= \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd \cos \frac{1}{2}(A+C)}.
\]

(4.5.3)

**FIGURE 4.5.13**

*Left: notation for a general quadrilateral; in addition \( s = \frac{1}{2}(a + b + c + d) \). Right: a parallelogram.*

Often, however, it is easiest to compute the area by dividing the quadrilateral into triangles. One can also divide the quadrilateral into triangles to compute one side given the other sides and angles, etc.

More formulas can be given for special cases of quadrilaterals. In a parallelogram, opposite sides are parallel and the diagonals intersect in the middle (Figure 4.5.13, right). It follows that opposite sides have the same length and that two consecutive angles add up to 180°. In the notation of the figure, we have

\[
A = C, \quad B = D, \quad A + B = 180^\circ, \\
h = a \sin A = a \sin B, \quad \text{area} = bh, \\
p = \sqrt{a^2 + b^2 - 2ab \cos A}, \quad q = \sqrt{a^2 + b^2 - 2ab \cos B}.
\]

(All this follows from the triangle formulas applied to the triangles \( ABD \) and \( ABC \).)

Two particular cases of parallelograms are

1. The rectangle \( \square \), where all angles equal 90°. The diagonals of a rectangle have the same length. The general formulas for parallelograms reduce to

\[
h = a, \quad \text{area} = ab, \quad \text{and} \quad p = q = \sqrt{a^2 + b^2}.
\]
2. The rhombus or diamond ♦, where adjacent sides have the same length \((a = b)\). The diagonals of a rhombus are perpendicular. In addition to the general formulas for parallelograms, we have area \(= \frac{1}{2}pq\) and \(p^2 + q^2 = 4a^2\).

The square or regular quadrilateral is both a rectangle and a rhombus. See page 276.

A quadrilateral is a trapezoid if two sides are parallel. In the notation of the figure on the right we have

\[ A + D = B + C = 180^\circ, \quad \text{area} = \frac{1}{2} (AB + CD)h. \]

The diagonals of a quadrilateral with consecutive sides \(a, b, c, d\) are perpendicular if and only if \(a^2 + c^2 = b^2 + d^2\).

A quadrilateral is cyclic if it can be inscribed in a circle, that is, if its four vertices belong to a single, circumscribed, circle. This is possible if and only if the sum of opposite angles is \(180^\circ\). If \(R\) is the radius of the circumscribed circle, we have (in the notation of Figure 4.5.13, left)

\[
\begin{align*}
\text{area} &= \sqrt{(s-a)(s-b)(s-c)(s-d)} = \frac{1}{2}(ac + bd) \sin \theta, \\
&= \frac{\sqrt{(ac + bd)(ad + bc)(ab + cd)}}{4R} \quad \text{(Brahmagupta)}, \\
p &= \sqrt{\frac{(ac + bd)(ab + cd)}{ad + bc}}, \\
R &= \frac{1}{4} \sqrt[4]{\frac{(ac + bd)(ad + bc)(ab + cd)}{(s-a)(s-b)(s-c)(s-d)}}, \\
\sin \theta &= \frac{2 \text{area}}{ac + bd}, \\
pq &= ac + bd \quad \text{(Ptolemy)}. 
\end{align*}
\]

A quadrilateral is circumscribable if it has an inscribed circle (that is, a circle tangent to all four sides). Its area is \(rs\), where \(r\) is the radius of the inscribed circle and \(s\) is as above.

For a quadrilateral that is both cyclic and circumscribable, we have the following additional equalities, where \(m\) is the distance between the centers of the inscribed and circumscribed circles:

\[
\begin{align*}
a + c &= b + d, \\
R &= \frac{1}{4} \sqrt[4]{\frac{(ac + bd)(ad + bc)(ab + cd)}{abcd}}, \\
\text{area} &= \sqrt{abcd} = rs, \\
\frac{1}{r^2} &= \frac{1}{(R-m)^2} + \frac{1}{(R+m)^2}.
\end{align*}
\]
4.5.3 REGULAR POLYGONS

A polygon is regular if all of its sides are equal and all its angles are equal. Either condition implies the other in the case of a triangle, but not in general. (A rhombus has equal sides but not necessarily equal angles, and a rectangle has equal angles but not necessarily equal sides).

For a k-sided regular polygon of side \( a \), let \( \theta \) be the angle at any vertex, and \( r \) and \( R \) the radii of the inscribed and circumscribed circles (\( r \) is called the apothem). As usual, let \( s = \frac{1}{2} ka \) be the half-perimeter. Then

\[
\theta = \left( \frac{k-2}{k} \right) 180^\circ, \\
a = 2r \tan \frac{180^\circ}{k} = 2R \sin \frac{180^\circ}{k}, \\
\text{area} = \frac{1}{4} ks^2 \cot \frac{180^\circ}{k} = kr^2 \tan \frac{180^\circ}{k} \\
= \frac{1}{2} kR^2 \sin \frac{360^\circ}{k}, \\
r = \frac{1}{2} s \cot \frac{180^\circ}{k}, \\
R = \frac{1}{2} s \csc \frac{180^\circ}{k}.
\]

<table>
<thead>
<tr>
<th>Name</th>
<th>( k )</th>
<th>( \text{Area} )</th>
<th>( r )</th>
<th>( R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equilateral triangle</td>
<td>3</td>
<td>( 0.43301 a^2 )</td>
<td>0.28868 ( a )</td>
<td>0.57735 ( a )</td>
</tr>
<tr>
<td>Square</td>
<td>4</td>
<td>( a^2 )</td>
<td>0.50000 ( a )</td>
<td>0.70711 ( a )</td>
</tr>
<tr>
<td>Regular pentagon</td>
<td>5</td>
<td>( 1.72048 a^2 )</td>
<td>0.68819 ( a )</td>
<td>0.85065 ( a )</td>
</tr>
<tr>
<td>Regular hexagon</td>
<td>6</td>
<td>( 2.59808 a^2 )</td>
<td>0.86603 ( a )</td>
<td>( a )</td>
</tr>
<tr>
<td>Regular heptagon</td>
<td>7</td>
<td>( 3.63391 a^2 )</td>
<td>1.03826 ( a )</td>
<td>1.15238 ( a )</td>
</tr>
<tr>
<td>Regular octagon</td>
<td>8</td>
<td>( 4.82843 a^2 )</td>
<td>1.20711 ( a )</td>
<td>1.30656 ( a )</td>
</tr>
<tr>
<td>Regular nonagon</td>
<td>9</td>
<td>( 6.18182 a^2 )</td>
<td>1.37374 ( a )</td>
<td>1.46190 ( a )</td>
</tr>
<tr>
<td>Regular decagon</td>
<td>10</td>
<td>( 7.69421 a^2 )</td>
<td>1.53884 ( a )</td>
<td>1.61803 ( a )</td>
</tr>
<tr>
<td>Regular undecagon</td>
<td>11</td>
<td>( 9.36564 a^2 )</td>
<td>1.70284 ( a )</td>
<td>1.77473 ( a )</td>
</tr>
<tr>
<td>Regular dodecagon</td>
<td>12</td>
<td>( 11.19625 a^2 )</td>
<td>1.86603 ( a )</td>
<td>1.93185 ( a )</td>
</tr>
</tbody>
</table>

If \( a_k \) denotes the side of a \( k \)-sided regular polygon inscribed in a circle of radius \( R \), we have

\[
a_{2k} = \sqrt{2R^2 - R\sqrt{4R^2 - a_k^2}}.
\]

If \( A_k \) denotes the side of a \( k \)-sided regular polygon circumscribed about the same circle,

\[
A_{2k} = \frac{2RA_k}{2R + \sqrt{4R^2 + A_k^2}}.
\]
In particular,
\[
A_{2k} = \frac{a_k A_k}{a_k + A_k}, \quad a_{2k} = \sqrt{\frac{a_k A_{2k}}{2}}.
\]

The areas \(s_k, s_{2k}, S_k\) and \(S_{2k}\) of the same polygons satisfy
\[
s_{2k} = \sqrt{s_k S_k}, \quad S_{2k} = \frac{2s_{2k} S_k}{s_{2k} + S_k}.
\]

## 4.6 CIRCLES

The set of points whose distance to a fixed point (the center) is a fixed positive number (the radius) is a circle. A circle of radius \(r\) and center \((x_0, y_0)\) is described by the equation
\[
(x - x_0)^2 + (y - y_0)^2 = r^2.
\]
or
\[
x^2 + y^2 - 2x x_0 - 2y y_0 + x_0^2 + y_0^2 - r^2 = 0.
\]

Conversely, an equation of the form
\[
x^2 + y^2 + 2dx + 2ey + f = 0
\]
defines a circle if \(d^2 + e^2 > f\); the center is \((-d, -e)\) and the radius is \(\sqrt{d^2 + e^2 - f}\).

Three points not on the same line determine a unique circle. If the points have coordinates \((x_1, y_1)\), \((x_2, y_2)\) and \((x_3, y_3)\), then the equation of the circle is
\[
\begin{vmatrix}
    x^2 + y^2 & x & y & 1 \\
    x_1^2 + y_1^2 & x_1 & y_1 & 1 \\
    x_2^2 + y_2^2 & x_2 & y_2 & 1 \\
    x_3^2 + y_3^2 & x_3 & y_3 & 1
\end{vmatrix} = 0.
\]

A chord of a circle is a line segment between two points. (Figure 4.6.14). A diameter is a chord that goes through the center, or the length of such a chord (therefore the diameter is twice the radius). Given two points \(P_1 = (x_1, y_1)\) and \(P_2 = (x_2, y_2)\), there is a unique circle whose diameter is \(P_1 P_2\); its equation is
\[
(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.
\]

The length or circumference of a circle of radius \(r\) is \(2\pi r\), and the area is \(\pi r^2\).

The length of the arc of circle subtended by an angle \(\theta\), shown as \(s\) in Figure 4.6.14, is \(r\theta\). Other relations between the radius, the arc length, the chord, and the areas of
the corresponding *sector* and *segment* are, in the notation of Figure 4.6.14,

\[
d = R \cos \frac{1}{2} \theta = \frac{1}{2} c \cot \frac{1}{2} \theta = \frac{1}{2} \sqrt{4R^2 - c^2},
\]

\[
c = 2R \sin \frac{1}{2} \theta = 2d \tan \frac{1}{2} \theta = 2\sqrt{R^2 - d^2} = \sqrt{4h(2R - h)},
\]

\[
\theta = \frac{s}{R} = 2 \cos^{-1} \frac{d}{R} = 2 \tan^{-1} \frac{c}{2d} = 2 \sin^{-1} \frac{c}{2R},
\]

area of sector = \(\frac{1}{2} RS = \frac{1}{2} R^2 \theta\),

area of segment = \(\frac{1}{2} R^2 (\theta - \sin \theta) = \frac{1}{2} (RS - cd) = R^2 \cos^{-1} \frac{d}{R} - d\sqrt{R^2 - d^2},\)

\[
= R^2 \cos^{-1} \frac{R - h}{R} - (R - h)\sqrt{2Rh - h^2}.
\]

**FIGURE 4.6.14**

The arc of a circle subtended by the angle \(\theta\) is \(s\); the chord is \(c\); the sector is the whole slice of the pie; the segment is the cap bounded by the arc and the chord (that is, the slice minus the triangle).

![Diagram of a circle with arc, chord, sector, and segment labeled](image)

**Other properties of circles:**

1. If the central angle \(AOB\) equals \(\theta\), the angle \(ACB\), where \(C\) is any point on the circle, equals \(\frac{1}{2} \theta\) or \(180^\circ - \frac{1}{2} \theta\) (Figure 4.6.15, left). Conversely, given a segment \(AB\), the set of points that “see” \(AB\) under a fixed angle is an arc of a circle (Figure 4.6.15, right). In particular, the set of points that see \(AB\) under a right angle is a circle with diameter \(AB\).

2. Let \(P_1, P_2, P_3, P_4\) be points in the plane, and let \(d_{ij}\), for \(1 \leq i, j \leq 4\), be the distance between \(P_i\) and \(P_j\). A necessary and sufficient condition for all of the points to lie on the same circle (or line) is that one of the following equalities be satisfied:

\[
\pm d_{12}d_{34} \pm d_{13}d_{24} \pm d_{14}d_{23} = 0.
\]

This is equivalent to Ptolemy’s formula for cyclic quadrilaterals (page 275).

3. In oblique coordinates with angle \(\omega\), a circle of center \((x_0, y_0)\) and radius \(r\) is described by the equation

\[
(x - x_0)^2 + (y - y_0)^2 + 2(x - x_0)(y - y_0) \cos \omega = r^2.
\]

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4. In polar coordinates, the equation for a circle centered at the pole and having radius $a$ is $r = a$. The equation for a circle of radius $a$ passing through the pole and with center at the point $(r, \theta) = (a, \theta_0)$ is $r = 2a \cos(\theta - \theta_0)$. The equation for a circle of radius $a$ and with center at the point $(r, \theta) = (r_0, \theta_0)$ is

$$r^2 - 2r_0r \cos(\theta - \theta_0) + r_0^2 - a^2 = 0.$$ 

5. If a line intersects a circle of center $O$ at points $A$ and $B$, the segments $OA$ and $OB$ make equal angles with the line. In particular, a tangent line is perpendicular to the radius that goes through the point of tangency.

6. Fix a circle and a point $P$ in the plane, and consider a line through $P$ that intersects the circle at $A$ and $B$ (with $A = B$ for a tangent). Then the product of the distances $PA \cdot PB$ is the same for all such lines. It is called the power of $P$ with respect to the circle.

4.7 CONICS

A conic (or conic section) is a plane curve that can be obtained by intersecting a right circular cone (page 312) with a plane that does not go through the vertex of the cone. There are three possibilities, depending on the relative positions of the cone and the plane (Figure 4.7.16). If no line of the cone is parallel to the plane, then the intersection is a closed curve, called an ellipse. If one line of the cone is parallel to the plane, the intersection is an open curve whose two ends are asymptotically parallel; this is called a parabola. Finally, there may be two lines in the cone parallel to the plane; the curve in this case has two open segments, and is called a hyperbola.
4.7.1 ALTERNATIVE CHARACTERIZATION

Assume given a point $F$ in the plane, a line $d$ not going through $F$, and a positive real number $e$. The set of points $P$ such that the distance $PF$ is $e$ times the distance from $P$ to $d$ (measured along a perpendicular) is a conic. We call $F$ the focus, $d$ the directrix, and $e$ the eccentricity of the conic. If $e < 1$ we have an ellipse, if $e = 1$ a parabola, and if $e > 1$ a hyperbola (Figure 4.7.17). This construction gives all conics except the circle, which is a particular case of the ellipse according to the earlier definition (we can recover it by taking the limit $e \to 0$).

For any conic, a line perpendicular to $d$ and passing through $F$ is an axis of symmetry. The ellipse and the hyperbola have an additional axis of symmetry, perpendicular to the first, so that there is an alternate focus and directrix, $F'$ and $d'$, obtained as the reflection of $F$ and $d$ with respect to this axis. (By contrast, the focus and directrix are uniquely defined for a parabola.)

The simplest analytic form for the ellipse and hyperbola is obtained when the two symmetry axes coincide with the coordinate axes. The ellipse in Figure 4.7.18
has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad (4.7.1)$$

with $b < a$. The $x$-axis is the major axis, and the $y$-axis is the minor axis. These names are also applied to the segments, determined on the axes by the ellipse, and to the lengths of these segments: $2a$ for the major axis and $2b$ for the minor. The vertices are the intersections of the major axis with the ellipse and have coordinates $(a, 0)$ and $(-a, 0)$. The distance from the center to either focus is $\sqrt{a^2 - b^2}$, and the sum of the distances from a point in the ellipse to the foci is $2a$. The *latus rectum* (in the singular, *latus rectum*) are the chords perpendicular to the major axis and going through the foci; their length is $2b^2/a$. The eccentricity is $\sqrt{a^2 - b^2}/a$. All ellipses of the same eccentricity are similar; in other words, the shape of an ellipse depends only on the ratio $b/a$. The distance from the center to either directrix is $a^2/\sqrt{a^2 - b^2}$.

**FIGURE 4.7.18**

Ellipse with major semiaxis $a$ and minor semiaxis $b$. Here $b/a = 0.6$.

The hyperbola in Figure 4.7.19 has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \quad (4.7.2)$$

The $x$-axis is the transverse axis, and the $y$-axis is the conjugate axis. The vertices are the intersections of the transverse axis with the ellipse and have coordinates $(a, 0)$ and $(-a, 0)$. The segment thus determined, or its length $2a$, is also called the transverse axis, while the length $2b$ is also called the conjugate axis. The distance from the center to either focus is $\sqrt{a^2 + b^2}$, and the difference between the distances from a point in the hyperbola to the foci is $2a$. The *latus recta* are the chords perpendicular to the transverse axis and going through the foci; their length is $2b^2/a$. The eccentricity is $\sqrt{a^2 + b^2}/a$. The distance from the center to either directrix is $a^2/\sqrt{a^2 + b^2}$. The
FIGURE 4.7.19
Hyperbola with transverse semiaxis \( a \) and conjugate semiaxis \( b \). Here \( b/a = 0.4 \).

legs of the hyperbola approach the asymptotes, lines of slope \( \pm b/a \) that cross at the center.

All hyperbolas of the same eccentricity are similar; in other words, the shape of a hyperbola depends only on the ratio \( b/a \). Unlike the case of the ellipse (where the major axis, containing the foci, is always longer than the minor axis), the two axes of a hyperbola can have arbitrary lengths. When they have the same length, so that \( a = b \), the asymptotes are perpendicular, and \( e = \sqrt{2} \), the hyperbola is called rectangular.

The simplest analytic form for the parabola is obtained when the axis of symmetry coincides with one coordinate axis, and the vertex (the intersection of the axis with the curve) is at the origin. The equation of the parabola on the right is

\[
y^2 = 4ax,
\]

where \( a \) is the distance from the vertex to the focus, or, which is the same, from the vertex to the directrix. The latus rectum is the chord perpendicular to the axis and going through the focus; its length is \( 4a \). All parabolas are similar: they can be made to look identical by scaling, translation, and rotation.

4.7.2 THE GENERAL QUADRATIC EQUATION

The analytic equation for a conic in arbitrary position is the following:

\[
Ax^2 + By^2 + Cxy + Dx + Ey + F = 0,
\]

where at least one of \( A, B, C \) is nonzero. To reduce this to one of the forms given previously, perform the following steps (note that the decisions are based on the most recent values of the coefficients, taken after all the transformations so far):
1. If $C \neq 0$, simultaneously perform the substitutions $x \mapsto qx + y$ and $y \mapsto qy - x$, where
\[ q = \sqrt{\left( \frac{B - A}{C} \right)^2 + 1 + \frac{B - A}{C}}. \] (4.7.5)

Now $C = 0$. (This step corresponds to rotating and scaling about the origin.)

2. If $B = 0$, interchange $x$ and $y$. Now $B \neq 0$.

3. If $E \neq 0$, perform the substitution $y \mapsto y - \frac{1}{2}(E/B)$. (This corresponds to translating in the $y$ direction.) Now $E = 0$.

4. If $A = 0$:
   
   (a) If $D \neq 0$, perform the substitution $x \mapsto x - \frac{1}{2}(D/A)$ (translation in the $x$ direction), and divide the equation by $B$ to get Equation (4.7.3). The conic is a parabola.
   
   (b) If $D = 0$, the equation gives a degenerate conic. If $F = 0$, we have the line $y = 0$ with multiplicity two. If $F < 0$, we have two parallel lines $y = \pm \sqrt{-B/A}x$; if $F > 0$ we have two imaginary lines; the equation has no solution within the real numbers.

5. If $A \neq 0$:
   
   (a) If $D \neq 0$, perform the substitution $x \mapsto x - \frac{1}{2}(D/A)$. Now $D = 0$. (This corresponds to translating in the $x$ direction.)
   
   (b) If $F \neq 0$, divide the equation by $F$ to get a form with $F = 1$.
      
      i. If $A$ and $B$ have opposite signs, the conic is a hyperbola; to get to Equation (4.7.2), interchange $x$ and $y$, if necessary, so that $A$ is positive; then make $a = 1/\sqrt{A}$ and $b = 1/\sqrt{B}$.
      
      ii. If $A$ and $B$ are both positive, the conic is an ellipse; to get to Equation (4.7.1), interchange $x$ and $y$, if necessary, so that $A \leq B$, then make $a = 1/\sqrt{A}$ and $b = 1/\sqrt{B}$. The circle is the particular case $a = b$.
      
      iii. If $A$ and $B$ are both negative, we have an imaginary ellipse; the equation has no solution in real numbers.
   
   (c) If $F = 0$, the equation again represents a degenerate conic: when $A$ and $B$ have different signs, we have a pair of lines $y = \pm \sqrt{-B/A}x$, and, when they have the same sign, we get a point (the origin).

We work out an example for clarity. Suppose the original equation is
\[ 4x^2 + y^2 - 4xy + 3x - 4y + 1 = 0. \] (4.7.6)

In step 1 we apply the substitutions $x \mapsto 2x + y$ and $y \mapsto 2y - x$. This gives $25x^2 + 10x - 5y + 1 = 0$. Next we interchange $x$ and $y$ (step 2) and get $25y^2 + 10y - 5x + 1 = 0$. Replacing $y$ by $y - \frac{1}{3}$ in step 3, we get $25y^2 - 5x = 0$. Finally, in step 4a we divide the equation by 25, thus giving it the form of Equation (4.7.3) with $a = \frac{1}{25}$. We have reduced the conic to a parabola with vertex at the origin and focus at
To locate the features of the original curve, we work our way back along the chain of substitutions (recall the convention about substitutions and transformations from Section 4.1.1):

<table>
<thead>
<tr>
<th>Substitution</th>
<th>$y \mapsto y - \frac{1}{5}$</th>
<th>$x \mapsto y$</th>
<th>$y \mapsto x$</th>
<th>$x \mapsto 2x + y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vertex</td>
<td>$(0, 0)$</td>
<td>$(0, -\frac{1}{5})$</td>
<td>$(-\frac{1}{5}, 0)$</td>
<td>$(-\frac{7}{5}, -\frac{1}{5})$</td>
</tr>
<tr>
<td>Focus</td>
<td>$(\frac{1}{20}, 0)$</td>
<td>$(\frac{1}{20}, -\frac{1}{5})$</td>
<td>$(-\frac{1}{5}, \frac{1}{20})$</td>
<td>$(-\frac{7}{20}, \frac{6}{20})$</td>
</tr>
</tbody>
</table>

We conclude that the original curve, Equation (4.7.6), is a parabola with vertex $(-\frac{2}{5}, -\frac{1}{5})$ and focus $(-\frac{7}{20}, \frac{6}{20})$.

If one just wants to know the type of the conic defined by Equation (4.7.4), an alternative analysis consists in forming the quantities

$$
\Delta = \begin{vmatrix}
A & \frac{1}{2}C & \frac{1}{2}D \\
\frac{1}{2}C & B & \frac{1}{2}E \\
\frac{1}{2}D & \frac{1}{2}E & F
\end{vmatrix}, \quad J = \begin{vmatrix}
A & \frac{1}{2}C \\
\frac{1}{2}C & B
\end{vmatrix}, \quad I = A + B,
$$

$$
K = \begin{vmatrix}
A & \frac{1}{2}D \\
\frac{1}{2}D & F
\end{vmatrix} + \begin{vmatrix}
B & \frac{1}{2}E \\
\frac{1}{2}E & F
\end{vmatrix},
$$

and finding the appropriate case in the following table, where an entry in parentheses indicates that the equation has no solution in real numbers:

<table>
<thead>
<tr>
<th>$\Delta$</th>
<th>$J$</th>
<th>$\Delta / I$</th>
<th>$K$</th>
<th>Type of conic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\neq 0$</td>
<td>$&lt; 0$</td>
<td></td>
<td></td>
<td>Hyperbola</td>
</tr>
<tr>
<td>$\neq 0$</td>
<td>$0$</td>
<td></td>
<td></td>
<td>Parabola</td>
</tr>
<tr>
<td>$\neq 0$</td>
<td>$&gt; 0$</td>
<td>$&lt; 0$</td>
<td></td>
<td>Ellipse</td>
</tr>
<tr>
<td>$\neq 0$</td>
<td>$&gt; 0$</td>
<td>$&gt; 0$</td>
<td></td>
<td>(Imaginary ellipse)</td>
</tr>
<tr>
<td>$0$</td>
<td>$&lt; 0$</td>
<td></td>
<td></td>
<td>Intersecting lines</td>
</tr>
<tr>
<td>$0$</td>
<td>$&gt; 0$</td>
<td></td>
<td></td>
<td>Point</td>
</tr>
<tr>
<td>$0$</td>
<td>$0$</td>
<td>$&lt; 0$</td>
<td></td>
<td>Distinct parallel lines</td>
</tr>
<tr>
<td>$0$</td>
<td>$0$</td>
<td>$&gt; 0$</td>
<td></td>
<td>(Imaginary parallel lines)</td>
</tr>
<tr>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td></td>
<td>Coincident lines</td>
</tr>
</tbody>
</table>

For the central conics (the ellipse, the point, and the hyperbola), the center $(x_0, y_0)$ is the solution of the system of equations

$$2Ax + Cy + D = 0,$$
$$Cx + 2By + E = 0,$$

and the axes have slope $q$ and $-1/q$, where $q$ is given by Equation (4.7.5).
4.7.3 ADDITIONAL PROPERTIES OF ELLIPSES

Let $C$ be the ellipse with equation $x^2/a^2 + y^2/b^2 = 1$, with $a > b$, and let $F, F' = (\pm \sqrt{a^2 - b^2}, 0)$ be its foci (see Figure 4.7.18).

1. A parametric representation for $C$ is given by $(a \cos \theta, b \sin \theta)$. The area of the shaded sector on the right is $\frac{1}{2}ab\theta = \frac{1}{2}ab \cos^{-1}(x/a)$. The length of the arc from $(a, 0)$ to the point $(a \cos \theta, b \sin \theta)$ is given by the elliptic integral

$$a \int_0^\theta \sqrt{1 - e^2 \cos^2 \phi} \, d\phi = a \, E(\pi/2 - \theta, e),$$

where $e$ is the eccentricity. (See page 523 for elliptic integrals.) Setting $\theta = 2\pi$ results in

area $C = \pi ab$,  \hspace{5mm} \text{perimeter } C = 4a \, E(0, e).$

2. A rational parametric representation for $C$ is given by $\left( a \frac{1 - t^2}{1 + t^2}, \frac{2bt}{1 + t^2} \right)$.

3. The polar equation for $C$ in the usual polar coordinate system is

$$r = \frac{ab}{\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}}.$$

With respect to a coordinate system with origin at a focus, the equation is

$$r = \frac{l}{1 \pm e \cos \theta},$$

where $l = b^2/a$ is half the latus rectum. (Use the $+$ sign for the focus with positive $x$-coordinate and the $-$ sign for the focus with negative $x$-coordinate.)

4. Let $P$ be any point of $C$. The sum of the distances $PF$ and $PF'$ is constant and equal to $2a$.

5. Let $P$ be any point of $C$. Then the rays $PF$ and $PF'$ make the same angle with the tangent to $C$ at $P$. Thus any light ray originating at $F$ and reflected in the ellipse will also go through $F'$.

6. Let $T$ be any line tangent to $C$. The product of the distances from $F$ and $F'$ to $T$ is constant and equals $b^2$.

7. Lahire’s theorem: Let $D$ and $D'$ be fixed lines in the plane, and consider a third moving line on which three points $P$, $P'$ and $P''$ are marked. If we constrain $P$ to lie in $D$ and $P'$ to lie in $D'$, then $P''$ describes an ellipse.
4.7.4 ADDITIONAL PROPERTIES OF HYPERBOLAS

Let $C$ be the hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, and let

$$F, F' = (\pm \sqrt{a^2 + b^2}, 0)$$

be its foci (see Figure 4.7.19). The conjugate hyperbola of $C$ is the hyperbola $C'$ with equation $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. It has the same asymptotes as $C$, the same axes (transverse and conjugate axes being interchanged), and its eccentricity $e'$ is related to that of $C$ by $e'^2 + e^2 = 1$.

1. A parametric representation for $C$ is given by $(a \sec \theta, b \tan \theta)$. A different parametric representation, which gives one branch only, is $(a \cosh \theta, b \sinh \theta)$. The area of the shaded sector on the right is

$$\frac{1}{2}ab\theta = \frac{1}{2}ab \cosh^{-1}\left(\frac{x}{a}\right) = \frac{1}{2}ab \log \frac{x + \sqrt{x^2 - a^2}}{a}.$$  

The length of the arc from $(a, 0)$ to the point $(a \cosh \theta, b \sinh \theta)$ is given by the elliptic integral

$$a \int_0^\theta \sqrt{e^2 \cosh^2 \phi - 1} \, d\phi = -bi \, E \left(\theta i, \frac{ea}{b}\right) = a \int_1^x \sqrt{\frac{e^2 \xi^2 - a^2}{\xi^2 - a^2}} \, d\xi,$$

where $e$ is the eccentricity, $i = \sqrt{-1}$, and $x = a \cosh \theta$.

2. A rational parametric representation for $C$ is given by

$$\left(a \frac{1 + t^2}{1 - t^2}, \frac{2bt}{1 - t^2}\right).$$

3. The polar equation for $C$ in the usual polar coordinate system is

$$r = \frac{ab}{\sqrt{a^2 \sin^2 \theta - b^2 \cos^2 \theta}}.$$  

With respect to a system with origin at a focus, the equation is

$$r = \frac{1}{1 \pm e \cos \theta},$$

where $l = b^2/a$ is half the latus rectum. (Use the $-$ sign for the focus with positive $x$-coordinate and the $+$ sign for the focus with negative $x$-coordinate.)

4. Let $P$ be any point of $C$. The unsigned difference between the distances $PF$ and $PF'$ is constant and equal to $2a$.  

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5. Let \( P \) be any point of \( C \). Then the rays \( PF \) and \( PF' \) make the same angle with the tangent to \( C \) at \( P \). Thus any light ray originating at \( F \) and reflected in the hyperbola will appear to emanate from \( F' \).

6. Let \( T \) be any line tangent to \( C \). The product of the distances from \( F \) and \( F' \) to \( T \) is constant and equals \( b^2 \).

7. Let \( P \) be any point of \( C \). The area of the parallelogram formed by the asymptotes and the parallels to the asymptotes going through \( P \) is constant and equals \( \frac{1}{2}ab \).

8. Let \( L \) be any line in the plane. If \( L \) intersects \( C \) at \( P \) and \( P' \) and intersects the asymptotes at \( Q \) and \( Q' \), the distances \( PQ \) and \( P'Q' \) are the same. If \( L \) is tangent to \( C \) we have \( P = P' \), so that the point of tangency bisects the segment \( QQ' \).

### 4.7.5 ADDITIONAL PROPERTIES OF PARABOLAS

Let \( C \) be the parabola with equation \( y^2 = 4ax \), and let \( F = (a, 0) \) be its focus.

1. Let \( P = (x, y) \) and \( P' = (x', y') \) be points on \( C \). The area bounded by the chord \( PP' \) and the corresponding arc of the parabola is

\[
\frac{|y' - y|^3}{24a}.
\]

It equals four-thirds of the area of the triangle \( PP'Q \), where \( Q \) is the point on \( C \) whose tangent is parallel to the chord \( PP' \) (formula due to Archimedes).

2. The length of the arc from \((0, 0)\) to the point \((x, y)\) is

\[
\frac{y}{4} \sqrt{4 + \frac{y^2}{a^2}} + a \sinh^{-1}\left(\frac{y}{2a}\right) = \frac{y}{4} \sqrt{4 + \frac{y^2}{a^2}} + a \log \left(\frac{y + \sqrt{y^2 + 4a^2}}{2a}\right).
\]

3. The polar equation for \( C \) in the usual polar coordinate system is

\[
r = \frac{4a \cos \theta}{\sin^2 \theta}.
\]

With respect to a coordinate system with origin at \( F \), the equation is

\[
r = \frac{l}{1 - \cos \theta},
\]

where \( l = 2a \) is half the latus rectum.

4. Let \( P \) be any point of \( C \). Then the ray \( PF \) and the horizontal line through \( P \) make the same angle with the tangent to \( C \) at \( P \). Thus light rays parallel to the axis and reflected in the parabola converge onto \( F \) (principle of the parabolic reflector).

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4.8 SPECIAL PLANE CURVES

4.8.1 ALGEBRAIC CURVES

Curves that can be given in implicit form as $f(x, y) = 0$, where $f$ is a polynomial, are called algebraic. The degree of $f$ is called the degree or order of the curve. Thus, conics (page 279) are algebraic curves of degree two. Curves of degree three already have a great variety of shapes, and only a few common ones will be given here.

The simplest case is the curve whose graph is of a polynomial of degree three:

$$y = ax^3 + bx^2 + cx + d,$$

with $a \neq 0$. This curve is a (general) cubic parabola (Figure 4.8.20), symmetric with respect to the point $B$ where $x = -b/3a$.

![Figure 4.8.20](image)

The general cubic parabola for $a > 0$. For $a < 0$, reflect in a horizontal line.

The equation of a semicubic parabola (Figure 4.8.21, left) is $y^2 = kx^3$; by proportional scaling one can take $k = 1$. This curve should not be confused with the cissoid of Diocles (Figure 4.8.21, middle), whose equation is $(a - x)y^2 = x^3$ with $a \neq 0$.

The latter is asymptotic to the line $x = a$, whereas the semicubic parabola has no asymptotes. The cissoid’s points are characterized by the equality $OP = AB$ in Figure 4.8.21, right. One can take $a = 1$ by proportional scaling.

More generally, any curve of degree three with equation $(x - x_0)y^2 = f(x)$, where $f$ is a polynomial, is symmetric with respect to the $x$-axis and asymptotic to the line $x = x_0$. In addition to the cissoid, the following particular cases are important:

1. The witch of Agnesi has equation $xy^2 = a^2(a-x)$, with $a \neq 0$, and is characterized by the geometric property shown in Figure 4.8.21, right. The same property provides the parametric representation $x = a \cos^2 \theta$, $y = a \tan \theta$. Once more, proportional scaling reduces to the case $a = 1$.

2. The folium of Descartes (Figure 4.8.22, left) is described by equation $(x - a)y^2 = -x^2(\frac{1}{4}x + a)$, with $a \neq 0$ (reducible to $a = 1$ by proportional scaling). By rotating $135^\circ$ (right) we get the alternative and more familiar equation $x^3 + y^3 = cxy$, where $c = \frac{1}{2} \sqrt{2a}$. The folium of Descartes is a rational curve, this is, it...
is parametrically represented by rational functions. In the tilted position, the equation is \( x = ct/(1 + t^3) \), \( y = ct^2/(1 + t^3) \) (so that \( t = y/x \)).

3. The strophoid’s equation is \( (x - a)y^2 = -x^2(x + a) \), with \( a \neq 0 \) (reducible to \( a = 1 \) by proportional scaling). It satisfies the property \( AP = AP' = OA \) in Figure 4.8.22, right; this means that \( PO P' \) is a right angle. The strophoid’s polar representation is \( r = -a \cos 2\theta \sec \theta \), and the rational parametric representation is \( x = a(t^2 - 1)/(t^2 + 1) \), \( y = at(t^2 - 1)/(t^2 + 1) \) (so that \( t = y/x \)).
the curve has one smooth segment and one with a self-intersection, or two segments depending on whether \( k \) is greater than, equal to, or smaller than \( a \) (Figure 4.8.23). The case \( k = a \) is also known as the lemniscate (of Jakob Bernoulli), the equation reduces to \((x^2 + y^2)^2 = a^2(x^2 - y^2)\), and upon a 45° rotation to \((x^2 + y^2)^2 = 2a^2 xy\). Each Cassini oval is the section of a torus of revolution by a plane parallel to the axis of revolution.

**FIGURE 4.8.23**
Cassini’s ovals for \( k = 0.5a, 0.9, a, 1.1a \) and \( 1.5a \) (from the inside to the outside). The foci (dots) are at \( x = a \) and \( x = -a \). The black curve, \( k = a \), is also called Bernoulli’s lemniscate.

2. A conchoid of Nichomedes is the set of points such that the signed distance \( AP \) in Figure 4.8.24, left, equals a fixed real number \( k \) (the line \( L \) and the origin \( O \) being fixed). If \( L \) is the line \( x = a \), the conchoid’s polar equation is \( r = a \sec \theta + k \). Once more, \( a \) is a scaling parameter, and the value of \( k/a \) controls the shape: when \( k > -a \) the curve is smooth, when \( k = -a \) there is a cusp, and when \( k < -a \) there is a self-intersection. The curves for \( k \) and \(-k\) can also be considered two leaves of the same conchoid, with Cartesian equation \((x - a)^2(x^2 + y^2) = k^2 x^2\).

**FIGURE 4.8.24**
Defining property of the conchoid of Nichomedes (left), and curves for \( k = \pm 0.5a, k = \pm a \), and \( k = \pm 1.5a \) (right).

3. A limaçon of Pascal is the set of points such that the distance \( AP \) in Figure 4.8.25, left, equals a fixed positive number \( k \) measured on either side (the circle \( C \) and
the origin \( O \) being fixed). If \( C \) has diameter \( a \) and center at \((0, \frac{1}{4}a)\), the limaçon’s polar equation is \( r = a \cos \theta + k \), and its Cartesian equation is

\[
(x^2 + y^2 - ax)^2 = k^2(x^2 + y^2).
\]

The value of \( k/a \) controls the shape, and there are two particularly interesting cases. For \( k = a \), we get a cardioid (see also page 293). For \( a = \frac{1}{2}k \), we get a curve that can be used to trisect an arbitrary angle \( \alpha \). If we draw a line \( L \) through the center of the circle \( C \) making an angle \( \alpha \) with the positive \( x \)-axis, and if we call \( P \) the intersection of \( L \) with the limaçon \( a = 2k \), the line from \( O \) to \( P \) makes an angle with \( L \) equal to \( \frac{1}{3}\alpha \).

**FIGURE 4.8.25**

Defining property of the limaçon of Pascal (left), and curves for \( k = 1.5a \), \( k = a \), and \( k = 0.5a \) (right). The middle curve is the cardioid; the one on the right a trisectrix.

Hypocycloids and epicycloids with rational ratios (see next section) are also algebraic curves, generally of higher degree.

### 4.8.2 ROULETTES (SPIROGRAPH CURVES)

Suppose given a fixed curve \( C \) and a moving curve \( M \), which rolls on \( C \) without slipping. The curve drawn by a point \( P \) kept fixed with respect to \( M \) is called a roulette, of which \( P \) is the pole.

The most important examples of roulettes arise when \( M \) is a circle and \( C \) is a straight line or a circle, but an interesting additional example is provided by the catenary \( y = a \cosh(x/a) \), which arises by rolling the parabola \( y = x^2/(4a) \) on the \( x \)-axis with pole the focus of the parabola (that is, \( P = (0, a) \) in the initial position). The catenary is the shape taken under the action of gravity by a chain or string of uniform density whose ends are held in the air.

A circle rolling on a straight line gives a trochoid, with the cycloid as a special case when the pole \( P \) lies on the circle (Figure 4.8.26). If the moving circle \( M \) has radius \( a \) and the distance from the pole \( P \) to the center of \( M \) is \( k \), the trochoid’s
The cycloid, therefore, has the parametric equation

\[ x = a(\phi - \sin \phi), \quad y = a(1 - \cos \phi). \]

One can eliminate \( \phi \) to get \( x \) as a (multivalued) function of \( y \), which takes the following form for the cycloid:

\[ x = \cos^{-1} \left( \frac{a - y}{a} \right) \pm \sqrt{2ay - y^2}. \]

The length of one arch of the cycloid is \( 8a \), and the area under the arch is \( 3\pi a^2 \).

A trochoid is also called a curtate cycloid when \( k < a \) (that is, when \( P \) is inside the circle) and a prolate cycloid when \( k > a \).

A circle rolling on another circle and exterior to it gives an epitrochoid. If \( a \) is the radius of the fixed circle, \( b \) that of the rolling circle, and \( k \) is the distance from \( P \) to the center of the rolling circle, the parametric equation of the epitrochoid is

\[ x = (a + b) \cos \theta - k \cos((1 + b/a)\theta), \quad y = (a + b) \sin \theta - k \sin((1 + b/a)\theta). \]

These equations assume that, at the start, everything is aligned along the positive \( x \)-axis, as in Figure 4.8.27, left. Usually one considers the case when \( a/b \) is a rational number, say \( a/b = p/q \) where \( p \) and \( q \) are relatively prime. Then the rolling circle returns to its original position after rotating \( q \) times around the fixed circle, and the epitrochoid is a closed curve—in fact, an algebraic curve. One also usually takes \( k = b \), so that \( P \) lies on the rolling circle; the curve in this case is called an epicycloid.

The middle diagram in Figure 4.8.27 shows the case \( b = k = \frac{1}{2}a \), called the nephroid; this curve is the cross section of the caustic of a spherical mirror. The diagram on the
Hypotrochoids and hypocycloids are defined in the same way as epitrochoids and epicycloids, but the rolling circle is inside the fixed one. The parametric equation of the hypotrochoid is

\[
x = (a - b) \cos \theta + k \cos((a/b - 1)\theta), \quad y = (a - b) \sin \theta - k \sin((a/b - 1)\theta),
\]

where the letters have the same meaning as for the epitrochoid. Usually one takes \(a/b\) rational and \(k = b\). There are several interesting particular cases:

- \(b = k = a\) gives a point.
- \(b = k = \frac{1}{2}a\) gives a diameter of the circle \(C\).
- \(b = k = \frac{1}{3}a\) gives the deltoid (Figure 4.8.28, left), whose algebraic equation is

\[
(x^2 + y^2)^2 - 8ax^3 + 24axy^2 + 18a^2(x^2 + y^2) - 27a^4 = 0.
\]
- \(b = k = \frac{1}{4}a\) gives the astroid (Figure 4.8.28, right), an algebraic curve of degree six whose equation can be reduced to \(x^{2/3} + y^{2/3} = a^{2/3}\). The figure illustrates another property of the astroid: its tangent intersects the coordinate axes at points that are always the same distance \(a\) apart. Otherwise said, the astroid is the envelope of a moving segment of fixed length whose endpoints are constrained to lie on the two coordinate axes.

### 4.8.3 Spirals

A number of interesting curves have polar equation \(r = f(\theta)\), where \(f\) is a monotonic function (always increasing or decreasing). This property leads to a spiral shape. The logarithmic spiral or Bernoulli spiral (Figure 4.8.29, left) is self-similar: by rotation the curve can be made to match any scaled copy of itself. Its equation is \(r = ke^{a\theta}\); the angle between the radius from the origin and the tangent to the curve is constant and...
equal to $\phi = \cot^{-1} a$. A curve parametrized by arc length and such that the curvature is proportional to the parameter at each point is a Bernoulli spiral.

In the Archimedean spiral or linear spiral (Figure 4.8.29, middle), the spacing between intersections along a ray from the origin is constant. The equation of this spiral is $r = a\theta$; by scaling one can take $a = 1$. It has an inner endpoint, in contrast with the logarithmic spiral, which spirals down to the origin without reaching it. The Cornu spiral or clothoid (Figure 4.8.29, right), important in optics and engineering, has the following parametric representation in Cartesian coordinates:

$$X = aC(t) = a\int_0^t \cos\left(\frac{1}{2}\pi s^2\right) ds, \quad y = aS(t) = a\int_0^t \sin\left(\frac{1}{2}\pi s^2\right) ds.$$ 

(C and $S$ are the so-called Fresnel integrals; see page 498). A curve parametrized by arclength and such that the curvature is inversely proportional to the parameter at each point is a Cornu spiral (compare the Bernoulli spiral).

4.8.4 THE PEANO CURVE AND FRACTAL CURVES

There are curves (in the sense of continuous maps from the real line to the plane) that completely cover a two-dimensional region of the plane. We give a construction
of such a Peano curve, adapted from David Hilbert’s example. The construction is inductive and is based on replacement rules. We consider building blocks of six shapes: , the length of the straight segments being twice the radius of the curved ones. A sequence of these patterns, end-to-end, represents a curve, if we disregard the gray and black half-disk. The replacement rules are the following:

The rules are applied taking into account the way each piece is turned. Here we apply the replacement rules to a particular initial pattern:

(We scale the result so it has the same size as the original.) Applying the process repeatedly gives, in the limit, the Peano curve. Here are the first five steps:

The same idea of replacement rules leads to many interesting fractal, and often self-similar, curves. For example, the substitution leads to the Koch snowflake when applied to an initial equilateral triangle, like this (the first three stages and the sixth are shown):
4.8.5 CLASSICAL CONSTRUCTIONS

The ancient Greeks used straightedges and compasses to find the solutions to numerical problems. For example, they found square roots by constructing the geometric mean of two segments. Three famous problems that have been proved intractable by this method are

- The trisection of an arbitrary angle
- The squaring of the circle (the construction of a square whose area is equal to that of a given circle)
- The doubling of the cube (the construction of a cube with double the volume of a given cube)

A regular $n$-gon inscribed in the unit circle can be constructed by straightedge and compass alone if, and only if, $n$ has the form $n = 2^\ell p_1 p_2 \ldots p_k$, where $\ell$ is a nonnegative integer and $\{p_i\}$ are distinct Fermat primes (primes of the form $2^{2^m} + 1$). The only known Fermat primes are for $m = 1, 2, 3, 4$. Thus, regular $n$-gons can be constructed for $n = 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, \ldots, 257, \ldots$.

4.9 COORDINATE SYSTEMS IN SPACE

Conventions

When we talk about “the point with coordinates $(x, y, z)$” or “the surface with equation $f(x, y, z)$”, we always mean Cartesian coordinates. If a formula involves another type of coordinates, this fact will be stated explicitly. Note that Section 4.1.1 has information on substitutions and transformations relevant to the three-dimensional case.

4.9.1 CARTESIAN COORDINATES IN SPACE

In Cartesian coordinates (or rectangular coordinates), a point $P$ is referred to by three real numbers, indicating the positions of the perpendicular projections from the point to three fixed, perpendicular, graduated lines, called the axes. If the coordinates are denoted $x, y, z$, in that order, the axes are called the $x$-axis, etc., and we write $P = (x, y, z)$. Often the $x$-axis is imagined to be horizontal and pointing roughly toward the viewer (out of the page), the $y$-axis also horizontal and pointing more or less to the right, and the $z$-axis vertical, pointing up. The system is called right-handed if it can be rotated so the three axes are in this position. Figure 4.9.30 shows a right-handed system. The point $x = 0, y = 0, z = 0$ is the origin, where the three axes intersect.
In Cartesian coordinates, \( P = (4.2, 3.4, 2.2) \).

4.9.2 CYLINDRICAL COORDINATES IN SPACE

To define cylindrical coordinates, we take an axis (usually called the \( z \)-axis) and a perpendicular plane, on which we choose a ray (the initial ray) originating at the intersection of the plane and the axis (the origin). The coordinates of a point \( P \) are the polar coordinates \((r, \theta)\) of the projection of \( P \) on the plane, and the coordinate \( z \) of the projection of \( P \) on the axis (Figure 4.9.31). See Section 4.1.3 for remarks on the values of \( r \) and \( \theta \).

Among the possible sets \((r, \theta, z)\) of cylindrical coordinates for \( P \) are \((10, 30^\circ, 5)\) and \((10, 390^\circ, 5)\).

4.9.3 SPHERICAL COORDINATES IN SPACE

To define spherical coordinates, we take an axis (the polar axis) and a perpendicular plane (the equatorial plane), on which we choose a ray (the initial ray) originating at the intersection of the plane and the axis (the origin \( O \)). The coordinates of a point \( P \) are the distance \( \rho \) from \( P \) to the origin, the angle \( \phi \) (zenith) between the line \( OP \)
and the positive polar axis, and the angle $\theta$ (*azimuth*) between the initial ray and the projection of $OP$ to the equatorial plane. See Figure 4.9.32. As in the case of polar and cylindrical coordinates, $\theta$ is only defined up to multiples of $360^\circ$, and likewise $\phi$. Usually $\phi$ is assigned a value between 0 and $180^\circ$, but values of $\phi$ between $180^\circ$ and $360^\circ$ can also be used; the triples $(\rho, \phi, \theta)$ and $(\rho, 360^\circ - \phi, 180^\circ + \theta)$ represent the same point. Similarly, one can extend $\rho$ to negative values; the triples $(\rho, \phi, \theta)$ and $(-\rho, 180^\circ - \phi, 180^\circ + \theta)$ represent the same point.

**FIGURE 4.9.32**
A set of spherical coordinates for $P$ is $(\rho, \theta, \phi) = (10, 60^\circ, 30^\circ)$.

**FIGURE 4.9.33**
Standard relations between Cartesian, cylindrical, and spherical coordinate systems. The origin is the same for all three. The positive $z$-axes of the Cartesian and cylindrical systems coincide with the positive polar axis of the spherical system. The initial rays of the cylindrical and spherical systems coincide with the positive $x$-axis of the Cartesian system, and the rays $\theta = 90^\circ$ coincide with the positive $y$-axis.
4.9.4 RELATIONS BETWEEN CARTESIAN, CYLINDRICAL, AND SPHERICAL COORDINATES

Consider a Cartesian, a cylindrical, and a spherical coordinate system, related as shown in Figure 4.9.33. The Cartesian coordinates \((x, y, z)\), the cylindrical coordinates \((r, \theta, z)\), and the spherical coordinates \((\rho, \phi, \theta)\) of a point are related as follows:

\[
\begin{align*}
\text{cart} & \leftrightarrow \text{cyl} & x &= r \cos \theta, & r &= \sqrt{x^2 + y^2}, \\
& & y &= r \sin \theta, & \theta &= \tan^{-1} \frac{y}{x}, \\
& & & & \sin \theta = \frac{y}{\sqrt{x^2 + y^2}}, \\
& & & & \cos \theta = \frac{x}{\sqrt{x^2 + y^2}}.
\end{align*}
\]

\[
\begin{align*}
\text{cyl} & \leftrightarrow \text{sph} & r &= \rho \sin \phi, & \rho &= \sqrt{r^2 + z^2}, \\
& & z &= \rho \cos \phi, & \phi &= \tan^{-1} \frac{r}{z}, \\
& & & & \sin \phi = \frac{r}{\sqrt{r^2 + z^2}}, \\
& & & & \cos \phi = \frac{z}{\sqrt{r^2 + z^2}}.
\end{align*}
\]

\[
\begin{align*}
\text{cart} & \leftrightarrow \text{sph} & x &= \rho \cos \theta \sin \phi, & \rho &= \sqrt{x^2 + y^2 + z^2}, \\
& & y &= \rho \sin \theta \sin \phi, & \theta &= \tan^{-1} \frac{y}{x}, \\
& & z &= \rho \cos \phi, & \phi &= \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z}, \\
& & & & \phi &= \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}}.
\end{align*}
\]

4.9.5 HOMOGENEOUS COORDINATES IN SPACE

A quadruple of real numbers \((x : y : z : t)\), with \(t \neq 0\), is a set of homogeneous coordinates for the point \(P\) with Cartesian coordinates \((x/t, y/t, z/t)\). Thus the same point has many sets of homogeneous coordinates: \((x : y : z : t)\) and \((x' : y' : z' : t')\) represent the same point if, and only if, there is some real number \(\alpha\) such that \(x' = \alpha x, y' = \alpha y, z' = \alpha z, t' = \alpha t\). If \(P\) has Cartesian coordinates \((x_0, y_0, z_0)\), one set of homogeneous coordinates for \(P\) is \((x_0, y_0, z_0, 1)\).

Section 4.1.4 has more information on the relationship between Cartesian and homogeneous coordinates. Section 4.10.2 has formulas for space transformations in homogeneous coordinates.

4.10 SPACE SYMMETRIES OR ISOMETRIES

A transformation of space (invertible map of space to itself) that preserves distances is called an isometry of space. Every isometry of space is a composition of transformations of the following types:

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• The identity (which leaves every point fixed)
• A translation by a vector \( \mathbf{v} \)
• A rotation through an angle \( \alpha \) around a line \( L \)
• A screw motion through an angle \( \alpha \) around a line \( L \), with displacement \( d \)
• A reflection in a line \( P \)
• A glide-reflection in a line \( P \) with displacement vector \( \mathbf{v} \)

The identity is a particular case of a translation and of a rotation; rotations are particular cases of screw motions; reflections are particular cases of glide-reflections. However, as in the plane case, it is more intuitive to consider each case separately.

### 4.10.1 FORMULAS FOR SYMMETRIES: CARTESIAN COORDINATES

In the formulas below, multiplication between a matrix and a triple of coordinates should be carried out regarding the triple as a column vector (or a matrix with three rows and one column).

**Translation** by \( (x_0, y_0, z_0) \):

\[
(x, y, z) \mapsto (x + x_0, y + y_0, z + z_0).
\]

**Rotation** through \( \alpha \) (counterclockwise) around the line through the origin with direction cosines \( a, b, c \) (see page 304): \( (x, y, z) \mapsto M(x, y, z) \), where \( M \) is the matrix,

\[
\begin{bmatrix}
a^2(1 - \cos \alpha) + \cos \alpha & ab(1 - \cos \alpha) - c \sin \alpha & ac(1 - \cos \alpha) + b \sin \alpha \\
ab(1 - \cos \alpha) + c \sin \alpha & b^2(1 - \cos \alpha) + \cos \alpha & bc(1 - \cos \alpha) - a \sin \alpha \\
ac(1 - \cos \alpha) - b \sin \alpha & bc(1 - \cos \alpha) + a \sin \alpha & c^2(1 - \cos \alpha) + \cos \alpha
\end{bmatrix}
\]

(4.10.1)

**Rotation** through \( \alpha \) (counterclockwise) around the line with direction cosines \( a, b, c \) through an arbitrary point \( (x_0, y_0, z_0) \):

\[
(x, y, z) \mapsto (x_0, y_0, z_0) + M(x - x_0, y - y_0, z - z_0),
\]

where \( M \) is given by Equation (4.10.1).

**Arbitrary rotations and Euler angles**: Any rotation of space fixing the origin can be decomposed as a rotation by \( \phi \) about the \( z \)-axis, followed by a rotation by \( \theta \) about the \( y \)-axis, followed by a rotation by \( \psi \) about the \( z \)-axis. The numbers \( \phi, \theta \) and \( \psi \) are called the Euler angles of the composite rotation, which acts as:

\[
(x, y, z) \mapsto M(x, y, z),
\]

where \( M \) is the matrix given by

\[
\begin{bmatrix}
\cos \phi \cos \theta \cos \psi - \sin \phi \sin \psi & -\sin \phi \cos \theta \cos \psi - \cos \phi \sin \psi & \sin \theta \cos \psi \\
\sin \phi \cos \psi + \cos \phi \cos \theta \sin \psi & -\sin \phi \cos \theta \sin \psi + \cos \phi \cos \psi & \sin \theta \sin \psi \\
-\cos \phi \sin \theta & \sin \theta \sin \phi & \cos \theta
\end{bmatrix}
\]

(4.10.2)
An alternative decomposition, more natural if we think of the coordinate system as a rigid trihedron that rotates in space, is the following: a rotation by $\psi$ about the $z$-axis, followed by a rotation by $\theta$ about the rotated $y$-axis, followed by a rotation by $\phi$ about the rotated $z$-axis. Note that the order is reversed.

Provided that $\theta$ is not a multiple of 180°, the decomposition of a rotation in this form is unique (apart from the ambiguity arising from the possibility of adding a multiple of 360° to any angle). Figure 4.10.34 shows how the Euler angles can be read off geometrically.

**FIGURE 4.10.34**
The coordinate rays $Ox$, $Oy$, $Oz$, together with their images $O\xi$, $O\eta$, $O\zeta$ under a rotation, fix the Euler angles associated with that rotation, as follows: $\theta = zO\zeta$, $\psi = xOr = yOs$, and $\phi = sO\eta$. (Here the ray Or is the projection of $O\zeta$ to the $xy$-plane. The ray Os is determined by the intersection of the $xy$- and $\xi\eta$-planes.)

**Warning:** Some references define Euler angles differently; the most common variation is that the second rotation is taken about the $x$-axis instead of about the $y$-axis.

**Screw motion** with angle $\alpha$ and displacement $d$ around the line with direction cosines $a$, $b$, $c$ through an arbitrary point $(x_0, y_0, z_0)$:

$$(x, y, z) \mapsto (x_0 + ad, y_0 + bd, z_0 + cd) + M(x - x_0, y - y_0, z - z_0),$$

where $M$ is given by (4.10.1).

**Reflection**

in the $xy$-plane: $(x, y, z) \mapsto (x, y, -z)$.

in the $xz$-plane: $(x, y, z) \mapsto (x, -y, z)$.

in the $yz$-plane: $(x, y, z) \mapsto (-x, y, z)$.

(4.10.3)

**Reflection** in a plane with equation $ax + by + cz + d = 0$:

$$(x, y, z) \mapsto \frac{1}{a^2 + b^2 + c^2} (M(x_0, y_0, z_0) - (2ad, 2bd, 2cd)).$$

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where $M$ is the matrix
\[
M = \begin{bmatrix}
-a^2 + b^2 + c^2 & -2ab & -2ac \\
-2ab & a^2 - b^2 + c^2 & -2bc \\
-2ac & -2bc & a^2 + b^2 - c^2
\end{bmatrix}.
\] (4.10.4)

**Reflection** in a plane going through $(x_0, y_0, z_0)$ and whose normal has direction cosines $a, b, c$:

\[
(x, y, z) \mapsto (x_0 + y_0 + z_0) + M(x - x_0, y - y_0, z - z_0),
\]

where $M$ is as in (4.10.4).

**Glide-reflection** in a plane $P$ with displacement vector $v$: Apply first a reflection in $P$, then a translation by the vector $v$.

### 4.10.2 Formulas for Symmetries: Homogeneous Coordinates

All isometries of space can be expressed in homogeneous coordinates in terms of multiplication by a matrix. As in the case of plane isometries (Section 4.2.2), this means that the successive application of transformations reduces to matrix multiplication. (In the formulas below, $\begin{bmatrix} M & 0 \\ \hline 0 & 1 \end{bmatrix}$ is the $4 \times 4$ projective matrix obtained from the $3 \times 3$ matrix $M$ by adding a row and a column as stated.)

**Translation** by $(x_0, y_0, z_0)$: \[
\begin{bmatrix}
1 & 0 & 0 & x_0 \\
0 & 1 & 0 & y_0 \\
0 & 0 & 1 & z_0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

**Rotation** through the origin: \[
\begin{bmatrix}
M & 0 \\
0 & 1
\end{bmatrix},
\]

where $M$ is given in (4.10.1) or (4.10.2), as the case may be.

**Reflection** in a plane through the origin: \[
\begin{bmatrix}
M & 0 \\
0 & 1
\end{bmatrix},
\]

where $M$ is given in (4.10.4).

From this one can deduce all other transformations, as in the case of plane transformations (see page 255).
4.11 OTHER TRANSFORMATIONS OF SPACE

4.11.1 SIMILARITIES
A transformation of space that preserves shapes is called a similarity. Every similarity of the plane is obtained by composing a proportional scaling transformation (also known as a homothety) with an isometry. A proportional scaling transformation centered at the origin has the form

\[(x, y, z) \mapsto (ax, ay, az),\]

where \(a \neq 0\) is the scaling factor (a real number). The corresponding matrix in homogeneous coordinates is

\[H_a = \begin{bmatrix}
a & 0 & 0 & 0 \\
0 & a & 0 & 0 \\
0 & 0 & a & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}.
\]

In cylindrical coordinates, the transformation is \((r, \theta, z) \mapsto (ar, \theta, az)\). In spherical coordinates, it is \((r, \phi, \theta) \mapsto (ar, \phi, \theta)\).

4.11.2 AFFINE TRANSFORMATIONS
A transformation that preserves lines and parallelism (maps parallel lines to parallel lines) is an affine transformation. There are two important particular cases of such transformations:

1. A nonproportional scaling transformation centered at the origin has the form \((x, y, z) \mapsto (ax, by, cz)\), where \(a, b, c \neq 0\) are the scaling factors (real numbers). The corresponding matrix in homogeneous coordinates is

\[H_{a,b,c} = \begin{bmatrix}
a & 0 & 0 & 0 \\
0 & b & 0 & 0 \\
0 & 0 & c & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}.
\]

2. A shear in the \(x\)-direction and preserving horizontal planes has the form \((x, y, z) \mapsto (x + rz, y, z)\), where \(r\) is the shearing factor. The corresponding matrix in homogeneous coordinates is

\[S_r = \begin{bmatrix}
1 & 0 & r & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}.
\]

Every affine transformation is obtained by composing a scaling transformation with an isometry, or one or two shears with a homothety and an isometry.
4.11.3 PROJECTIVE TRANSFORMATIONS

A transformation that maps lines to lines (but does not necessarily preserve parallelism) is a projective transformation. Any spatial projective transformation can be expressed by an invertible $4 \times 4$ matrix in homogeneous coordinates; conversely, any invertible $4 \times 4$ matrix defines a projective transformation of space. Projective transformations (if not affine) are not defined on all of space, but only on the complement of a plane (the missing plane is “mapped to infinity”).

The following particular case is often useful, especially in computer graphics, in projecting a scene from space to the plane. Suppose an observer is at the point $E = (x_0, y_0, z_0)$ of space, looking toward the origin $O = (0, 0, 0)$. Let $P$, the screen, be the plane through $O$ and perpendicular to the ray $EO$. Place a rectangular coordinate system $\xi\eta$ on $P$ with origin at $O$ so that the positive $\eta$-axis lies in the half-plane determined by $E$ and the positive $z$-axis of space (that is, the $z$-axis is pointing “up” as seen from $E$). Then consider the transformation that associates with a point $X = (x, y, z)$ the triple $(\xi, \eta, \zeta)$, where $(\xi, \eta)$ are the coordinates of the point, where the line $EX$ intersects $P$ (the screen coordinates of $X$ as seen from $E$), and $\zeta$ is the inverse of the signed distance from $X$ to $E$ along the line $EO$ (this distance is the depth of $X$ as seen from $E$). This is a projective transformation, given by the matrix

$$
\begin{bmatrix}
-r^2y_0 & r^2x_0 & 0 & 0 \\
-rx_0z_0 & -ry_0z_0 & \rho^2 & 0 \\
0 & 0 & 0 & \rho \\
-\rho x_0 & -\rho y_0 & -\rho z_0 & r^2\rho
\end{bmatrix}
$$

with $\rho = \sqrt{x_0^2 + y_0^2}$ and $r = \sqrt{x_0^2 + y_0^2 + z_0^2}$.

4.12 DIRECTION ANGLES AND DIRECTION COSINES

Given a vector $(a, b, c)$ in three-dimensional space, the direction cosines of this vector are

$$
\cos \alpha = \frac{a}{\sqrt{a^2 + b^2 + c^2}},
$$

$$
\cos \beta = \frac{b}{\sqrt{a^2 + b^2 + c^2}},
$$

$$
\cos \gamma = \frac{c}{\sqrt{a^2 + b^2 + c^2}}. 
$$

(4.12.1)

Here the direction angles $\alpha$, $\beta$, $\gamma$ are the angles that the vector makes with the positive $x$-, $y$- and $z$-axes, respectively. In formulas, usually the direction cosines appear, rather than the direction angles. We have

$$
\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.
$$
4.13 PLANES

The (Cartesian) equation of a plane is linear in the coordinates \( x, y, \) and \( z \):

\[
ax + by + cz + d = 0. \tag{4.13.1}
\]

The normal direction to this plane is \((a, b, c)\). The intersection of this plane with the \( x \)-axis, or \( x \)-intercept, is \( x = -d/a \), the \( y \)-intercept is \( y = -d/b \), and the \( z \)-intercept is \( z = -d/c \). The plane is vertical (perpendicular to the \( xy \)-plane) if \( c = 0 \). It is perpendicular to the \( x \)-axis if \( b = c = 0 \), and likewise for the other coordinates.

When \( a^2 + b^2 + c^2 = 1 \) and \( d \leq 0 \) in the equation \( ax + by + cz + d = 0 \), the equation is said to be in normal form. In this case \( d \) is the distance of the plane to the origin, and \((a, b, c)\) are the direction cosines of the normal.

To reduce an arbitrary equation \( ax + by + cz + d = 0 \) to normal form, divide by \( \pm \sqrt{a^2 + b^2 + c^2} \), where the sign of the radical is chosen opposite the sign of \( d \) when \( d \neq 0 \), the same as the sign of \( c \) when \( d = 0 \) and \( c \neq 0 \), and the same as the sign of \( b \) otherwise.

Planes with prescribed properties

- Plane through \((x_0, y_0, z_0)\) and perpendicular to the direction \((a, b, c)\):
  \[
a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \tag{4.13.2}
  \]

- Plane through \((x_0, y_0, z_0)\) and parallel to the directions \((a_1, b_1, c_1)\) and \((a_2, b_2, c_2)\):
  \[
  \begin{vmatrix}
  x - x_0 & y - y_0 & z - z_0 \\
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 
  \end{vmatrix} = 0. \tag{4.13.3}
  \]

- Plane through \((x_0, y_0, z_0)\) and \((x_1, y_1, z_1)\) and parallel to the direction \((a, b, c)\):
  \[
  \begin{vmatrix}
  x - x_0 & y - y_0 & z - z_0 \\
  x_1 - x_0 & y_1 - y_0 & z_1 - z_0 \\
  a & b & c 
  \end{vmatrix} = 0. \tag{4.13.4}
  \]

- Plane going through \((x_0, y_0, z_0), (x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\):
  \[
  \begin{vmatrix}
  x & y & z & 1 \\
  x_0 & y_0 & z_0 & 1 \\
  x_1 & y_1 & z_1 & 1 \\
  x_2 & y_2 & z_2 & 1
  \end{vmatrix} = 0 \text{ or } \begin{vmatrix}
  x - x_0 & y - y_0 & z - z_0 \\
  x_1 - x_0 & y_1 - y_0 & z_1 - z_0 \\
  x_2 - x_0 & y_2 - y_0 & z_2 - z_0
  \end{vmatrix} = 0. \tag{4.13.5}
  \]

(The last three formulas remain true in oblique coordinates.)

- The distance from the point \((x_0, y_0, z_0)\) to the plane \( ax + by + cz + d = 0 \) is
  \[
  \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}. \tag{4.13.6}
  \]

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The angle between two planes 
\[ a_0 x + b_0 y + c_0 z + d_0 = 0 \] and 
\[ a_1 x + b_1 y + c_1 z + d_1 = 0 \] is
\[
\cos^{-1} \frac{a_0 a_1 + b_0 b_1 + c_0 c_1}{\sqrt{a_0^2 + b_0^2 + c_0^2} \sqrt{a_1^2 + b_1^2 + c_1^2}}. \tag{4.13.7}
\]
In particular, the two planes are parallel when 
\[ a_0 : b_0 : c_0 = a_1 : b_1 : c_1, \] and perpendicular when 
\[ a_0 a_1 + b_0 b_1 + c_0 c_1 = 0. \]

**Concurrence and coplanarity**

Four planes 
\[ a_0 x + b_0 y + c_0 = 0, \]
\[ a_1 x + b_1 y + c_1 = 0, \]
\[ a_2 x + b_2 y + c_2 = 0 \]
and 
\[ a_3 x + b_3 y + c_3 = 0 \] are concurrent if and only if
\[
\begin{vmatrix}
1 & a_0 & b_0 & c_0 \\
1 & a_1 & b_1 & c_1 \\
1 & a_2 & b_2 & c_2 \\
1 & a_3 & b_3 & c_3
\end{vmatrix} = 0.
\]

Four points \((x_0, y_0, z_0), (x_1, y_1, z_1), (x_2, y_2, z_2)\) and \((x_3, y_3, z_3)\) are coplanar if and only if
\[
\begin{vmatrix}
x_0 & y_0 & z_0 & 1 \\
x_1 & y_1 & z_1 & 1 \\
x_2 & y_2 & z_2 & 1 \\
x_3 & y_3 & z_3 & 1
\end{vmatrix} = 0.
\]
(Both of these assertions remain true in oblique coordinates.)

### 4.14 LINES

Two planes that are not parallel or coincident intersect in a straight line, such that one can express a line by a pair of linear equations
\[
\begin{cases}
ax + by + cz + d = 0 \\
a'x + b'y + c'z + d' = 0
\end{cases}
\]
such that \(bc' - cb', ca' - ac', \) and \(ab' - ba'\) are not all zero. The line thus defined is parallel to the vector \((bc' - cb', ca' - ac', ab' - ba')\). The direction cosines of the line are those of this vector. See Equation (4.12.1). (The direction cosines of a line are only defined up to a simultaneous change in sign, because the opposite vector still gives the same line.)

The following particular cases are important:

- **Line through \((x_0, y_0, z_0)\) parallel to the vector \((a, b, c)\):**
\[
\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.
\]
• Line through \((x_0, y_0, z_0)\) and \((x_1, y_1, z_1)\):

\[
\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}
\]

This line is parallel to the vector \((x_1 - x_0, y_1 - y_0, z_1 - z_0)\).

**Distances**

• The *distance* between two points in space is the *length of the line segment* joining them. The distance between the points \((x_0, y_0, z_0)\) and \((x_1, y_1, z_1)\) is

\[
\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}.
\]

• The point \(k\%\) of the way from \(P_0 = (x_0, y_0, z_0)\) to \(P_1 = (x_1, y_1, z_1)\) is

\[
\left(\frac{100 k}{100}, \frac{y_0 + (100 - k) y_1}{100}, \frac{z_0 + (100 - k) z_1}{100}\right).
\]

(The same formula also applies in oblique coordinates.) This point divides the segment \(P_0P_1\) in the ratio \(k : (100 - k)\). As a particular case, the *midpoint* of \(P_0P_1\) is given by

\[
\left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2}, \frac{z_0 + z_1}{2}\right).
\]

• The *distance* between the point \((x_0, y_0, z_0)\) and the line through \((x_1, y_1, z_1)\) in direction \((a, b, c)\):

\[
\sqrt{\frac{(y_0 - y_1)^2}{a^2} + \frac{(z_0 - z_1)^2}{b^2} + \frac{(x_0 - x_1)^2}{c^2}}.
\]

• The *distance* between the line through \((x_0, y_0, z_0)\) in direction \((a_0, b_0, c_0)\) and the line through \((x_1, y_1, z_1)\) in direction \((a_1, b_1, c_1)\):

\[
\sqrt{\frac{b_0^2 c_0^2}{a_0^2} + \frac{c_0^2 a_0^2}{b_0^2} + \frac{a_0^2 b_0^2}{c_0^2}}.
\]

\[
(4.14.1)
\]

**Angles**

*Angle* between lines with directions \((x_0, y_0, z_0)\) and \((x_1, y_1, z_1)\):

\[
\cos^{-1} \frac{a_0 a_1 + b_0 b_1 + c_0 c_1}{\sqrt{a_0^2 + b_0^2 + c_0^2} \sqrt{a_1^2 + b_1^2 + c_1^2}}.
\]

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In particular, the two lines are parallel when \( a_0 : b_0 : c_0 = a_1 : b_1 : c_1 \), and perpendicular when \( a_0a_1 + b_0b_1 + c_0c_1 = 0 \).

Angle between lines with direction angles \( \alpha_0, \beta_0, \gamma_0 \) and \( \alpha_1, \beta_1, \gamma_1 \):

\[
\cos^{-1}(\cos \alpha_0 \cos \alpha_1 + \cos \beta_0 \cos \beta_1 + \cos \gamma_0 \cos \gamma_1)
\]

### Concurrence, coplanarity, parallelism

Two lines specified by point and direction are coplanar if, and only if, the determinant in the numerator of Equation (4.14.1) is zero. In this case they are concurrent (if the denominator is nonzero) or parallel (if the denominator is zero).

Three lines with directions \((a_0, b_0, c_0)\), \((a_1, b_1, c_1)\) and \((a_2, b_2, c_2)\) are parallel to a common plane if and only if

\[
\begin{vmatrix}
  a_0 & b_0 & c_0 \\
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
\end{vmatrix} = 0.
\]

### 4.15 POLYHEDRA

For any polyhedron topologically equivalent to a sphere—in particular, for any convex polyhedron—the Euler formula holds:

\[
v - e + f = 2,
\]

where \( v \) is the number of vertices, \( e \) is the number of edges, and \( f \) is the number of faces.

Many common polyhedra are particular cases of cylinders (Section 4.16) or cones (Section 4.17). A cylinder with a polygonal base (the base is also called a directrix) is called a prism. A cone with a polygonal base is called a pyramid. A frustum of a cone with a polygonal base is called a truncated pyramid. Formulas (4.16.1), (4.17.1), and (4.17.2) give the volumes of a general prism, pyramid, and truncated pyramid.

A prism whose base is a parallelogram is a parallelepiped. The volume of a parallelepiped with one vertex at the origin and adjacent vertices at \((x_1, y_1, z_1)\), \((x_2, y_2, z_2)\), and \((x_3, y_3, z_3)\) is given by

\[
\begin{vmatrix}
  x_1 & y_1 & z_1 \\
  x_2 & y_2 & z_2 \\
  x_3 & y_3 & z_3 \\
\end{vmatrix}
\]

The rectangular parallelepiped is a particular case: all of its faces are rectangles. If the side lengths are \( a, b, c \), the volume is \( abc \), the total area is \( 2(ab + ac + bc) \), and each diagonal has length \( \sqrt{a^2 + b^2 + c^2} \). When \( a = b = c \) we get the cube. See Section 4.15.1.
A pyramid whose base is a triangle is a tetrahedron. The volume of a tetrahedron with one vertex at the origin and the other vertices at \((x_1, y_1, z_1)\), \((x_2, y_2, z_2)\), and \((x_3, y_3, z_3)\) is given by

\[
\text{Volume} = \frac{1}{6} \left| \begin{array}{ccc} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{array} \right|.
\]

In a tetrahedron with vertices \(P_0, P_1, P_2, P_3\), let \(d_{ij}\) be the distance (edge length) from \(P_i\) to \(P_j\). Form the determinants

\[
\Delta = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & d_{01}^2 & d_{02}^2 \\ 1 & d_{01}^2 & 0 & d_{12}^2 \\ 1 & d_{03}^2 & d_{13}^2 & 0 \end{vmatrix}
\]

and

\[
\Gamma = \begin{vmatrix} 0 & d_{01}^2 & d_{02}^2 & d_{03}^2 \\ d_{01}^2 & 0 & d_{12}^2 & d_{13}^2 \\ d_{02}^2 & d_{12}^2 & 0 & d_{23}^2 \\ d_{03}^2 & d_{13}^2 & d_{23}^2 & 0 \end{vmatrix}.
\]

Then the volume of the tetrahedron is \(\sqrt{|\Delta|}/288\), and the radius of the circumscribed sphere is \(\frac{1}{2}\sqrt{|\Delta|}/\sqrt{T72\Delta}|\).

### 4.15.1 REGULAR POLYHEDRA

Figure 4.15.35 shows the five regular polyhedra, or Platonic solids. In the following tables and formulas, \(a\) is the length of an edge, \(\theta\) the dihedral angle at each edge, \(R\) the radius of the circumscribed sphere, \(r\) the radius of the inscribed sphere, \(V\) the volume, \(S\) the total surface area, \(v\) the total number of vertices, \(e\) the total number of edges, \(f\) the total number of faces, \(p\) the number of edges in a face (3 for equilateral triangles, 4 for squares, 5 for regular pentagons), and \(q\) the number of edges meeting at a vertex.

\[
\theta = 2 \sin^{-1} \frac{\cos(180^\circ/p)}{\sin(180^\circ/q)},
\]

\[
R = \tan \frac{180^\circ}{p} \tan \frac{180^\circ}{q},
\]

\[
R = \frac{1}{2} \sin(180^\circ/q) \cos \frac{\theta}{2},
\]

\[
S = \frac{fp}{4} \cot \frac{180^\circ}{p},
\]

\[
V = \frac{1}{3} r S.
\]

<table>
<thead>
<tr>
<th>Name</th>
<th>(v)</th>
<th>(e)</th>
<th>(f)</th>
<th>(p)</th>
<th>(q)</th>
<th>(\sin \theta)</th>
<th>(\theta)</th>
</tr>
</thead>
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<tr>
<td>Regular tetrahedron</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>(2\sqrt{2}/3)</td>
<td>70°31’44”</td>
</tr>
<tr>
<td>Cube</td>
<td>8</td>
<td>12</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>90°</td>
</tr>
<tr>
<td>Regular octahedron</td>
<td>6</td>
<td>12</td>
<td>8</td>
<td>3</td>
<td>3</td>
<td>(2\sqrt{3}/3)</td>
<td>109°28’16”</td>
</tr>
<tr>
<td>Regular dodecahedron</td>
<td>20</td>
<td>30</td>
<td>12</td>
<td>5</td>
<td>3</td>
<td>(2\sqrt{5})</td>
<td>116°33’54”</td>
</tr>
<tr>
<td>Regular icosahedron</td>
<td>12</td>
<td>30</td>
<td>20</td>
<td>3</td>
<td>5</td>
<td>(2/3)</td>
<td>138°11’23”</td>
</tr>
</tbody>
</table>
FIGURE 4.15.35
The Platonic solids. Top: the tetrahedron (self-dual). Middle: the cube and the octahedron (dual to one another). Bottom: the dodecahedron and the icosahedron (dual to one another).
### 4.16 CYLINDERS

Given a line $L$ and a curve $C$ in a plane $P$, the cylinder with generator $L$ and directrix $C$ is the surface obtained by moving $L$ parallel to itself, so that a point of $L$ is always on $C$. If $L$ is parallel to the $z$-axis, the surface’s implicit equation does not involve the variable $z$. Conversely, any implicit equation that does not involve one of the variables (or that can be brought to that form by a change of coordinates) represents a cylinder.

If $C$ is a simple closed curve, we also apply the word cylinder to the solid enclosed by the surface generated in this way (Figure 4.16.36, left). The volume

![Figure 4.16.36](image)

*Left: an oblique cylinder with generator $L$ and directrix $C$. Right: a right circular cylinder.*
where $A$ is the area in the plane $P$ enclosed by $C$, $h$ is the distance between $P$ and $P'$ (measured perpendicularly), $l$ is the length of the segment of $L$ contained between $P$ and $P'$, and $\theta$ is the angle that $L$ makes with $P$. When $\theta = 90^\circ$ we have a right cylinder, and $h = l$. For a right cylinder, the lateral area between $P$ and $P'$ is $hs$, where $s$ is the length (circumference) of $C$.

The most important particular case is the right circular cylinder (often simply called a cylinder). If $r$ is the radius of the base and $h$ is the altitude (Figure 4.16.36, right), the lateral area is $2\pi rh$, the total area is $2\pi r(r + h)$, and the volume is $\pi r^2h$. The implicit equation of this surface can be written $x^2 + y^2 = r^2$; see also Section 4.20.

## 4.17 CONES

Given a curve $C$ in a plane $P$ and a point $O$ not in $P$, the cone with vertex $O$ and directrix $C$ is the surface obtained as the union of all lines that join $O$ with points of $C$. If $O$ is the origin and the surface is given implicitly by an algebraic equation, that equation is homogeneous (all terms have the same total degree in the variables). Conversely, any homogeneous implicit equation (or one that can be made homogeneous by a change of coordinates) represents a cone.

If $C$ is a simple closed curve, we also apply the word cone to the solid enclosed by the surface generated in this way (Figure 4.17.37, top). The volume contained between $P$ and the vertex $O$ is

$$V = \frac{1}{3}Ah,$$

where $A$ is the area in the plane $P$ enclosed by $C$ and $h$ is the distance from $O$ and $P$ (measured perpendicularly).

The solid contained between $P$ and a plane $P'$ parallel to $P$ (on the same side of the vertex) is called a frustum. Its volume is

$$V = \frac{1}{3}h(A + A' + \sqrt{AA'}),$$

where $A$ and $A'$ are the areas enclosed by the sections of the cone by $P$ and $P'$ (often called the bases of the frustum).

The most important particular case of a cone is the right circular cone (often simply called a cone). If $r$ is the radius of the base, $h$ is the altitude, and $l$ is the length between the vertex and a point on the base circle (Figure 4.17.37, bottom left), the
FIGURE 4.17.37
Top: a cone with vertex \( O \) and directrix \( C \). Bottom left: a right circular cone. Bottom right: A frustum of the latter.

Following relationships apply:

\[
\begin{align*}
l &= \sqrt{r^2 + h^2}, \\
\text{Lateral area} &= \pi rl = \pi r \sqrt{r^2 + h^2}, \\
\text{Total area} &= \pi r(l + r) = \pi r(\sqrt{r^2 + h^2}), \text{ and} \\
\text{Volume} &= \frac{1}{3} \pi r^2 h.
\end{align*}
\]

The implicit equation of this surface can be written \( x^2 + y^2 + z^2 = r^2 \); see also Section 4.20.

For a frustum of a right circular cone (Figure 4.17.37, bottom right),

\[
\begin{align*}
l &= \sqrt{(r_1 - r_2)^2 + h^2}, \\
\text{Lateral area} &= \pi (r_1 + r_2)l, \\
\text{Total area} &= \pi (r_1^2 + r_2^2 + (r_1 + r_2)l), \text{ and} \\
\text{Volume} &= \frac{1}{3} \pi h(r_1^2 + r_2^2 + r_1 r_2).
\end{align*}
\]

4.18 SPHERES

The set of points in space whose distance to a fixed point (the center) is a fixed positive number (the radius) is a sphere. A circle of radius \( r \) and center \((x_0, y_0, z_0)\) is defined
by the equation
\[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2,\]
or
\[x^2 + y^2 + z^2 - 2xx_0 - 2yy_0 - 2zz_0 + x_0^2 + y_0^2 + z_0^2 - r^2 = 0.\]

Conversely, an equation of the form
\[x^2 + y^2 + z^2 + 2dx + 2ey + 2fz + g = 0\]
defines a sphere if \(d^2 + e^2 + f^2 > g\); the center is \((-d, -e, -f)\) and the radius is \(\sqrt{d^2 + e^2 + f^2 - g}\).

Four points not on the same line determine a unique sphere. If the points have coordinates \((x_1, y_1, z_1)\), \((x_2, y_2, z_2)\), \((x_3, y_3, z_3)\) and \((x_4, x_4, z_4)\), the equation of the sphere is
\[
\begin{vmatrix}
  x^2 + y^2 + z^2 & x & y & z & 1 \\
  x_1^2 + y_1^2 + z_1^2 & x_1 & y_1 & z_1 & 1 \\
  x_2^2 + y_2^2 + z_2^2 & x_2 & y_2 & z_2 & 1 \\
  x_3^2 + y_3^2 + z_3^2 & x_3 & y_3 & z_3 & 1 \\
  x_4^2 + y_4^2 + z_4^2 & x_4 & y_4 & z_4 & 1
\end{vmatrix} = 0.
\]

Given two points \(P_1 = (x_1, y_1, z_1)\) and \(P_2 = (x_2, y_2, z_2)\), there is a unique sphere whose diameter is \(P_1P_2\); its equation is
\[(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0.\]

The area of a sphere of radius \(r\) is \(4\pi r^2\), and the volume is \(\frac{4}{3}\pi r^3\).

The area of a spherical polygon (that is, of a polygon on the sphere whose sides are arcs of great circles) is
\[S = \left(\sum_{i=1}^{n} \theta_i - (n - 2) - \pi\right)r^2,\]  \hspace{1cm} (4.18.1)
where \(r\) is the radius of the sphere, \(n\) is the number of vertices, and \(\theta_i\) are the internal angles of the polygons in radians. In particular, the sum of the angles of a spherical triangle is always greater than \(\pi = 180^\circ\), and the excess is proportional to the area.

**Spherical cap**

Let the radius be \(r\) (Figure 4.18.38, left). The area of the curved region is \(2\pi rh = \pi p^2\). The volume of the cap is \(\frac{1}{6}h^2(3r - h) = \frac{1}{6}\pi h(3a^2 + h^2)\).

**Spherical zone (of two bases)**

Let the radius be \(r\) (Figure 4.18.38, middle). The area of the curved region is \(2\pi rh\).
The volume of the zone is \(\frac{1}{3}\pi h(3a^2 + 3b^2 + h^2)\).
Spherical segment and lune
Let the radius be \( r \) (Figure 4.18.38, right). The area of the curved region (lune) is \( 2r^2\theta \), the angle being measured in radians. The volume of the segment is \( \frac{2}{3}r^3\theta \).

Spheres in arbitrary dimensions
If the volume of an \( n \)-dimensional sphere of radius \( r \) is \( V_n(r) \) and its surface area is \( S_n(r) \), then

\[
V_n(r) = \frac{2\pi r^2}{n} V_{n-2} = \frac{2\pi^{n/2}r^n}{n\Gamma\left(\frac{n}{2}\right)},
\]

\[
S_n(r) = \frac{n}{r} V_n = \frac{d}{dr}[V_n(r)].
\]

Hence, \( V_2 = \pi r^2 \), \( V_3 = \frac{4}{3}\pi r^3 \), \ldots, \( S_2 = 2\pi r \), \( S_3 = 4\pi r^2 \), \ldots.

4.19 SURFACES OF REVOLUTION: THE TORUS

A surface of revolution is formed by the rotation of a planar curve \( C \) about an axis in the plane of the curve and not cutting the curve. The Pappus–Guldinus theorem says that:

- The area of the surface of revolution on a curve \( C \) is equal to the product of the length of \( C \) and the length of the path traced by the centroid of \( C \) (which is \( 2\pi \) the distance from this centroid to the axis of revolution).
- The volume bounded by the surface of revolution on a simple closed curve \( C \) is equal to the product of the area bounded by \( C \) and the length of the path traced by the centroid of the area bounded by \( C \).

When \( C \) is a circle, the surface obtained is a circular torus or torus of revolution (Figure 4.19.39). Let \( r \) be the radius of the revolving circle and let \( R \) be the distance
from its center to the axis of rotation. The *area* of the torus is $4\pi^2 Rr$, and its *volume*

is $2\pi^2 Rr^2$.

**FIGURE 4.19.39**

*A torus of revolution.*

---

**4.20 QUADRICS**

A surface defined by an algebraic equation of degree two is called a *quadric*. Spheres, circular cylinders, and circular cones are quadrics. By means of a rigid motion, any quadric can be transformed into a quadric having one of the following equations (where $a, b, c \neq 0$):

1. Real ellipsoid: \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \]
2. Imaginary ellipsoid: \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1. \]
3. Hyperboloid of one sheet: \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1. \]
4. Hyperboloid of two sheets: \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1. \]
5. Real quadric cone: \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0. \]
6. Imaginary quadric cone: \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0. \]
7. Elliptic paraboloid: \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + 2z = 0. \]
8. Hyperbolic paraboloid: \[ \frac{x^2}{a^2} - \frac{y^2}{b^2} + 2z = 0. \]
9. Real elliptic cylinder: \[ \frac{x^2}{a^2} + y^2/b^2 = 1. \]
10. Imaginary elliptic cylinder: \[ \frac{x^2}{a^2} + y^2/b^2 = -1. \]
11. Hyperbolic cylinder: \[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \]
12. Real intersecting planes: \[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0. \]
13. Imaginary intersecting planes: \[ \frac{x^2}{a^2} + y^2/b^2 = 0. \]
14. Parabolic cylinder: \[ x^2 + 2y = 0. \]
15. Real parallel planes: \[ x^2 = 1. \]
16. Imaginary parallel planes: \[ x^2 = -1. \]
17. Coincident planes: \[ x^2 = 0. \]

Surfaces with Equations 9–17 are cylinders over the plane curves of the same equation (Section 4.16). Equations 2, 6, 10, and 16 have no real solutions, so that they
FIGURE 4.20.40
do not describe surfaces in real three-dimensional space. A surface with Equation 5 can be regarded as a cone (Section 4.17) over a conic $C$ (any ellipse, parabola or hyperbola can be taken as the directrix; there is a two-parameter family of essentially distinct cones over it, determined by the position of the vertex with respect to $C$). The surfaces with Equations 1, 3, 4, 7, and 8 are shown in Figure 4.20.40.

The surfaces with Equations 1–6 are central quadrics; in the form given, the center is at the origin. The quantities $a, b, c$ are the semiaxes.

The volume of the ellipsoid with semiaxes $a, b, c$ is $\frac{4}{3}\pi abc$. When two of the semiaxes are the same, we can also write the area of the ellipsoid in closed form. Suppose $b = c$, so the ellipsoid $\frac{x^2}{a^2} + \frac{y^2 + z^2}{b^2} = 1$ is the surface of revolution obtained by rotating the ellipse $x^2/a^2 + y^2/b^2 = 1$ around the $x$-axis. Its area is

\[
2\pi b^2 + \frac{2\pi a^2 b}{\sqrt{a^2 - b^2}} \sin^{-1} \frac{\sqrt{a^2 - b^2}}{a} = 2\pi b^2 + \frac{\pi a^2 b}{\sqrt{b^2 - a^2}} \log \left( b + \sqrt{b^2 - a^2} \right).
\]

The two quantities are equal, but only one avoids complex numbers, depending on whether $a > b$ or $a < b$. When $a > b$, we have a prolate spheroid, that is, an ellipse rotated around its major axis; when $a < b$ we have an oblate spheroid, which is an ellipse rotated around its minor axis.

Given a general quadratic equation in three variables,

\[
ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy + 2px + 2qy + 2rz + d = 0, \tag{4.20.1}
\]

one can determine the type of conic by consulting the table:

<table>
<thead>
<tr>
<th>$\rho_1$</th>
<th>$\rho_4$</th>
<th>$\Delta$</th>
<th>$k$ signs</th>
<th>$K$ signs</th>
<th>Type of quadric</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
<td>$&lt; 0$</td>
<td>Same</td>
<td>Real ellipsoid</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>$&gt; 0$</td>
<td>Opp</td>
<td>Imaginary ellipsoid</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>$&gt; 0$</td>
<td>Opp</td>
<td>Hyperboloid of one sheet</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>$&lt; 0$</td>
<td>Opp</td>
<td>Hyperboloid of two sheets</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>Same</td>
<td>Opp</td>
<td>Real quadric cone</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>Same</td>
<td>Same</td>
<td>Imaginary quadric cone</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>$&lt; 0$</td>
<td>Same</td>
<td>Elliptic paraboloid</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>$&gt; 0$</td>
<td>Opp</td>
<td>Hyperbolic paraboloid</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>Same</td>
<td>Opp</td>
<td>Real elliptic cylinder</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>Same</td>
<td>Same</td>
<td>Imaginary elliptic cylinder</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>Opp</td>
<td>Opp</td>
<td>Hyperbolic cylinder</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>Same</td>
<td>Opp</td>
<td>Real intersecting planes</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>Same</td>
<td>Opp</td>
<td>Real parallel planes</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>Opp</td>
<td>Opp</td>
<td>Parabolic cylinder</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>Same</td>
<td>Opp</td>
<td>Imaginary intersecting planes</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>Same</td>
<td>Opp</td>
<td>Imaginary parallel planes</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>Same</td>
<td>Opp</td>
<td>Coincident planes</td>
<td></td>
</tr>
</tbody>
</table>

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The columns have the following meaning. Let

\[
\begin{pmatrix}
a & h & g \\
h & b & f \\
g & f & c
\end{pmatrix}
\]
and

\[
\begin{pmatrix}
a & h & g & p \\
h & b & f & q \\
g & f & c & r \\
p & q & r & d
\end{pmatrix}
\].

Let \( \rho_1 \) and \( \rho_4 \) be the ranks of \( e \) and \( E \), and let \( \Delta \) be the determinant of \( E \). The column “\( k \) signs” refers to the nonzero eigenvalues of \( e \), that is, the roots of

\[
\begin{vmatrix}
a - x & h & g \\
h & b - x & f \\
g & f & c - x
\end{vmatrix} = 0;
\]

if all nonzero eigenvalues have the same sign, choose “same”, otherwise “opposite”. Similarly, “\( K \) signs” refers to the sign of the nonzero eigenvalues of \( E \).
## 4.21 Knots Up to Eight Crossings

<table>
<thead>
<tr>
<th>$n$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of knots With $n$ crossings</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>7</td>
<td>21</td>
<td>49</td>
<td>165</td>
<td>552</td>
</tr>
</tbody>
</table>

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4.22 DIFFERENTIAL GEOMETRY

4.22.1 CURVES

Definitions

A regular parametric representation of class \( C^k \), \( k \geq 1 \), is a vector valued function \( f : I \rightarrow \mathbb{R}^n \), where \( I \subset \mathbb{R} \) is an interval that satisfies (i) \( f \) is of class \( C^k \), and (ii) \( f'(t) \neq 0 \), for all \( t \in I \).

In terms of a standard basis of \( \mathbb{R}^3 \), we write \( x = f(t) = (f_1(t), f_2(t), f_3(t)) \), where the real valued functions \( f_i, i = 1, 2, 3 \) are the component functions of \( f \).

An allowable change of parameter of class \( C^k \) is any \( C^k \) function \( \phi : J \rightarrow I \), where \( J \) is an interval and \( \phi(J) \subset I \), that satisfies \( \phi'(\tau) \neq 0 \), for all \( \tau \in J \).

A \( C^k \) regular parametric representation \( f \) is equivalent to a \( C^k \) regular parametric representation \( g \) if and only if an allowable change of parameter \( \phi \) exists so that \( f(I_g) = I_f \), and \( g(\tau) = f(\phi(\tau)) \), for all \( \tau \in I_g \).

A regular curve \( C \) of class \( C^k \) is an equivalence class of \( C^k \) regular parametric representation under the equivalence relation on the set of regular parametric representations defined in Section 4.22.1.

The arc length of any regular curve \( C \) defined by the regular parametric representation \( f \), with \( I_f = [a, b] \), is defined by

\[
L = \int_a^b |f'(u)| \, du. 
\]  

(4.22.1)

An arc length parameter along \( C \) is defined by

\[
s = \alpha(t) = \pm \int_c^t |f'(u)| \, du. 
\]  

(4.22.2)

The choice of sign is arbitrary and \( c \) is any number in \( I_f \).

A natural representation of class \( C^k \) of the regular curve defined by the regular parametric representation \( f \) is defined by \( g(s) = f(\alpha^{-1}(s)) \), for all \( s \in [0, L] \).

A property of a regular curve \( C \) is any property of a regular parametric representation that defines \( C \) as invariant under any allowable change of parameter.

Let \( g \) be a natural representation of a regular curve \( C \). The following quantities may be defined at each point \( x = g(s) \) of \( C \):

- **Unit tangent vector** \( t(s) = \dot{g}(s) = \frac{dg}{ds} \)
- **Curvature vector** \( k(s) = t(s) \)
- **Principal normal** \( n(s) = \pm k(s)/|k(s)| \), for \( k(s) \neq 0 \) defined to be continuous along \( C \)
- **Unit binormal vector** \( b(s) = t(s) \times n(s) \)
- **Moving trihedron** \( \{t(s), n(s), b(s)\} \)
- **Curvature** \( \kappa(s) = n(s) \cdot k(s) \)
Radius of curvature  $\rho(s) = 1/|\kappa(s)|$, when $\kappa(s) \neq 0$

Torsion  $\tau(s) = -n(s) \cdot b(s)$

Tangent line  $y = \lambda t(s) + x$

Normal plane  $(y - x) \cdot t(s) = 0$

Principal normal line  $y = \lambda n(s) + x$

Rectifying plane  $(y - x) \cdot n(s) = 0$

Binormal line  $y = \lambda b(s) + x$

Osculating plane  $(y - x) \cdot b(s) = 0$

Osculating sphere  $c = x + \rho(s)n(s) - (\kappa(s)/(\kappa^2(s)\tau(s)))b(s)$, and $r^2 = \rho^2(s) + \kappa^2(s)/(\kappa^2(s)\tau^2(s))$

Results

The arc length $L$ and the arc length parameter $s$ of any regular parametric representation $f$ are invariant under any allowable change of parameter. Thus, $L$ is a property of the regular curve $C$ defined by $f$.

The arc length parameter satisfies $\frac{ds}{dt} = \alpha'(t) = \pm |f'(t)|$, which implies that $|f'(s)| = 1$, if and only if $t$ is an arc length parameter. Thus, arc length parameters are uniquely determined up to the transformation $s \rightarrow \tilde{s} = \pm s + s_0$, where $s_0$ is any constant.

The curvature, torsion, tangent line, normal plane, principal normal line, rectifying plane, binormal line, and osculating plane are properties of the regular curve $C$ defined by any regular parametric representation $f$.

If $x = f(t)$ is any regular representation of a regular curve $C$, the following results hold at point $f(t)$ of $C$:

$$|\kappa| = \frac{\left| x'' \times x' \right|}{|x'|^3}, \quad \tau = \frac{x' \cdot (x'' \times x''')}{|x' \times x''|^2}. \tag{4.22.3}$$

The vectors of the moving trihedron satisfy the Serret–Frenet equations

$$\vec{i} = \kappa \vec{n}, \quad \vec{n} = -\kappa \vec{t} + \tau \vec{b}, \quad \vec{b} = -\tau \vec{n}. \tag{4.22.4}$$

For any plane curve represented parametrically by $x = f(t) = (t, f(t), 0)$,

$$|\kappa| = \frac{\left| \frac{d^2x}{dt^2} \right|}{\left(1 + \left(\frac{dx}{dt}\right)^2\right)^{3/2}}. \tag{4.22.5}$$

Expressions for the curvature vector and curvature of a plane curve corresponding to different representations are given in the following table:

<table>
<thead>
<tr>
<th>Representation</th>
<th>$x = f(t), y = g(t)$</th>
<th>$y = f(x)$</th>
<th>$\tau = f(\theta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Curvature vector $k$</td>
<td>$\frac{(y - y)x - (x - x)y}{(x^2 + y^2)^{3/2}}$</td>
<td>$y''/(1 + y'^2)^{3/2}$</td>
<td>$\frac{r^2 + 2r^2 - r r''}{(r^2 + r^3)^2}$</td>
</tr>
<tr>
<td>Curvature $</td>
<td>\kappa</td>
<td>= \rho^{-1}$</td>
<td>$\frac{</td>
</tr>
</tbody>
</table>
The equation of the osculating circle of a plane curve is given by

\[(y - c) \cdot (y - c) = \rho^2,\]

where \(c = x + \rho^2 k\) is the center of curvature.

**THEOREM 4.22.1** (Fundamental existence and uniqueness theorem)

Let \(\kappa(s)\) and \(\tau(s)\) be any continuous functions defined for all \(s \in [a, b]\). Then there exists, up to a congruence, a unique space curve \(C\) for which \(\kappa\) is the curvature function, \(\tau\) is the torsion function, and \(s\) an arc length parameter along \(C\).

**Example**

A regular parametric representation of the circular helix is given by \(x = f(t) = (a \cos t, a \sin t, bt)\), for all \(t \in \mathbb{R}\), where \(a > 0\) and \(b \neq 0\) are constant. By successive differentiation,

\[x' = (-a \sin t, a \cos t, b),\]
\[x'' = (-a \cos t, -a \sin t, 0),\]
\[x''' = (a \sin t, -a \cos t, 0),\]

so that \(\frac{dx}{dt} = |x'| = \sqrt{a^2 + b^2}\). Hence,

- **Arc length parameter**: \(s = a(t) = t(a^2 + b^2)^{\frac{1}{2}}\)
- **Unit tangent vector**: \(\mathbf{t} = \frac{x}{|x'|} = (a^2 + b^2)^{-\frac{1}{2}}(-a \sin t, a \cos t, b)\)
- **Curvature vector**: \(\mathbf{k} = \frac{dx}{dt} = \frac{dx}{ds} \frac{ds}{dt} = (a^2 + b^2)^{-1}(-a \cos t, -a \sin t, 0)\)
- **Curvature**: \(\kappa = |\mathbf{k}| = a(a^2 + b^2)^{-1}\)
- **Principal normal unit vector**: \(\mathbf{n} = \mathbf{k}/|\mathbf{k}| = (-\cos t, -\sin t, 0)\)
- **Unit binormal vector**:

\[\mathbf{b} = \mathbf{t} \times \mathbf{n} = (a^2 + b^2)^{-\frac{1}{2}}(b \sin t, b \cos t, a)\]
\[\dot{\mathbf{b}} = \frac{dt}{ds} \frac{d\mathbf{b}}{dt} = b(a^2 + b^2)^{-1}(\cos t, \sin t, 0)\]

- **Torsion**: \(\tau = -\mathbf{n} \cdot \dot{\mathbf{b}} = b(a^2 + b^2)^{-1}\)

The values of |\(\kappa|\) and \(\tau\) can be verified using the formulae in (4.22.3). The sign of (the invariant) \(\tau\) determines whether the helix is right handed, \(\tau > 0\), or left handed, \(\tau < 0\).
4.22.2 SURFACES

Definitions

A coordinate patch of class $C^k$, $k \geq 1$ on a surface $S \subset \mathbb{R}^3$ is a vector valued function $f : U \rightarrow S$, where $U \subset \mathbb{R}^2$ is an open set, that satisfies (i) $f$ is class $C^k$ on $U$, (ii) $\frac{\partial f}{\partial u}(u, v) \times \frac{\partial f}{\partial v}(u, v) \neq 0$, for all $(u, v) \in U$, and (iii) $f$ is one-to-one and bi-continuous on $U$.

In terms of a standard basis of $\mathbb{R}^3$ we write $x = f(u, v) = (f_1(u, v), f_2(u, v), f_3(u, v))$, where the real valued functions $\{f_1, f_2, f_3\}$ are the component functions of $f$. The notation $x_1 = x_u = \frac{\partial f}{\partial u}(u, v)$, $x_2 = x_v = \frac{\partial f}{\partial v}(u, v)$, $u^1 = u$, $u^2 = v$, is frequently used.

A Monge patch is a coordinate patch where $f$ has the form $f(u, v) = (u, v, f(u, v))$, where $f$ is a real valued function of class $C^k$.

The $u$-parameter curves $v = v_0$ on $S$ are the images of the lines $v = v_0$ in $U$. They are parametrically represented by $x = f(u, v_0)$. The $v$-parameter curves $u = u_0$ are defined similarly.

An allowable parametric transformation of class $C^k$ is a one-to-one function $\phi : U \rightarrow V$, where $U, V \subset \mathbb{R}^2$ are open, that satisfies

$$\det \left[ \begin{array}{cc} \frac{\partial \phi^1}{\partial u}(u, v) & \frac{\partial \phi^1}{\partial v}(u, v) \\ \frac{\partial \phi^2}{\partial u}(u, v) & \frac{\partial \phi^2}{\partial v}(u, v) \end{array} \right] \neq 0,$$

for all $(u, v) \in U$, whose the real valued functions, $\phi^1$ and $\phi^2$, defined by $\phi(u, v) = [\phi^1(u, v), \phi^2(u, v)]$, are the component functions of $\phi$. One may also write the parametric transformation as $\tilde{u}^1 = \phi^1(u^1, u^2)$, $\tilde{u}^2 = \phi^2(u^1, u^2)$.

A local property of surface $S$ is any property of a coordinate patch that is invariant under any allowable parametric transformation.

Let $f$ define a coordinate patch on a surface $S$. The following quantities may be defined at each point $x = f(u, v)$ on the patch:

Normal vector $\mathbf{N} = x_u \times x_v$

Unit normal vector $\mathbf{N} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}$

Normal line $y = \lambda \mathbf{N} + \mathbf{x}$

Tangent plane $(y - x) \cdot \mathbf{N} = 0$, or $y = x + \lambda \mathbf{x}_u + \mu \mathbf{x}_v$

Fundamental differential $dx = x_u du^\alpha = x_u du + x_v dv$ (a repeated upper and lower index signifies a summation over the range $\alpha = 1, 2$)

First fundamental form $I = dx \cdot dx = g_{\alpha\beta}(u, v) du^\alpha du^\beta$

$$= E(u, v) du^2 + 2F(u, v) du dv + G(u, v) dv^2$$

First fundamental metric coefficients $\begin{cases} E(u, v) = g_{11}(u, v) = x_1 \cdot x_1 \\ F(u, v) = g_{12}(u, v) = x_1 \cdot x_2 \\ G(u, v) = g_{22}(u, v) = x_2 \cdot x_2 \end{cases}$

Second fundamental form $II = -d\mathbf{x} \cdot d\mathbf{N} = b_{\alpha\beta}(u, v) du^\alpha du^\beta$

$$= e(u, v) du^2 + 2f(u, v) du dv + g(u, v) dv^2$$

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Second fundamental metric coefficients
\[
\begin{align*}
  e(u, v) &= b_{11}(u, v) = x_{11} \cdot N \\
  f(u, v) &= b_{12}(u, v) = x_{12} \cdot N \\
  g(u, v) &= b_{22}(u, v) = x_{22} \cdot N
\end{align*}
\]

Normal curvature vector of curve C on S through x
\[\mathbf{k}_n = (\mathbf{k} \cdot \mathbf{N}) \mathbf{N}\]

Normal curvature in the \(du : dv\) direction
\[\kappa_n = \mathbf{k} \cdot \mathbf{N} = \frac{II}{I}\]

Dupin’s indicatrix
\[e^2 + 2f x_1 x_2 + gx_2^2 = \pm 1\]

Elliptic point
\[eg - f^2 > 0\]

Hyperbolic point
\[eg - f^2 < 0,\]

Parabolic point
\[eg - f^2 = 0\] not all of \(e, f, g = 0\)

Planar point
\[e = f = g = 0\]

Umbilical point
\[\kappa_n = \text{constant for all directions } du : dv\]

Principal directions
The perpendicular directions \(du : dv\) in which \(\kappa_n\) attains its extreme values

Principal curvatures
The extreme values \(\kappa_1\) and \(\kappa_2\) of \(\kappa_n\)

Line of curvature
A curve on S whose tangent line at each point coincides with an asymptotic direction

Asymptotic direction
A direction \(du : dv\) for which \(\kappa_n = 0\)

Asymptotic line
A curve on S whose tangent line at each point coincides with a principal direction

Gaussian curvature
\[K = \kappa_1\kappa_2 = \frac{eg - f^2}{EG - F^2}\]

Mean curvature
\[H = \frac{\kappa_1 + \kappa_2}{2} = \frac{gE + eG - 2fF}{2(EG - F^2)}\]

Geodesic curvature vector of a curve C on S through x
\[\mathbf{k}_g = \mathbf{k} - (\mathbf{k} \cdot \mathbf{N}) \mathbf{N} = [\ddot{u}^a + \Gamma^a_{\beta\gamma} \dot{u}^\beta \dot{u}^\gamma] x_a\]

where \(\Gamma^a_{\beta\gamma}\) denote the Christoffel symbols of the second kind for the metric \(g_{a\beta}\), defined in Section 5.10

Geodesic on S
A curve on S which satisfies
\[\mathbf{k}_g = 0\] at each point

**Results**

The tangent plane, normal line, first fundamental form, second fundamental form and all derived quantities thereof are local properties of any surface \(S\).

The transformation laws for the first and second fundamental metric coefficients and any allowable parametric transformation are given respectively by
\[
\tilde{g}_{a\beta} = g_{\gamma\delta} \frac{\partial u^\gamma}{\partial \tilde{u}^a} \frac{\partial u^\delta}{\partial \tilde{u}^\beta}, \quad \text{and} \quad \tilde{b}_{a\beta} = b_{\gamma\delta} \frac{\partial u^\gamma}{\partial \tilde{u}^a} \frac{\partial u^\delta}{\partial \tilde{u}^\beta}.
\]

Thus \(g_{a\beta}\) and \(b_{a\beta}\) are the components of type \((0, 2)\) tensors.

\[I \geq 0\] for all directions \(du : dv\); \(I = 0\) if and only if \(du = dv = 0\).
The angle $\theta$ between two tangent lines to $S$ at $x = f(u, v)$ defined by the directions $du : dv$ and $\delta u : \delta v$, is given by

$$
\cos \theta = \frac{g_{\alpha\beta} du^\alpha \delta u^\beta}{(g_{\alpha\beta} du^\alpha du^\beta)^{1/2} (g_{\alpha\beta} \delta u^\alpha \delta u^\beta)^{1/2}}.
$$

The angle between the $u$-parameter curves and the $v$-parameter curves is given by $\cos \theta = F(u, v)/(E(u, v)G(u, v))^{1/2}$. The $u$-parameter curves and $v$-parameter curves are orthogonal if and only if $F(u, v) = 0$.

The arc length of a curve $C$ on $S$, defined by $x = f(u^1(t), u^2(t)), a \leq t \leq b$, is given by

$$
L = \int_a^b \sqrt{g_{\alpha\beta}(u^1(t), u^2(t))\dot{u}^\alpha \dot{u}^\beta} \, dt,
$$

$$
= \int_a^b \sqrt{E(u(t), v(t))\dot{u}^2 + 2F(u(t), v(t))\dot{u} \dot{v} + G(u(t), v(t))\dot{v}^2} \, dt.
$$

The area of $S = f(U)$ is given by

$$
A = \iint_U \sqrt{\det(g_{\alpha\beta}(u^1, u^2))} \, du^1 \, du^2,
$$

$$
= \iint_U \sqrt{E(u, v)G(u, v) - F^2(u, v)} \, du \, dv.
$$

The principal curvatures are the roots of the characteristic equation, $\det(b_{\alpha\beta} - \lambda g_{\alpha\beta}) = 0$, which may be written as $\lambda^2 - b_{\alpha\beta}g^{\alpha\beta}\lambda + b/g = 0$, where $g^{\alpha\beta}$ is the inverse of $g_{\alpha\beta}$, $b = \det(b_{\alpha\beta})$, and $g = \det(g_{\alpha\beta})$. The expanded form of the characteristic equation is

$$(EG - F^2)\lambda^2 - (eG - 2fF + gE)\lambda + eg - f^2 = 0. \quad (4.22.6)$$

The principal directions $du : dv$ are obtained by solving the homogeneous equation,

$$
b_1_{\alpha} g_{2\beta} du^\alpha du^\beta - b_{2\alpha} g_{1\beta} du^\alpha du^\beta = 0,
$$

or

$$(eF - fE) du^2 + (eG - gE) du dv + (fG - gF) dv^2 = 0.
$$

Rodrigues formula: $du : dv$ is a principal direction with principal curvature $\kappa$ if, and only if, $dN + \kappa dx = 0$.

A point $x = f(u, v)$ on $S$ is an umbilical point if and only if there exists a constant $k$ such that $b_{\alpha\beta}(u, v) = kg_{\alpha\beta}(u, v)$. The principal directions at $x$ are orthogonal if $x$ is not an umbilical point.

The $u$- and $v$-parameter curves at any nonumbilical point $x$ are tangent to the principal directions if and only if $f(u, v) = F(u, v) = 0$. If $f$ defines a coordinate patch without umbilical points, the $u$- and $v$-parameter curves are lines of curvature if and only if $f = F = 0$. 

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If \( f = F = 0 \) on a coordinate patch, the principal curvatures are given by \( \kappa_1 = e/E, \kappa_2 = g/G \). It follows that the Gaussian and mean curvatures have the forms

\[
K = \frac{eg}{EG}, \quad \text{and} \quad H = \frac{1}{2} \left( \frac{e}{E} + \frac{g}{G} \right). \tag{4.22.7}
\]

The Gauss equation: \( x_{\alpha\beta} = \Gamma_{\alpha\beta}^r x_r + b_{\alpha\beta} \mathbf{N} \).

The Weingarten equation: \( \mathbf{N}_\alpha = -b_{\alpha\beta} g^\beta r x_r \).

The Gauss–Mainardi–Codazzi equations: \( b_{\alpha\beta} b_{\gamma\delta} - b_{\alpha\gamma} b_{\beta\delta} = R_{\alpha\beta\gamma\delta}, b_{\alpha\beta,\gamma} - b_{\alpha\gamma,\beta} + \Gamma_{\alpha\beta}^\delta b_{\delta\gamma} - \Gamma_{\alpha\gamma}^\delta b_{\delta\beta} = 0 \), where \( R_{\alpha\beta\gamma\delta} \) denotes the Riemann curvature tensor defined in Section 5.10.

**THEOREM 4.22.2** (Gauss’s theorema egregium)

The Gaussian curvature \( K \) depends only on the components of the first fundamental metric \( g_{\alpha\beta} \) and their derivatives.

**THEOREM 4.22.3** (Fundamental theorem of surface theory)

If \( g_{\alpha\beta} \) and \( b_{\alpha\beta} \) are sufficiently differentiable functions of \( u \) and \( v \) which satisfy the Gauss–Mainardi–Codazzi equations, \( \det(g_{\alpha\beta}) > 0, g_{11} > 0, \) and \( g_{22} > 0 \), then a surface exists with \( I = g_{\alpha\beta} du^\alpha du^\beta \) and \( II = b_{\alpha\beta} du^\alpha du^\beta \) as its first and second fundamental forms. This surface is unique up to a congruence.

**Examples**

A Monge patch for a paraboloid of revolution is given by \( x = f(u,v) = (u,v,u^2 + v^2) \), for all \((u,v) \in U = \mathbb{R}^2\). By successive differentiation one obtains \( x_u = (1,0,2u), x_v = (0,1,2v), x_{uu} = (0,0,2), x_{uv} = (0,0,0), \) and \( x_{vv} = (0,0,2) \).

- **Unit normal vector:** \( \mathbf{N} = (1 + 4u^2 + 4v^2)^{-\frac{1}{2}} (-2u, -2v, 1) \).
- **First fundamental coefficients:** \( E(u,v) = g_{11} (u,v) = 1 + 4u^2, \quad F(u,v) = g_{12} (u,v) = 4uv, \quad G(u,v) = g_{22} (u,v) = 1 + 4v^2 \).

- **First fundamental form:** \( I = (1 + 4u^2) du^2 + 8uv du dv + (1 + 4v^2) dv^2 \). Since \( F(u,v) = 0 \Rightarrow u = 0 \) or \( v = 0 \), it follows that the \( u \)-parameter curve \( v = 0 \) is orthogonal to any \( u \)-parameter curve, and the \( v \)-parameter curve \( u = 0 \) is orthogonal to any \( v \)-parameter curve. Otherwise the \( u \)- and \( v \)-parameter curves are not orthogonal.
- **Second fundamental coefficients:** \( e(u,v) = b_{11} (u,v) = 2(1 + 4u^2 + 4v^2)^{-\frac{1}{2}}, \quad f(u,v) = b_{12} (u,v) = 0, \quad g(u,v) = b_{22} (u,v) = 2(1 + 4u^2 + 4v^2)^{-\frac{1}{2}}. \)
- **Second fundamental form:** \( II = 2(1 + 4u^2 + 4v^2)^{-\frac{1}{2}} (du^2 + dv^2) \).

- **Classification of points:** \( e(u,v)g(u,v) = 4(1 + 4u^2 + 4v^2) > 0 \) implies that all points on \( S \) are elliptic points. The point \((0,0,0)\) is the only umbilical point.

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• **Equation for the principal directions:** \(uv du^2 + (v^2 - u^2) du dv + u v dv^2 = 0\) factors to read \((u du + v dv)(v du - u dv) = 0\).

• **Lines of curvature:** Integrate the differential equations, \(u dv + v dv = 0\), and \(v du - v du = 0\), to obtain, respectively, the equations of the lines of curvature, \(u^2 + v^2 = r^2\), and \(u/v = \cot \theta\), where \(r\) and \(\theta\) are constant.

• **Characteristic equation:** 
  
  \[(1 + 4u^2 + 4v^2)\lambda^2 - 4(1 + 2u^2 + 2v^2) (1 + 4u^2 + 4v^2)^{-\frac{1}{2}}\lambda + 4(1 + 4u^2 + 4v^2)^{-1} = 0.\]

• **Principal curvatures:**
  
  \(\kappa_1 = 2(1 + 4u^2 + 4v^2)^{-\frac{1}{2}}, \ \kappa_2 = 2(1 + 4u^2 + 4v^2)^{-\frac{1}{2}}\).
  
  The paraboloid of revolution may also be represented by \(x = \hat{F}(r, \theta) = (r \cos \theta, r \sin \theta, r^2)\). In this representation the \(r\)- and \(\theta\)-parameter curves are lines of curvature.

• **Gaussian curvature:** \(K = 4(1 + 4u^2 + 4v^2)^{-2}\).

• **Mean curvature:** \(H = 2(1 + 2u^2 + 2v^2)(1 + 4u^2 + 4v^2)^{-\frac{1}{2}}\).

#### 4.23 ANGLE CONVERSION

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5.1 DIFFERENTIAL CALCULUS

Limits
If \( \lim_{x \to a} f(x) = A \) and \( \lim_{x \to a} g(x) = B \) then

- \( \lim_{x \to a} (f(x) \pm g(x)) = A \pm B \)
- \( \lim_{x \to a} f(x)g(x) = AB \)
- \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{A}{B} \) (if \( B \neq 0 \))
- \( \lim_{x \to a} [f(x)]^{g(x)} = A^B \) (if \( A > 0 \))
- \( \lim_{x \to a} h(f(x)) = h(A) \)
  (if \( h \) continuous)
- if \( f(x) \leq g(x) \), then \( A \leq B \)
- if \( A = B \) and \( f(x) \leq h(x) \leq g(x) \), then \( \lim_{x \to a} h(x) = A \)

Examples:

- \( \lim_{x \to \infty} \left(1 + \frac{t}{x}\right)^x = e^t \)
- \( \lim_{x \to \infty} x^{1/x} = 1 \)
- \( \lim_{x \to \infty} \frac{(\log x)^p}{x^q} = 0 \) (if \( q > 0 \))
- \( \lim_{x \to 0} x^p |\log x|^q = 0 \) (if \( p > 0 \))
- \( \lim_{x \to 0} \frac{\sin ax}{x} = a \)
- \( \lim_{x \to 0} \frac{a^x - 1}{x} = \log a \)
- \( \lim_{x \to 0} \frac{\log(1 + x)}{x} = 1 \)
Derivatives

The derivative of the function \( f(x) \), written \( f'(x) \), is defined as

\[
\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{5.1.1}
\]

if the limit exists. If \( y = f(x) \), then \( \frac{dy}{dx} = f'(x) \). The \( n \)th derivative is

\[
y^{(n)} = \frac{d^n y}{dx^n} = \frac{d}{dx} \left( \frac{d^{n-1} y}{dx^{n-1}} \right) = \frac{d^n y}{dx^n} = f^{(n)}(x)
\]

Sometimes the fourth (or fifth) derivative is written \( y^{(iv)} \) (or \( y^{(v)} \)).

The partial derivative of \( f(x, y) \) with respect to \( x \), \( \frac{\partial f}{\partial x} \) or \( f_x \), is defined as

\[
\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \tag{5.1.2}
\]

Derivatives of common functions

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† Use the – sign when \( x > 0 \), use the + sign when \( x < 0 \).

Derivative formulae

Let \( u, v, w \) be functions of \( x \), and let \( a, c, \) and \( n \) be constants.

\[
\begin{align*}
\frac{d}{dx} (a) & = 0 \\
\frac{d}{dx} (x) & = 1 \\
\frac{d}{dx} (au) & = a \frac{du}{dx} \\
\frac{d}{dx} (u + v) & = \frac{du}{dx} + \frac{dv}{dx}
\end{align*}
\]
\[
\frac{d}{dx} (uv) = \frac{du}{dx} + \frac{dv}{dx}
\]
\[
\frac{d}{dx} (uvw) = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx}
\]
\[
\frac{d}{dx} \left( \frac{u}{v} \right) = -\frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}
\]
\[
\frac{d}{dx} (u^n) = nu^{n-1} \frac{du}{dx}
\]
\[
\frac{d}{dx} (\sqrt{u}) = \frac{1}{2\sqrt{u}} \frac{du}{dx}
\]
\[
\frac{d}{dx} \left( \frac{1}{u} \right) = -\frac{1}{u^2} \frac{du}{dx}
\]
\[
\frac{d}{dx} \left( \frac{1}{u^n} \right) = -\frac{n}{u^{n+1}} \frac{du}{dx}
\]
\[
\frac{d}{dx} \left( \frac{u^n}{v^m} \right) = \frac{u^{n-1}v^{m-1} \left( n \frac{du}{dx} - mu \frac{dv}{dx} \right)}{v^{m+1}}
\]
\[
\frac{d}{dx} (u^n v^m) = u^{n-1}v^{m-1} \left( n \frac{du}{dx} + mu \frac{dv}{dx} \right)
\]
\[
\frac{d}{dx} (f(u)) = \frac{df}{du} (u) \frac{du}{dx}
\]
\[
\frac{d^2}{dx^2} (f(u)) = \frac{d^2 f}{du^2} (u) \left( \frac{du}{dx} \right)^2 + \frac{df}{du} (u) \left( \frac{du}{dx} \right)^2
\]
\[
\frac{d^n}{dx^n} (uv) = \binom{n}{0} v \frac{d^nu}{dx^n} + \binom{n}{1} \frac{dv}{dx} \frac{d^{n-1}u}{dx^{n-1}} + \cdots + \binom{n}{n} \frac{d^nv}{dx^n}
\]
\[
\frac{d}{dx} \int_c^x f(t) dt = f(x)
\]
\[
\frac{d}{dx} \int_c^x f(t) dt = -f(x)
\]
\[
\frac{dx}{dy} = \left( \frac{dy}{dx} \right)^{-1} \quad \text{and} \quad \frac{d^2 x}{dy^2} = -\frac{d^2 y}{d^2 x} = \left( \frac{dy}{dx} \right)^3
\]
\[
\text{If } F(x, y) = 0, \text{ then } \frac{dy}{dx} = -\frac{F_x}{F_y} \quad \text{and} \quad \frac{d^2 y}{d^2 x} = -\frac{(F_{xx} F_y^2 - 2F_{xy} F_x F_y + F_{yy} F_x^2)}{F_y^3}
\]
\[
\text{Leibniz’s rule gives the derivative of an integral:}
\]
\[
\frac{d}{dx} \left( \int_{f(x)}^{g(x)} h(x, t) dt \right) = g'(x)h(x, g(x)) - f'(x)h(x, f(x)) + \int_{f(x)}^{g(x)} \frac{\partial h}{\partial x} (x, t) dt.
\]
\[
\text{If } x = x(t) \text{ and } y = y(t) \text{ then (the dots denote differentiation with respect to } t):\]
\[
\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}, \quad \frac{d^2 y}{dx^2} = \frac{\ddot{y} - \ddot{x}}{(\dot{x})^2}
\]

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Derivative theorems

- **Fundamental theorem of calculus:**
  Suppose \( f \) is continuous on \([a, b]\).

  - If \( G \) is defined as \( G(x) = \int_a^x f(t) \, dt \) for all \( x \) in \([a, b]\), then \( G \) is an antiderivative of \( f \) on \([a, b]\).

  - If \( F \) is any antiderivative of \( f \), then \( \int_a^b f(t) \, dt = F(b) - F(a) \). (Recall that \( \int_a^b f(x) \, dx \) is defined as a limit of Riemann sums.)

- **Intermediate value theorem:**
  If \( f(x) \) is continuous on \([a, b]\) and if \( f(a) \neq f(b) \), then \( f \) takes on every value between \( f(a) \) and \( f(b) \) in the interval \((a, b)\).

- **Rolle’s theorem:**
  If \( f(x) \) is continuous on \([a, b]\) and differentiable on \((a, b)\), and if \( f(a) = f(b) \), then \( f'(c) = 0 \) for at least one number \( c \) in \((a, b)\).

- **Mean value theorem:**
  If \( f(x) \) is continuous on \([a, b]\) and differentiable on \((a, b)\), then a number \( c \) exists in \((a, b)\) such that \( f(b) - f(a) = (b - a)f'(c) \).

The chain rule

If \( x = x(t) \), \( y = y(t) \), and \( z = z(x, y) \), then

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt},
\]

and

\[
\frac{d^2z}{dt^2} = \frac{\partial^2 z}{\partial x^2} \left( \frac{dx}{dt} \right)^2 + \frac{\partial^2 z}{\partial x \partial y} \frac{dx}{dt} \frac{dy}{dt} + \frac{\partial^2 z}{\partial y^2} \left( \frac{dy}{dt} \right)^2 + \frac{\partial^2 z}{\partial y \partial x} \frac{dx}{dt} \frac{dy}{dt}.
\]

If \( x = x(u, v) \), \( y = y(u, v) \), and \( z = z(x, y) \), then

\[
\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u},
\]
and

\[
\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}.
\]

If \( u = u(x, y) \), \( v = v(x, y) \), and \( f = f(x, y) \), then the partial derivative of \( f \) with respect to \( u \), holding \( v \) constant, written \( \left( \frac{\partial f}{\partial u} \right)_v \), can be expressed as

\[
\left( \frac{\partial f}{\partial u} \right)_v = \left( \frac{\partial f}{\partial x} \right)_y \left( \frac{\partial x}{\partial u} \right)_v + \left( \frac{\partial f}{\partial y} \right)_x \left( \frac{\partial y}{\partial u} \right)_v.
\] (5.1.3)

If \( x, y, \) and \( z \) all depend on one another (say, through \( f(x, y, z) = 0 \)), then the
partial derivative of \( x \) with respect to \( y \), holding \( z \) constant, written \( \left( \frac{\partial x}{\partial y} \right)_z \), is

\[
\left( \frac{\partial x}{\partial y} \right)_z = \left( \frac{\partial y}{\partial x} \right)_z^{-1} = -\frac{(\partial f/\partial y)_{x,z}}{(\partial f/\partial x)_{y,z}},
\]

\[
\left( \frac{\partial x}{\partial y} \right)_z \left( \frac{\partial y}{\partial z} \right)_x \left( \frac{\partial z}{\partial x} \right)_y = -1.
\] (5.1.4)

### L'Hôpital's rule

If \( f(x) \) and \( g(x) \) are continuous in the neighborhood of point \( a \), and if \( f(x) \) and \( g(x) \) both tend to 0 or \( \infty \) as \( x \to a \), then

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}
\] (5.1.5)

if the right hand side exists. For example

\[
\lim_{x \to 0} \frac{x - \sin x}{x^3} = \lim_{x \to 0} \frac{1 - \cos x}{3x^2} = \lim_{x \to 0} \frac{\sin x}{6x} = \lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6},
\]

and

\[
\lim_{x \to \infty} \frac{x^n}{e^x} = \lim_{x \to \infty} \frac{nx^{n-1}}{e^x} = \lim_{x \to \infty} \frac{n(n-1)x^{n-2}}{e^x} = \cdots = \lim_{x \to \infty} \frac{n!}{e^x} = 0.
\]

### 5.1.1 MAXIMA AND MINIMA OF FUNCTIONS

- If a function \( f(x) \) has a local extremum at a number \( c \), then either \( f'(c) = 0 \) or \( f'(c) \) does not exist.
- If \( f'(c) = 0 \), \( f(x) \) is differentiable on an open interval containing \( c \), and
  - if \( f''(c) < 0 \), then \( f \) has a local maximum at \( c \);
  - if \( f''(c) > 0 \), then \( f \) has a local minimum at \( c \).

### Lagrange multipliers

To extremize the function \( f(x_1, x_2, \ldots, x_n) = f(x) \) subject to the \( n - 1 \) side constraints \( g(x) = 0 \), introduce an \( (n - 1) \)-dimensional vector of Lagrange multipliers \( \lambda \) and define \( F(x, \lambda) = f(x) + \lambda^T g(x) \). Then extremize \( F \) with respect to all of its arguments:

\[
\frac{\partial F}{\partial x_i} = \frac{\partial f}{\partial x_i} + \lambda^T \frac{\partial g}{\partial x_i} = 0, \quad \text{and} \quad \frac{\partial F}{\partial \lambda_j} = g_j = 0.
\]

For the simplest case, extremizing \( f(x, y) \) subject to \( g(x, y) = 0 \),

\[
f_x + \lambda g_x = 0, \quad f_y + \lambda g_y = 0, \quad \text{and} \quad g = 0.
\]
For example, finding the points on the unit circle, \( g(x, y) = (x - 1)^2 + (y - 2)^2 - 1 = 0 \), that are closest and furthest from the origin, the distance squared is \( f(x, y) = x^2 + y^2 \), can be determined by solving the three (nonlinear) algebraic equations:

\[
\begin{align*}
2x + 2\lambda(x - 1) &= 0, \\
2y + 2\lambda(y - 2) &= 0, \\
(x - 1)^2 + (y - 2)^2 &= 0.
\end{align*}
\]

The solutions are \( x = 1 + \sqrt{5}, y = 2 + 2/\sqrt{5}, \lambda = -1 - \sqrt{5} \) (furthest), and \( x = 1 - \sqrt{5}, y = 2 - 2/\sqrt{5}, \lambda = \sqrt{5} - 1 \) (closest).

### 5.1.2 VECTOR CALCULUS

In rectilinear coordinates, \( \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \). If \( u \) and \( v \) are scalars and \( F \) and \( G \) are vectors, then

\[
\nabla(u + v) = \nabla u + \nabla v, \\
\nabla(uv) = u\nabla v + v\nabla u, \\
\nabla(F + G) = \nabla F + \nabla G, \\
\nabla(F \cdot G) = (F \cdot \nabla)G + (G \cdot \nabla)F + F \times (\nabla \times G) + G \times (\nabla \times F), \\
\n\nabla \cdot \nabla u = u(\nabla \cdot \nabla) + \nabla \times \nabla \times F, \\
\n\nabla \times \nabla \times F = \nabla(\nabla \cdot \nabla) - \nabla^2 F, \\
\n\nabla \cdot (F \times G) = F(\nabla \cdot G) - G(\nabla \cdot F) - (\nabla \cdot F) + (\nabla \cdot G)F, \\
\n\nabla \times (F \times G) = F(\nabla \cdot G) - G(\nabla \cdot F) + (\nabla \cdot F) - (\nabla \cdot G)F.
\]

A vector field \( F \) is irrotational if \( \nabla \times F = 0 \). A vector field \( F \) is solenoidal if \( \nabla \cdot F = 0 \).

If \( r = |r|, a \) is a constant vector, and \( n \) is an integer, then

\[
\begin{array}{|c|c|c|}
\hline
\Phi & \nabla \Phi & \nabla^2 \Phi \\
\hline
a \cdot r & \frac{a}{r^n} & n\frac{a}{r^{n+2}} \\
\log r & \frac{a}{r^2} & \frac{a}{r^2} \\
\hline
\end{array}
\]

<table>
<thead>
<tr>
<th>( F )</th>
<th>( \nabla \cdot F )</th>
<th>( \nabla \times F )</th>
<th>( (\nabla \cdot F) )</th>
<th>( (\nabla \times F) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>r</td>
<td>3</td>
<td>0</td>
<td>2a</td>
<td>a \times G</td>
</tr>
<tr>
<td>a \times r</td>
<td>0</td>
<td>0</td>
<td>2a</td>
<td>a \times G</td>
</tr>
<tr>
<td>ar^n</td>
<td>nr^{n-2}(r \cdot a)</td>
<td>nr^{n-2}(r \times a)</td>
<td>nr^{n-2}(r \cdot G)</td>
<td>nr^{n-2}(r \cdot G)</td>
</tr>
<tr>
<td>rr^n</td>
<td>(n + 3)r^n</td>
<td>0</td>
<td>r^nG + nr^{n-2}(r \cdot G)</td>
<td>r^nG + nr^{n-2}(r \cdot G)</td>
</tr>
<tr>
<td>a \log r</td>
<td>r \cdot a/r^2</td>
<td>r \times a/r^2</td>
<td>(\nabla \cdot F)</td>
<td>(G \cdot r)a/r^2</td>
</tr>
</tbody>
</table>

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\[
\begin{array}{|c|c|c|}
\hline
F & \nabla^2 F & \nabla \cdot F \\
\hline
\mathbf{r} & 0 & 0 \\
\mathbf{a} \times \mathbf{r} & 0 & n r^{n-2} \mathbf{a} \\
\mathbf{a} r^n & n(n+1) r^{n-2} \mathbf{a} + n(n-2) r^{n-4} (\mathbf{r} \cdot \mathbf{a}) \mathbf{r} & n(n-2) r^n \\
\mathbf{a} \log r & \frac{\mathbf{a}}{r^2} & \frac{[r^2 \mathbf{a} - 2(\mathbf{r} \cdot \mathbf{a}) \mathbf{r}]}{r^4} \\
\hline
\end{array}
\]

\[
\frac{d}{dt}(\mathbf{F} + \mathbf{G}) = \frac{d\mathbf{F}}{dt} + \frac{d\mathbf{G}}{dt}, \\
\frac{d}{dt}(\mathbf{F} \cdot \mathbf{G}) = \mathbf{F} \cdot \frac{d\mathbf{G}}{dt} + \frac{d\mathbf{F}}{dt} \cdot \mathbf{G}, \\
\frac{d}{dt}(\mathbf{F} \times \mathbf{G}) = \mathbf{F} \times \frac{d\mathbf{G}}{dt} + \frac{d\mathbf{F}}{dt} \times \mathbf{G}, \\
\frac{d}{dt}[\mathbf{V}_1 \mathbf{V}_2 \mathbf{V}_3] = \left[ \frac{d\mathbf{V}_1}{dt} \right] \mathbf{V}_2 \mathbf{V}_3 + \mathbf{V}_1 \left( \frac{d\mathbf{V}_2}{dt} \right) \mathbf{V}_3 + \mathbf{V}_1 \mathbf{V}_2 \left( \frac{d\mathbf{V}_3}{dt} \right), \\
\frac{d}{dt}(\mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3) = \left( \frac{d\mathbf{V}_1}{dt} \right) \times (\mathbf{V}_2 \times \mathbf{V}_3) + \mathbf{V}_1 \times \left( \left( \frac{d\mathbf{V}_2}{dt} \right) \times \mathbf{V}_3 \right) + \mathbf{V}_1 \times \left( \mathbf{V}_2 \times \left( \frac{d\mathbf{V}_3}{dt} \right) \right),
\]

where \([\mathbf{V}_1 \mathbf{V}_2 \mathbf{V}_3] = \mathbf{V}_1 \cdot (\mathbf{V}_2 \times \mathbf{V}_3)\) is the scalar triple product.

### 5.1.3 MATRIX AND VECTOR DERIVATIVES

#### Definitions

1. The derivative of the row vector \(\mathbf{y}\) with respect to the scalar \(x\) is
   \[
   \frac{\partial \mathbf{y}}{\partial x} = \begin{bmatrix}
   \frac{\partial y_1}{\partial x} \\
   \frac{\partial y_2}{\partial x} \\
   \vdots \\
   \frac{\partial y_m}{\partial x}
\end{bmatrix}.
   \]

2. The derivative of a scalar \(y\) with respect to the vector \(\mathbf{x}\) is
   \[
   \frac{\partial y}{\partial \mathbf{x}} = \begin{bmatrix}
   \frac{\partial y}{\partial x_1} \\
   \frac{\partial y}{\partial x_2} \\
   \vdots \\
   \frac{\partial y}{\partial x_n}
\end{bmatrix}.
   \]

3. Let \(\mathbf{x}\) be a \(n \times 1\) vector and let \(\mathbf{y}\) be a \(m \times 1\) vector. The derivative of \(\mathbf{y}\) with respect to \(\mathbf{x}\) is the matrix
   \[
   \frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{bmatrix}
   \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_1} \\
   \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_2} \\
   \vdots & \vdots & \ddots & \vdots \\
   \frac{\partial y_1}{\partial x_n} & \frac{\partial y_2}{\partial x_n} & \cdots & \frac{\partial y_m}{\partial x_n}
\end{bmatrix}.
   \]

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In multivariate analysis, if \( x \) and \( y \) have the same length, then the absolute value of the determinant of \( \frac{\partial y}{\partial x} \) is called the Jacobian of the transformation determined by \( y = y(x) \).

4. The derivative of the matrix \( A(t) = (a_{ij}(t)) \), with respect to the scalar \( t \), is the matrix \( \frac{dA(t)}{dt} = \left( \frac{da_{ij}(t)}{dt} \right) \).

5. If \( X = (x_{ij}) \) is a \( n \times m \) matrix and if \( y \) is a scalar function of \( X \) given by \( y = f(X) \), then the derivative of \( y \) with respect to \( X \) is

\[
\frac{\partial y}{\partial X} = \sum_{ij} E_{ij} \frac{\partial y}{\partial x_{ij}}.
\]

6. If \( Y = (y_{ij}) \) is a \( p \times q \) matrix and \( X \) is a \( m \times n \) matrix, then the derivative of \( Y \) with respect to \( X \) is

\[
\frac{\partial Y}{\partial X} = \sum_{rs} E_{rs} \otimes \frac{\partial Y}{\partial x_{rs}}.
\]

**Properties**

<table>
<thead>
<tr>
<th>( y ) (scalar or a vector)</th>
<th>( \frac{\partial y}{\partial X} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>( A )</td>
</tr>
<tr>
<td>( x^T A )</td>
<td>( A^T )</td>
</tr>
<tr>
<td>( x^T x )</td>
<td>( 2x )</td>
</tr>
<tr>
<td>( x^T A x )</td>
<td>( A x + A^T x )</td>
</tr>
</tbody>
</table>

2. If the matrices \( A \) and \( B \) can be multiplied, then \( \frac{d(AB)}{dt} = \frac{dA}{dt} B + A \frac{dB}{dt} \).

3. If \( C = A \otimes B \), then \( \frac{dC}{dt} = \frac{dA}{dt} \otimes B + A \otimes \frac{dB}{dt} \).

4. If \( z = y(x) \), then \( \frac{\partial z}{\partial y} = \frac{\partial y}{\partial X} \frac{\partial y}{\partial z} \).

5. If \( y = \text{tr} \), then \( \frac{\partial y}{\partial X} = I_n \).

6. The derivative of the determinant of a matrix can be written:
   - If \( Y_{ij} \) is the cofactor of element \( y_{ij} \) in \( |Y| \), then \( \frac{\partial |Y|}{\partial y_{rs}} = \sum_i \sum_j Y_{ij} \frac{\partial y_{ij}}{\partial y_{rs}} \).
   - If all of the components \( (x_{ij}) \) of \( X \) are independent, then \( \frac{\partial |X|}{\partial X} = |X| (X^{-1})^T \).
   - If \( X \) is a symmetric matrix, then \( \frac{\partial |X|}{\partial X} = 2X - \sum_i x_{ii} \).

7. \( \frac{\partial (AXB)}{\partial x_{rs}} = AE_{rs}B \) where \( E_{rs} \) is an elementary matrix of the same size as \( X \).

8. If \( Y = AX^T B \), then \( \frac{\partial y_{ij}}{\partial X} = BE^T_{ij} A \).
9. If \( Y = AX^{-1}B \), then
   \[ \frac{\partial Y}{\partial x_{rs}} = -AX^{-1}E_{rs}X^{-1}B. \]
   \[ \frac{\partial y_{ij}}{\partial x} = -(X^{-1})^T A^T E_{ij} B^T (X^{-1})^T. \]

10. If \( Y = X^T AX \), then
    \[ \frac{\partial Y}{\partial x_{rs}} = E_{rs}^T AX + X^T A E_{rs}. \]
    \[ \frac{\partial y_{ij}}{\partial x} = A X E_{ij} + A^T X E_{ij}. \]

11. Derivatives of powers of matrices are obtained as follows:
    \[ \begin{align*}
    &\text{If } Y = X^r, \quad \frac{\partial Y}{\partial x_{rs}} = \sum_{k=0}^{r-1} X^k E_{rs} X^{n-k-1}. \\
    &\text{If } Y = X^{-r}, \quad \frac{\partial Y}{\partial x_{rs}} = -X^{-r} \left( \sum_{k=0}^{r-1} X^k E_{rs} X^{n-k-1} \right) X^{-r}.
    \end{align*} \]

12. If \( y = \text{Vec} \ Y \) and \( x = \text{Vec} \ X \), then
    \[ \begin{align*}
    &\text{If } Y = AX, \quad \frac{\partial y}{\partial x} = I \otimes A^T. \\
    &\text{If } Y = XA, \quad \frac{\partial y}{\partial x} = A \otimes I. \\
    &\text{If } Y = AX^{-1}B, \quad \frac{\partial y}{\partial x} = -(X^{-1} B) \otimes (X^{-1})^T A^T.
    \end{align*} \]

13. Derivative formulae:
    \[ \begin{align*}
    &\frac{\partial \log |X|}{\partial x} = (X^{-1})^T. \\
    &\frac{\partial \text{tr}(AX)}{\partial x} = A^T. \\
    &\frac{\partial \text{tr}(XA)}{\partial x} = A. \\
    &\frac{\partial \text{tr}(XAXB)}{\partial x} = AXB + A^T XB^T. \\
    &\text{If } X \text{ and } Y \text{ are matrices, then } \left( \frac{\partial y}{\partial x} \right)^T = \frac{\partial y^T}{\partial x^T}. \\
    &\text{If } X, Y, \text{ and } Z \text{ are matrices of size } m \times n, n \times v, \text{ and } p \times q, \text{ then } \frac{\partial (XY)}{\partial Z} = \frac{\partial X}{\partial Z} (I_q \otimes Y) + (I_p \otimes X) \frac{\partial Y}{\partial Z}. 
    \end{align*} \]

5.2 Differential Forms

For the vector \( \mathbf{a} = (a_1, \ldots, a_k, \ldots, a_n) \), define \( dx_k \) to be the function that assigns to the vector \( \mathbf{a} \) its \( k \)th coordinate (that is, \( dx_k(\mathbf{a}) = a_k \)). Geometrically, \( dx_k(\mathbf{a}) \) is the length, with appropriate sign, of the projection of \( \mathbf{a} \) on the \( k \)th coordinate axis. When \( \{F_i\} \) are functions, the linear combination of the functions \( \{dx_k\} \)
\[ \omega_\mathbf{x} = F_1(\mathbf{x}) \, dx_1 + F_2(\mathbf{x}) \, dx_2 + \cdots + F_n(\mathbf{x}) \, dx_n \quad (5.2.1) \]
produces a new function $\omega_x$. This function acts on vectors $a$ as

$$
\omega_x(a) = F_1(x) \, dx_1(a) + F_2(x) \, dx_2(a) + \cdots + F_n(x) \, dx_n(a).
$$

Such a function is a “differential 1-form” or a “1-form”. For example:

1. If $a = (-2, 0, 4)$ then $dx_1(a) = -2$, $dx_2(a) = 0$, and $dx_3(a) = 4$.
2. If in $\mathbb{R}^2$, $\omega_x = \omega(x,y) = x^2 \, dx + y^2 \, dy$, then $\omega_{(x,y)}(a, b) = ax^2 + by^2$ and $\omega_{(1,-3)}(a, b) = a + 9b$.
3. If $f(x)$ is a differentiable function, then $\nabla_x f$, the differential of $f$ at $x$, is a 1-form. Note that $\nabla_x f$ acting on $a = (a_1, a_2, a_3)$ is

$$
\nabla_x f(a) = \frac{\partial f}{\partial x_1}(x) \, dx_1(a) + \frac{\partial f}{\partial x_2}(x) \, dx_2(a) + \frac{\partial f}{\partial x_3}(x) \, dx_3(a),
$$

$$
= \frac{\partial f}{\partial x_1}(x)a_1 + \frac{\partial f}{\partial x_2}(x)a_2 + \frac{\partial f}{\partial x_3}(x)a_3.
$$

### Products of 1-forms

The basic 1-forms in $\mathbb{R}^3$ are $dx_1$, $dx_2$, and $dx_3$. The wedge product $dx_1 \wedge dx_2$ is defined so that it is a function of ordered pairs in $\mathbb{R}^2$. Geometrically, $dx_1 \wedge dx_2(a, b)$ will be the area of the parallelogram spanned by the projections of $a$ and $b$ into the $(x_1, x_2)$-plane. The sign of the area is determined so that if the projections of $a$ and $b$ have the same orientation as the positive $x_1$ and $x_2$ axes, then the area is positive; it is negative when these orientations are opposite. Thus, if $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$, then

$$
dx_1 \wedge dx_2(a, b) = \det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = a_1 b_2 - a_2 b_1,
$$

and the determinant automatically gives the correct sign. This generalizes to

$$
dx_i \wedge dx_j(a, b) = \det \begin{bmatrix} dx_i(a) & dx_i(b) \\ dx_j(a) & dx_j(b) \end{bmatrix} = \det \begin{bmatrix} a_i & b_i \\ a_j & b_j \end{bmatrix}.
$$

(5.2.2)

- If $\omega$ and $\mu$ are 1-forms, and $f$ and $g$ are real-valued functions, then $f \omega + g \mu$ is a 1-form.

- If $\omega$, $\nu$, and $\mu$ are 1-forms, then $(f \omega + g \nu) \wedge \mu = f \omega \wedge \mu + g \nu \wedge \mu$.

- $dx_i \wedge dx_j = -dx_j \wedge dx_i$

- $dx_i \wedge dx_i = 0$

- $dx_i \wedge dx_j(b, a) = -dx_i \wedge dx_j(a, b)$
Differential 2-forms

In \( \mathbb{R}^3 \), the most general linear combination of the functions \( dx_i \wedge dx_j \) has the form
\[
c_1 dx_2 \wedge dx_3 + c_2 dx_3 \wedge dx_1 + c_3 dx_1 \wedge dx_2.
\]
If \( F = (F_1, F_2, F_3) \) is a vector field, then the function of ordered pairs,
\[
r_X(a, b) = F_1(x) dx_2 \wedge dx_3 + F_2(x) dx_3 \wedge dx_1 + F_3(x) dx_1 \wedge dx_2 \quad (5.2.3)
\]
is a “differential 2-form” or “2-form”.

1. For the specific 2-form \( r_X = 2 dx_2 \wedge dx_3 + dx_3 \wedge dx_1 + 5 dx_1 \wedge dx_2 \), if \( a = (1, 2, 3) \) and \( b = (0, 1, 1) \), then
\[
r_X(a, b) = 2 \det \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} + \det \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} + 5 \det \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}
= 2 \cdot (-1) + 1 \cdot (-1) + 5 \cdot (1) = 2
\]

independent of \( x \). Note that \( a \times b = \det \begin{bmatrix} i & j & k \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} = (-1, -1, 1) \), and so
\[
r_X(a, b) = (2, 1, 5) \cdot (a \times b).
\]

2. When changing from Cartesian coordinates to polar coordinates, the element of area \( dA \) can be written
\[
dA = dx \wedge dy
= (-r \sin \theta \, d\theta + \cos \theta \, dr) \wedge (r \cos \theta \, d\theta + \sin \theta \, dr)
= -r^2 \sin \theta \cos \theta \, d\theta \wedge dr + \sin \theta \cos \theta \, dr \wedge d\theta
- r \sin^2 \theta \, d\theta \wedge dr + r \cos^2 \theta \, dr \wedge d\theta
= r \, dr \wedge d\theta
\]

The 2-forms in \( \mathbb{R}^n \)

Every 2-form can be written in terms of a set of “basic 2-forms”. For example, in \( \mathbb{R}^2 \) there is only one basic 2-form (which may be taken to be \( dx_1 \wedge dx_2 \)) and in \( \mathbb{R}^3 \) there are 3 basic 2-forms (possibly the set \( \{dx_1 \wedge dx_2, dx_2 \wedge dx_3, dx_3 \wedge dx_1 \} \)). The exterior product of any two 1-forms (in, say, \( \mathbb{R}^n \)) is found by multiplying the 1-forms as if there were ordinary polynomials in the variables \( dx_1, \ldots, dx_n \), and then simplifying using the rules for \( dx_i \wedge dx_j \).

For example, denoting the basic 1-forms in \( \mathbb{R}^3 \) as \( dx, dy, \) and \( dz \) then
\[
(x \, dx + y^2 \, dy) \wedge (dx + x \, dy) = x \, dx \wedge dx + y^2 \, dy \wedge dx,
+ x^2 \, dx \wedge dy + xy^2 \, dy \wedge dy,
= 0 - y^2 \, dx \wedge dy + x^2 \, dx \wedge dy + 0,
= (x^2 - y^2) \, dx \wedge dy.
\]
Higher dimensional forms

The meaning of the basic 3-form \(dx_1 \wedge dx_2 \wedge dx_3\) is that of a signed volume function. Thus, if \(a = (a_1, a_2, a_3)\), \(b = (b_1, b_2, b_3)\), and \(c = (c_1, c_2, c_3)\) then

\[
dx_1 \wedge dx_2 \wedge dx_3(a, b, c) = \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\]

which is a 3-dimensional oriented volume of the parallelepiped defined by the vectors \(a, b,\) and \(c\).

For an ordered \(p\)-tuple of vectors in \(\mathbb{R}^n\) \((a_1, a_2, \ldots, a_p)\), where \(p \geq 1\)

\[
dx_k \wedge dx_{k+1} \wedge \cdots \wedge dx_p(a_1, \ldots, a_p) = \det(dx_k(a_j))\] \[i = 1, \ldots, p \] (5.2.4)

This equation defines the basic \(p\)-forms in \(\mathbb{R}^n\), of which the general \(p\)-forms are linear combinations. Properties include:

1. The interchange of adjacent factors in a basic \(p\)-form changes the sign of the form.
2. A basic \(p\)-form with a repeated factor is zero.
3. The general \(p\)-form can be written \(\omega^p = \sum_{i_1 < \cdots < i_p} \omega_{i_1 \cdots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}\), where \(1 \leq i_k \leq n\) for \(k = 1, \ldots, p\). This sum has \(\binom{n}{p}\) distinct nonzero terms in it.
4. If \(p > n\), then \(\omega^p\) is identically zero.
5. If \(\omega^p\) is a \(p\)-form in \(\mathbb{R}^n\) and \(\omega^q\) is a \(q\)-form in \(\mathbb{R}^n\), then \(\omega^p \wedge \omega^q = (-1)^{pq} \omega^q \wedge \omega^p\).

The exterior derivative

The exterior differentiation operator is denoted by \(d\). When \(d\) is applied to a scalar function \(f(x)\), the result is the 1-form that is equivalent to the usual “total differential” \(df = \sum \frac{\partial f}{\partial x_k} dx_k\). For the 1-form \(\omega^1 = f_1 dx + \cdots + f_n dx_n\) the exterior derivative is \(d\omega^1 = (df_1) \wedge dx_1 + \cdots + (df_n) \wedge dx_n\). This generalizes to higher dimensional forms.

1. If \(f(x_1, x_2) = x_1^2 + x_2^2\), then \(df = d(x_1^2 + x_2^2) = 2x_1 dx_1 + 2x_2 dx_2\).
2. If \(\omega^1(x_1, x_2) = x_1 x_2 dx_1 + (x_1^2 + x_2^2) dx_2\), then \(d\omega^1\) is given by

\[
d\omega^1 = d(x_1 x_2 dx_1 + (x_1^2 + x_2^2) dx_2) = (x_2 dx_1 + x_1 dx_2) \wedge dx_1 + (2x_1 dx_1 + 2x_2 dx_2) \wedge dx_2 = x_1 dx_1 \wedge dx_2.
\]
Properties of the exterior derivative

1. If \( f_1(x_1, x_2) \) and \( f_2(x_1, x_2) \) are differentiable functions, then

\[
df_1 \wedge df_2 = \frac{\partial (f_1, f_2)}{\partial (x_1, x_2)} \, dx_1 \wedge dx_2.
\]

2. If \( \omega^p \) and \( \omega^q \) represent a \( p \)-form and a \( q \)-form, then

\[
d(\omega^p \wedge \omega^q) = (d\omega^p) \wedge \omega^q + (-1)^{pq} \omega^p \wedge (d\omega^q).
\]

3. If \( \omega^p \) is a \( p \)-form with at least two derivatives, then \( d(d\omega^p) = 0 \).
   - The relation \( d(d\omega^0) = 0 \) is equivalent to \( \text{curl}(\text{grad} \, F) = 0 \).
   - The relation \( d(d\omega^1) = 0 \) is equivalent to \( \text{div}(\text{curl} \, f) = 0 \).

---

5.3 INTEGRATION

5.3.1 DEFINITIONS

The following definitions apply to the expression \( I = \int_a^b f(x) \, dx \):

- The integrand is \( f(x) \).
- The upper limit is \( b \).
- The lower limit is \( a \).
- \( I \) is “the integral of \( f(x) \) from \( a \) to \( b \)”.

It is conventional to indicate the indefinite integral of a function represented by a lowercase letter by the corresponding uppercase letter. For example, \( F(x) = \int_a^x f(t) \, dt \) and \( G(x) = \int_a^x g(t) \, dt \). Note that all functions that differ from \( F(x) \) by a constant are also indefinite integrals of \( f(x) \).

- \( \int f(x) \, dx \) indefinite integral of \( f(x) \) (also written \( \int_a^x f(t) \, dt \))
- \( \int_a^b f(x) \, dx \) definite integral of \( f(x) \), defined as
  \[
  \lim_{n \to \infty} \left( \frac{b-a}{n} \sum_{k=1}^n f \left( a + \frac{k}{n} (b-a) \right) \right)
  \]
- \( \int_C f(x) \, dx \) definite integral of \( f(x) \), taken along the contour \( C \)
- \( \int_a^\infty f(x) \, dx \) defined as \( \lim_{R \to \infty} \int_a^R f(x) \, dx \)
• $\int_{-\infty}^{\infty} f(x) \, dx$ defined as the limit of $\int_{-S}^{R} f(x) \, dx$ as $R$ and $S$ independently go to $\infty$

• $\int f$ shorthand for $\int f(x) \, dx$

• Improper integral integral for which the region of integration is not bounded, or the integrand is not bounded

• Cauchy principal value

  – The Cauchy principal value of the integral $\int_{a}^{b} f(x) \, dx$, denoted $\int_{a}^{b} f(x) \, dx$, is defined as $\lim_{\epsilon \to 0^+} \left( \int_{a}^{c-\epsilon} f(x) \, dx + \int_{c+\epsilon}^{b} f(x) \, dx \right)$, assuming that $f$ is singular only at $c$.

  – The Cauchy principal value of the integral $\int_{-\infty}^{\infty} f(x) \, dx$ is defined as the limit of $\int_{-R}^{R} f(x) \, dx$ as $R \to \infty$.

• If, at the complex point $z = a$, $f(z)$ is either analytic or has an isolated singularity, then the residue of $f(z)$ at $z = a$ is given by the complex integral $\text{Res}_f(a) = \frac{1}{2\pi i} \oint_{C} f(\xi) \, d\xi$.

5.3.2 PROPERTIES OF INTEGRATION

Indefinite integrals have the properties:

1. $\int [a f(x) + b g(x)] \, dx = a \int f(x) \, dx + b \int g(x) \, dx$ (linearity).
2. $\int f(x) g(x) \, dx = F(x) g(x) - \int F(x) g'(x) \, dx$ (integration by parts).
3. $\int f(g(x)) g'(x) \, dx = F(g(x))$ (substitution).
4. $\int f(ax + b) \, dx = \frac{1}{a} F(ax + b)$.
5. If $f(x)$ is an odd function, then $F(x)$ is an even function.
6. If $f(x)$ is an even function and $F(0) = 0$, then $F(x)$ is an odd function.
7. If $f(x)$ has a finite number of discontinuities, then the integral $\int f(x) \, dx$ is the sum of the integrals over those subintervals where $f(x)$ is continuous (provided they exist).
8. Fundamental theorem of integral calculus If $f(x)$ is single-valued, bounded, and integrable on $[a, b]$, and there exists a function $F(x)$ such that $F'(x) = f(x)$ for $a \leq x \leq b$ then

   $$\int_{a}^{b} f(x) \, dx = F(x) \bigg|_{a}^{b} = F(b) - F(a)$$

   for $a \leq x \leq b$.

Definite integrals have the properties:

1. $\int_{a}^{a} f(x) \, dx = 0$. ©1996 CRC Press LLC
2. \( f_a^b f(x) \,dx = -f_a^b f(x) \,dx \).
3. \( f_a^b f(x) \,dx + f_a^c f(x) \,dx = f_a^c f(x) \,dx \) (additivity).
4. \( f_a^b [cf(x) + dg(x)] \,dx = c f_a^b f(x) \,dx + d f_a^b g(x) \,dx \) (linearity).

### 5.3.3 INEQUALITIES

1. **Schwarz’ inequality:**
   \[ f_a^b |fg| \leq \left( f_a^b |f|^2 \right)^{1/2} \left( f_a^b |g|^2 \right)^{1/2} \]
2. **Minkowski’s inequality:**
   \[ \left( f_a^b |f + g|^p \right)^{1/p} \leq \left( f_a^b |f|^p \right)^{1/p} + \left( f_a^b |g|^p \right)^{1/p} \] when \( p \geq 1 \).
3. **Hölder’s inequality:**
   \[ f_a^b |fg| \leq \left[ f_a^b |f|^p \right]^{1/p} \left[ f_a^b |g|^q \right]^{1/q} \] when \( \frac{1}{p} + \frac{1}{q} = 1 \), \( p > 1 \), and \( q > 1 \).
4. \( f_a^b f(x) \,dx \leq f_a^b |f(x)| \,dx \leq \left( \max_{x \in [a,b]} |f(x)| \right) (b-a) \) assuming \( a \leq b \).
5. If \( f(x) \leq g(x) \) on the interval \([a,b]\), then \( f_a^b f(x) \,dx \leq f_a^b g(x) \,dx \).

### 5.3.4 CONVERGENCE TESTS

1. If \( f_a^b |f(x)| \,dx \) is convergent, then \( f_a^b f(x) \,dx \) is convergent.
2. If \( 0 \leq f(x) \leq g(x) \) and \( f_a^b g(x) \,dx \) is convergent, then \( f_a^b f(x) \,dx \) is convergent.
3. If \( 0 \leq g(x) \leq f(x) \) and \( f_a^b g(x) \,dx \) is divergent, then \( f_a^b f(x) \,dx \) is divergent.

The following integrals may be used, for example, with the above tests:

- \( \int_1^\infty \frac{dx}{x^p} \) and \( \int_1^\infty \frac{dx}{x^p} \) are convergent when \( p > 1 \), and divergent when \( p \leq 1 \).
- \( \int_0^1 \frac{dx}{x^p} \) is convergent when \( p < 1 \), and divergent when \( p \geq 1 \).
5.3.5 SUBSTITUTION

Substitution can be used to change integrals to simpler forms. When the transform \( t = g(x) \) is chosen, the integral \( I = \int f(t) \, dt \) becomes \( I = \int f(g(x)) g'(x) \, dx \). Several precautions must be observed when using substitutions:

- Be sure to make the substitution in the \( dx \) term, as well as everywhere else in the integral.
- Be sure that the function substituted is one-to-one and continuous. If this is not the case, then the integral must be restricted in such a way as to make it true.
- With definite integrals, the limits should also be expressed in terms of the new dependent variables. With indefinite integrals, it is necessary to perform the reverse substitution to obtain the answer in terms of the original independent variable. This may also be done for definite integrals, but it is usually easier to change the limits.

Example

Consider the integral

\[ I = \int \frac{x^4}{\sqrt{a^2 - x^2}} \, dx \]

Here we choose to make the substitution \( x = |a| \sin \theta \). From this we find \( dx = |a| \cos \theta \, d\theta \) and

\[
\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 \theta} = |a| \sqrt{1 - \sin^2 \theta} = |a| |\cos \theta|
\]

Note the absolute value signs. It is very important to interpret the square root radical consistently as the positive square root. Thus \( \sqrt{x^2} = |x| \). Failure to observe this is a common cause of errors in integration.

Note that the substitution used above is not a one-to-one function, that is, it does not have a unique inverse. Thus the range of \( \theta \) must be restricted in such a way as to make the function one-to-one. In this case we can solve for \( \theta \) to obtain

\[ \theta = \sin^{-1} \frac{x}{|a|} \]

This will be unique if we restrict the inverse sine to the principal values \(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\).

Thus, the integral becomes (with \( dx = |a| \cos \theta \, d\theta \))

\[ I = \int \frac{a^4 \sin^4 \theta}{|a| |\cos \theta|} \, |a| \cos \theta \, d\theta. \]

Now, however, in the range of values chosen for \( \theta \), we find that \( \cos \theta \) is always positive. Thus, we may remove the absolute value signs from \( \cos \theta \) in the denominator. Then the \( \cos \theta \) terms cancel and the integral becomes

\[ I = a^4 \int \sin^4 \theta \, d\theta. \]
By application of the integration formula on page 378 this is integrated to obtain

\[ I = -\frac{a^4}{4} \sin^3 \theta \cos \theta - \frac{3a^4}{8} \sin \theta \cos \theta + \frac{3a^4}{8} \theta + C. \tag{5.3.1} \]

To obtain an evaluation of \( I \) as a function of \( x \), we must transform variables from \( \theta \) to \( x \). We have

\[ \cos \theta = \pm \sqrt{1 - \sin^2 \theta} = \pm \sqrt{1 - \frac{x^2}{a^2}} = \pm \frac{\sqrt{a^2 - x^2}}{|a|}. \]

Because of the previously recorded fact that \( \cos \theta \) is positive for our range of \( \theta \), we may omit the \( \pm \) sign. Using \( \sin \theta = \frac{x}{|a|} \) and \( \cos \theta = \frac{\sqrt{a^2 - x^2}}{|a|} \) we can evaluate Equation (5.3.1) to obtain the final result,

\[ I = \int \frac{x^4}{\sqrt{a^2 - x^2}} = -\frac{x^3}{4} \sqrt{a^2 - x^2} - \frac{3a^2 x}{8} \sqrt{a^2 - x^2} + \frac{3a^4}{8} \sin^{-1} \frac{x}{|a|} + C. \]

**Useful transformations**

The following transformations may make evaluation of an integral easier:

1. \( \int f \left( x, \sqrt{x^2 + a^2} \right) \, dx = a \int f(a \tan u, a \sec u) \sec^2 u \, du \)
   when \( u = \tan^{-1} \frac{x}{a} \) and \( a > 0 \).
2. \( \int f \left( x, \sqrt{x^2 - a^2} \right) \, dx = a \int f(a \sec u, a \tan u) \sec u \tan u \, du \)
   when \( u = \sec^{-1} \frac{x}{a} \) and \( a > 0 \).
3. \( \int f \left( x, \sqrt{a^2 - x^2} \right) \, dx = a \int f(a \sin u, a \cos u) \cos u \, du \)
   when \( u = \sin^{-1} \frac{x}{a} \) and \( a > 0 \).
4. \( \int f(\sin x) \, dx = 2 \int f \left( \frac{2z}{1+z^2} \right) \frac{dz}{1+z^2} \)
   when \( z = \tan \frac{x}{2} \).
5. \( \int f(\cos x) \, dx = 2 \int f \left( \frac{1-z^2}{1+z^2} \right) \frac{dz}{1+z^2} \)
   when \( z = \tan \frac{x}{2} \).
6. \( \int f(\cos x) \, dx = -\int f(v) \frac{dv}{\sqrt{1-v^2}} \)
   when \( v = \cos x \).
7. \( \int f(\sin x) \, dx = \int f(u) - \frac{du}{\sqrt{1-u^2}} \)
   when \( u = \sin x \).
8. \( \int f(\sin x, \cos x) \, dx = \int \int \left( u, \sqrt{1-u^2} \right) \frac{du}{\sqrt{1-u^2}} \)
   when \( u = \sin x \).
9. \( \int f(\sin x, \cos x) \, dx = 2 \int \int \left( \frac{2z}{1+z^2}, \frac{1-z^2}{1+z^2} \right) \frac{dz}{1+z^2} \)
   when \( z = \tan \frac{x}{2} \).
10. \( \int_{-\infty}^{\infty} F(u) \, du = \int_{-\infty}^{\infty} F(x) \, dx \)
    when \( u = x - \sum_{j=1}^{n} \frac{a_j}{x - c_j} \) where \( \{a_i\} \) is any sequence of positive constants and the \( \{c_j\} \) are any real constants whatsoever.

Several transformations of the integral \( \int_{0}^{\infty} f(x) \, dx \), with an infinite integration range, to an integral with a finite integration range, are shown:
Finite interval integral
\[
\int_{1/2}^{1/7} f\left(\frac{1}{2} \log \frac{1 + x}{1 - x}\right) \frac{dt}{\left(1 - \frac{1}{t}\right)^2}
\]

5.3.6 PARTIAL FRACTION DECOMPOSITION

Every integral of the form \( \int R(x) \, dx \), where \( R \) is a rational function, can be evaluated (in principle) in terms of elementary functions. The technique is to factor the denominator of \( R \) and create a partial fraction decomposition. Then each resulting subintegral is elementary.

Example

Consider the integral
\[
I = \int \frac{2x^3 - 10x^2 + 13x - 4}{x^2 - 5x + 6} \, dx.
\]
This can be written as
\[
I = \int \left( 2x + \frac{x - 4}{x^2 - 5x + 6} \right) \, dx = \int \left( 2x + \frac{2}{x - 2} - \frac{1}{x - 3} \right) \, dx
\]
which can be readily integrated
\[
I = x^2 + 2 \ln(x - 2) - \ln(x - 3).
\]

5.3.7 INTEGRATION BY PARTS

In one dimension, the integration by parts formula is
\[
\int u \, dv = uv - \int v \, du \quad \text{for indefinite integrals,} \quad (5.3.2)
\]
\[
\int_{a}^{b} u \, dv = uv|_{a}^{b} - \int_{a}^{b} v \, du \quad \text{for definite integrals.} \quad (5.3.3)
\]

When evaluating a given integral by this method, \( u \) and \( v \) must be chosen so that the form \( \int u \, dv \) becomes identical to the given integral. This is usually accomplished by specifying \( u \) and \( dv \) and deriving \( du \) and \( v \). Then the integration by parts formula will produce a boundary term and another integral to be evaluated. If \( u \) and \( v \) were well chosen, then this second integral may be easier to evaluate.

Example

Consider the integral
\[
I = \int x \sin x \, dx.
\]
Two obvious choices for the integration by parts formula are \( \{u = x, \, dv = \sin x \, dx\} \) and \( \{u = \sin x, \, dv = x \, dx\} \). We will try each of them in turn.

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• Using \( u = x, dv = \sin x \, dx \), we compute \( du = dx \) and \( v = \int dv = \int \sin x \, dx = -\cos x \). Hence, we can represent \( I \) in the alternative form as

\[
I = \int x \sin x \, dx = \int u \, dv = uv - \int v \, du = -x \cos x + \int \cos x \, dx.
\]

In this representation of \( I \), we must evaluate the last integral. Because we know \( \int \cos x \, dx = \sin x \), the final result is \( I = \sin x - x \cos x \).

• Using \( u = \sin x, dv = x \, dx \) we compute \( du = \cos x \, dx \) and \( v = \int dv = \int x \, dx = x^2/2 \). Hence, we can represent \( I \) in the alternative form as

\[
I = \int x \sin x \, dx = \int u \, dv = uv - \int v \, du = \frac{x^2}{2} \cos x - \int \frac{x^2}{2} \cos x \, dx.
\]

In this case, we have actually made the problem “worse” since the remaining integral appearing in \( I \) is “harder” than the one we started with.

**Example**

Consider the integral

\[
I = \int e^x \sin x \, dx.
\]

We choose to use the integration by parts formula with \( u = e^x \) and \( dv = \sin x \, dx \).

From these we compute \( du = e^x \, dx \) and \( v = \int dv = \int \sin x \, dx = -\cos x \). Hence, we can represent \( I \) in the alternative form as

\[
I = \int e^x \sin x \, dx = \int u \, dv = uv - \int v \, du = -e^x \cos x + \int e^x \cos x \, dx
\]

If we write this as

\[
I = -e^x \cos x + J \quad \text{with} \quad J = \int e^x \cos x \, dx, \quad (5.3.4)
\]

then we can apply integration by parts to \( J \) using \( \{u = e^x, dv = \cos x \, dx\} \). From these we compute \( du = e^x \, dx \) and \( v = \int dv = \int \cos x \, dx = \sin x \). Hence, we can represent \( J \) in the alternative form as

\[
J = \int e^x \cos x \, dx = \int u \, dv = uv - \int v \, du = e^x \sin x - \int e^x \sin x \, dx
\]

If we write this as

\[
J = e^x \sin x - I, \quad (5.3.5)
\]

then we can solve the linear equations (5.3.4) and (5.3.5) simultaneously to determine both \( I \) and \( J \). We find

\[
I = \int e^x \sin x \, dx = \frac{1}{2} \left( e^x \sin x - e^x \cos x \right), \quad \text{and}
\]

\[
J = \int e^x \cos x \, dx = \frac{1}{2} \left( e^x \sin x + e^x \cos x \right).
\]
Extended integration by parts rule

The following rule is obtained by \( n + 1 \) successive applications of integration by parts. Let

\[
\begin{align*}
g_1(x) &= \int g(x) \, dx, & g_2(x) &= \int g_1(x) \, dx, \\
g_3(x) &= \int g_2(x) \, dx, & \ldots, & g_n(x) &= \int g_{n-1}(x) \, dx. & (5.3.6)
\end{align*}
\]

Then

\[
\int f(x)g(x) \, dx = f(x)g_1(x) - f'(x)g_2(x) + f''(x)g_3(x) - \ldots \\
+ (-1)^n f^{(n)}(x)g_{n+1}(x) + (-1)^{n+1} \int f^{(n+1)}(x)g_{n+1}(x) \, dx. & (5.3.7)
\]

5.3.8 SPECIAL FUNCTIONS DEFINED BY INTEGRALS

Not all integrals of elementary functions (sines, cosines, rational functions, and others) can be evaluated in terms of elementary functions. For example, the integral \( \int e^{-x^2} \, dx \) is represented by the special function “erf\( (x) \)” (see page 498). Other useful functions include dilogarithms (see page 506) and elliptic integrals (see page 522).

The dilogarithm function is defined by \( \text{Li}_2(x) = -\int_0^x \frac{\ln(1-t)}{t} \, dt \). All integrals of the form \( \int P(x, \sqrt{R}) \log Q(x, \sqrt{R}) \, dx \), where \( P \) and \( Q \) are rational functions and \( R = A^2 + Bx + Cx^2 \), can be evaluated in terms of elementary functions and dilogarithms.

All integrals of the form \( \int R(x, \sqrt{T(x)}) \, dx \), where \( R \) is a rational function of its arguments and \( T(x) \) is a third or fourth order polynomial, can be integrated in terms of elementary functions and elliptic functions.

5.3.9 VARIATIONAL PRINCIPLES

If \( J \) depends on a function \( g(x) \) and its derivatives through an integral of the form \( J[g] = \int F(g, g', \ldots) \, dx \), then \( J \) will be stationary to small perturbations if \( F \) satisfies the corresponding Euler–Lagrange equation.
Function | Euler–Lagrange equation
--- | ---
$\int_{R} F(x, y, y') \, dx$ | $\frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) = 0$
$\int_{R} F(x, y, y', \ldots, y^{(n)}) \, dx$ | $\frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) - \ldots + (-1)^n \frac{d^n}{dx^n} \left( \frac{\partial F}{\partial y^{(n)}} \right) = 0$
$\iint_{R} \left[ a \left( \frac{\partial u}{\partial x} \right)^2 + b \left( \frac{\partial u}{\partial y} \right)^2 + c u^2 + 2 f u \right] \, dx \, dy, \quad \frac{\partial}{\partial x} \left( a \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( b \frac{\partial u}{\partial y} \right) - cu = f$
$\iint_{R} F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) \, dx \, dy \quad \frac{\partial F}{\partial u} - \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial u_x} \right) - \frac{\partial}{\partial y} \left( \frac{\partial F}{\partial u_y} \right) + \frac{\partial^2}{\partial x^2} \left( \frac{\partial F}{\partial u_{xx}} \right) + \frac{\partial^2}{\partial x \partial y} \left( \frac{\partial F}{\partial u_{xy}} \right) + \frac{\partial^2}{\partial y^2} \left( \frac{\partial F}{\partial u_{yy}} \right) = 0$

5.3.10 **LINE AND SURFACE INTEGRALS**

A line integral is a definite integral whose path of integration is along a specified curve; it can be evaluated by reducing it to ordinary integrals. If $f(x, y)$ is continuous on $C$, and the integration contour $C$ is parameterized by $(\phi(t), \psi(t))$ as $t$ varies from $a$ to $b$, then

$$\int_{C} f(x, y) \, dx = \int_{a}^{b} f(\phi(t), \psi(t)) \phi'(t) \, dt, \quad (5.3.8)$$

$$\int_{C} f(x, y) \, dy = \int_{a}^{b} f(\phi(t), \psi(t)) \psi'(t) \, dt. \quad (5.3.9)$$

In a simply connected domain, the line integral $I = \int_{C} X \, dx + Y \, dy + Z \, dz$ is independent of the path $C$ (beginning and ending at the same place) if, and only if, $u = (X, Y, Z)$ is a gradient vector, $u = \text{grad} \, F$ (that is, $F_x = X$, $F_y = Y$, and $F_z = Z$).

**Green’s theorem**: Let $D$ be a domain of the $xy$ plane, and let $C$ be a piecewise smooth, simple closed curve in $D$ whose interior $R$ is also in $D$. Let $P(x, y)$ and $Q(x, y)$ be functions defined in $D$ with continuous first partial derivatives in $D$. Then

$$\oint_{C} (P \, dx + Q \, dy) = \iint_{R} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy. \quad (5.3.10)$$

The above theorem may be written in the two alternative forms (using $u = P(x, y)i + Q(x, y)j$ and $v = Q(x, y)i - P(x, y)j$),

$$\oint_{C} u \, ds = \iint_{R} \text{curl} \, u \, dxdy \quad \text{and} \quad \oint_{C} v \, ds = \iint_{R} \text{div} \, v \, dxdy. \quad (5.3.11)$$

The first equation above is a simplification of Stokes’s theorem, the second equation is the divergence theorem.

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Stokes’s theorem: Let $S$ be a piecewise smooth oriented surface in space, whose boundary $C$ is a piecewise smooth simple closed curve, directed in accordance with the given orientation of $S$. Let $\mathbf{u} = Li + Mj + Nk$ be a vector field with continuous and differentiable components in a domain $D$ of space including $S$. Then,\[
\int_C \mathbf{u} \cdot d\mathbf{s} = \iint_S (\text{curl} \, \mathbf{u}) \cdot \mathbf{n} \, d\sigma, \tag{5.3.12}\]
where $\mathbf{n}$ is the chosen unit normal vector on $S$, that is \[
\int_C L \, dx + M \, dy + N \, dz = \iint_S \left( \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} \right) \, dy \, dz + \frac{\partial L}{\partial z} \, dx + \frac{\partial M}{\partial x} \, dy. \tag{5.3.12}
\]

Divergence theorem: Let $\mathbf{v} = Li + Mj + Nk$ be a vector field in a domain $D$ of space. Let $L$, $M$, and $N$ be continuous with continuous derivatives in $D$. Let $S$ be a piecewise smooth surface in $D$ that forms the complete boundary of a bounded closed region $R$ in $D$. Let $\mathbf{n}$ be the outer normal of $S$ with respect to $R$. Then \[
\iint_S \mathbf{v} \cdot d\mathbf{S} = \iiint_R \text{div} \, \mathbf{v} \, dV, \tag{5.3.13}
\]
that is \[
\iint_S L \, dy \, dz + M \, dz \, dx + N \, dx \, dy = \iiint_R \left( \frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z} \right) \, dx \, dy \, dz. \tag{5.3.13}
\]
If $D$ is a three-dimensional domain with boundary $B$, let $dV$ represent the volume element of $D$, let $dS$ represent the surface element of $B$, and let $dS = \mathbf{n} \, dS$, where $\mathbf{n}$ is the outer normal vector of the surface $B$. Then Gauss’s formulae are \[
\iiint_D \nabla \cdot \mathbf{A} \, dV = \iiint_B \mathbf{A} \cdot d\mathbf{S} = \int_B (\mathbf{n} \cdot \mathbf{A}) \, dS, \tag{5.3.14}
\]
\[
\iiint_D \nabla \times \mathbf{A} \, dV = \iiint_B \mathbf{A} \times d\mathbf{S} = \int_B (\mathbf{n} \times \mathbf{A}) \, dS, \tag{5.3.15}
\]
and \[
\iiint_D \nabla \phi \, dV = \iint_B \phi \, dS, \tag{5.3.16}
\]
where $\phi$ is an arbitrary scalar and $\mathbf{A}$ is an arbitrary vector.

Green’s theorems also relate a volume integral to a surface integral: Let $V$ be a volume with surface $S$, which we assume is simple and closed. Define $n$ as the outward normal to $S$. Let $\phi$ and $\psi$ be scalar functions which, together with $\nabla^2 \phi$ and $\nabla^2 \psi$, are defined in $V$ and on $S$. Then

1. Green’s first theorem states that \[
\int_S \phi \frac{\partial \psi}{\partial n} \, dS = \int_V \left( \phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi \right) \, dV. \tag{5.3.17}
\]
2. Green’s second theorem states that
\[ \int_S \left( \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) dS = \int_V \left( \phi \nabla^2 \psi - \psi \nabla^2 \phi \right) dV. \]
\[ (5.3.18) \]

### 5.3.11 CONTOUR INTEGRALS

If \( f(z) \) is analytic in the region inside of the simple closed curve \( C \) (with proper orientation), then

1. The Cauchy–Goursat integral theorem is \( \oint_C f(\xi) d\xi = 0 \).
2. Cauchy’s integral formula is
   \[ f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - z} d\xi \quad \text{and} \quad f'(z) = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - z)^2} d\xi. \]

In general, \( f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi \).

The residue theorem: For every simple closed contour \( C \) enclosing at most a finite number of (necessarily isolated) singularities \( \{z_1, z_2, \ldots, z_n\} \) of a single-valued function \( f(z) \) continuous on \( C \),
\[ \frac{1}{2\pi i} \oint_C f(\xi) d\xi = \sum_{k=1}^{n} \text{Res}_f(z_k) \]

### 5.3.12 CONTINUITY OF INTEGRAL ANTIDERIVATIVES

Consider the following different evaluations of an integral
\[ F(x) = \int f(x) \, dx = \int \frac{3}{5 - 4 \cos x} \, dx = \begin{cases} 2 \tan^{-1}(3 \tan(x/2)) \\ 2 \tan^{-1}(3 \sin x / (\cos x + 1)) \\ -\tan^{-1}(-3 \sin x / (5 \cos x - 4)) \\ 2 \tan^{-1} \left( 3 \tan \left( \frac{x}{2} \right) \right) + 2\pi \left\lfloor \frac{x}{2\pi} + \frac{1}{2} \right\rfloor \end{cases} \]
where \( \lfloor \cdot \rfloor \) denotes the floor function. These evaluations are all “correct” because differentiating any of them results in the original integrand. However, if we expect that \( \int_0^{4\pi} f(x) \, dx = F(4\pi) - F(0) \) is correct, then only the last evaluation is correct. This is true because the other evaluations of \( F(x) \) are discontinuous when \( x \) is a multiple of \( \pi \).

In general, if \( \hat{F}(x) = \int x \, f(x) \, dx \) is a discontinuous evaluation (with \( \hat{F}(x) \) discontinuous at the single point \( x = b \)), then a continuous evaluation on a finite interval is given by \( \int_a^b f(x) \, dx = F(c) - F(a) \), where
\[ F(x) = \hat{F}(x) - \hat{F}(a) + H(x - b) \left[ \lim_{x \to b^-} \hat{F}(x) - \lim_{x \to b^+} \hat{F}(x) \right] \]
and where \( H(\cdot) \) is the Heaviside function. For functions with an infinite number of discontinuities, note that \( \sum_{n=1}^{\infty} H(x - pn - q) = \left\lfloor \frac{x - q}{p} \right\rfloor \).

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Applications of integration

- Using Green’s theorems, the area bounded by the simple, closed, positively oriented contour $C$ is

$$\text{area} = \oint_C x \, dy = - \oint_C y \, dx. \quad (5.3.19)$$

- Arc length:
  1. $s = \int_{x_1}^{x_2} \sqrt{1 + y'^2} \, dx$ for $y = f(x)$.
  2. $s = \int_{t_1}^{t_2} \sqrt{\phi'^2 + \psi'^2} \, dt$ for $x = \phi(t)$, $y = \psi(t)$.
  3. $s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta = \int_{r_1}^{r_2} \sqrt{1 + r^2 \left(\frac{dr}{d\phi}\right)^2} \, dr$ for $r = f(\theta)$.

- Surface area for surfaces of revolution:
  1. $A = 2\pi \int_{x_1}^{x_2} f(x) \sqrt{1 + (f'(x))^2} \, dx$ when $y = f(x)$ is rotated about the $x$-axis.
  2. $A = 2\pi \int_{y_1}^{y_2} x \sqrt{1 + (f'(x))^2} \, dy$ when $y = f(x)$ is rotated about the $y$-axis.
  3. $A = 2\pi \int_{t_1}^{t_2} \phi \sqrt{\phi'^2 + \psi'^2} \, dt$ for $x = \phi(t)$, $y = \psi(t)$ rotated about the $x$-axis.
  4. $A = 2\pi \int_{t_1}^{t_2} \phi \sqrt{\phi'^2 + \psi'^2} \, dt$ for $x = \phi(t)$, $y = \psi(t)$ rotated about the $y$-axis.
  5. $A = 2\pi \int_{\phi_1}^{\phi_2} r \sin \phi \sqrt{r^2 + \left(\frac{dr}{d\phi}\right)^2} \, d\phi$ for $r = r(\phi)$ rotated about the $x$-axis.
  6. $A = 2\pi \int_{\phi_1}^{\phi_2} r \cos \phi \sqrt{r^2 + \left(\frac{dr}{d\phi}\right)^2} \, d\phi$ for $r = r(\phi)$ rotated about the $y$-axis.

- Volumes of revolution:
  1. $V = \pi \int_{x_1}^{x_2} f^2(x) \, dx$ for $y = f(x)$ rotated about the $x$-axis.
  2. $V = \pi \int_{y_1}^{y_2} x^2 f'(x) \, dx$ for $y = f(x)$ rotated about the $y$-axis.
  3. $V = \pi \int_{y_1}^{y_2} g^2(y) \, dy$ for $x = g(y)$ rotated about the $y$-axis.

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4. \( V = \pi \int_{t_1}^{t_2} \psi^2 \dot{\phi} \, dt \) for \( x = \phi(t), \ y = \psi(t) \) rotated about the \( x \)-axis.

5. \( V = \pi \int_{t_1}^{t_2} \phi^2 \psi \, dt \) for \( x = \phi(t), \ y = \psi(t) \) rotated about the \( y \)-axis.

6. \( V = \pi \int_{\phi_1}^{\phi_2} \sin^2 \phi \left( \frac{dr}{d\phi} \cos \phi - r \sin \phi \right) \, d\phi \) for \( r = f(\phi) \) rotated about the \( x \)-axis.

7. \( V = \pi \int_{\phi_1}^{\phi_2} \cos^2 \phi \left( \frac{dr}{d\phi} \sin \phi - r \cos \phi \right) \, d\phi \) for \( r = f(\phi) \) rotated about the \( y \)-axis.

• The area enclosed by the curve defined by the equation \( x^{b/c} + y^{b/c} = a^{b/c} \), where \( a > 0 \), \( c \) is an odd integer and \( b \) is an even integer, is given by \( A = \frac{2ca^2}{b} \left[ \Gamma \left( \frac{b}{c} \right) \right]^2 \).

• The integral \( I = \iiint_{R} x^{h-1} y^{m-1} z^{n-1} \, dV \), where \( R \) is the region of space bounded by the coordinate planes and that portion of the surface \( \left( \frac{x}{a} \right)^p + \left( \frac{y}{b} \right)^q + \left( \frac{z}{c} \right)^k = 1 \), in the first octant, and where \( \{ h, m, n, p, q, k, a, b, c \} \) are all positive real numbers, is given by

\[
\int_{0}^{a} x^{h-1} \, dx \int_{0}^{b} \left[ 1 - \left( \frac{y}{b} \right)^q \right]^{\frac{1}{p}} \, dy \int_{0}^{c} \left[ 1 - \left( \frac{z}{c} \right)^k \right]^{\frac{1}{q}} \, dz = \frac{a^h b^m c^n}{p q k} \Gamma \left( \frac{h}{p} \right) \Gamma \left( \frac{m}{q} \right) \Gamma \left( \frac{n}{k} \right) \Gamma \left( \frac{1}{p} + \frac{1}{q} + \frac{1}{k} \right).
\]

5.3.13 ASYMPTOTIC INTEGRAL EVALUATION

**Laplace’s method:** If \( f'(x_0) = 0, \ f''(x_0) < 0 \), and \( \lambda \to \infty \), then

\[
I_{x_0, \epsilon}(\lambda) \equiv \int_{x_0 - \epsilon}^{x_0 + \epsilon} g(x) e^{i\lambda f(x)} \, dx \sim g(x_0) e^{i\lambda f(x_0)} \left[ \frac{2\pi}{\lambda |f''(x_0)|} \right] + \ldots \tag{5.3.20}
\]

Hence, if points of local maximum \( \{ x_i \} \) satisfy \( f'(x_i) = 0 \) and \( f''(x_i) < 0 \), then

\[
\int_{-\infty}^{\infty} g(x) e^{i\lambda f(x)} \, dx \sim \sum_{i} I_{x_i, \epsilon}(\lambda).
\]

**Method of stationary phase:** If \( f(x_0) \neq 0, \ f'(x_0) = 0, \ f''(x_0) \neq 0, \ g(x_0) \neq 0 \), and \( \lambda \to \infty \), then

\[
J_{x_0, \epsilon}(\lambda) = \int_{x_0 - \epsilon}^{x_0 + \epsilon} g(x) e^{i\lambda f(x)} \, dx
\]

\[
\sim g(x_0) \left[ \frac{2\pi}{\lambda |f''(x_0)|} \right] \left[ i\lambda f(x_0) - \frac{i\pi}{4} \text{sgn} f''(x_0) \right] + \ldots \tag{5.3.21}
\]

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### 5.3.14 MOMENTS OF INERTIA FOR VARIOUS BODIES

<table>
<thead>
<tr>
<th>Body</th>
<th>Axis</th>
<th>Moment of inertia</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform thin rod</td>
<td>Normal to the length, at one end</td>
<td>$m \frac{l^2}{3}$</td>
</tr>
<tr>
<td>Uniform thin rod</td>
<td>Normal to the length, at the center</td>
<td>$m \frac{l^2}{12}$</td>
</tr>
<tr>
<td>Thin rectangular sheet, sides $a$ and $b$</td>
<td>Through the center parallel to $b$</td>
<td>$m \frac{a^2}{12}$</td>
</tr>
<tr>
<td>Thin rectangular sheet, sides $a$ and $b$</td>
<td>Through the center perpendicular to the sheet</td>
<td>$m \frac{a^2 + b^2}{12}$</td>
</tr>
<tr>
<td>Thin circular sheet of radius $r$</td>
<td>Normal to the plate through the center</td>
<td>$m \frac{r^2}{2}$</td>
</tr>
<tr>
<td>Thin circular sheet of radius $r$</td>
<td>Along any diameter</td>
<td>$m \frac{r^2}{4}$</td>
</tr>
<tr>
<td>Thin circular ring, radii $r_1$ and $r_2$</td>
<td>Through center normal to plane of ring</td>
<td>$m \frac{r_1^2 + r_2^2}{2}$</td>
</tr>
<tr>
<td>Thin circular ring, radii $r_1$ and $r_2$</td>
<td>Along any diameter</td>
<td>$m \frac{r_1^2 + r_2^2}{4}$</td>
</tr>
<tr>
<td>Rectangular parallelopiped, edges $a$, $b$, and $c$</td>
<td>Through center perpendicular to face $ab$ (parallel to edge $c$)</td>
<td>$m \frac{a^2 + b^2}{12}$</td>
</tr>
<tr>
<td>Sphere, radius $r$</td>
<td>Any diameter</td>
<td>$m \frac{2}{3}r^2$</td>
</tr>
<tr>
<td>Spherical shell, external radius $r_1$, internal radius $r_2$</td>
<td>Any diameter</td>
<td>$m \frac{r_1^2 + r_2^2}{2}$</td>
</tr>
<tr>
<td>Spherical shell, very thin, mean radius $r$</td>
<td>Any diameter</td>
<td>$m \frac{2}{3}r^3$</td>
</tr>
<tr>
<td>Right circular cylinder of radius $r$, length $l$</td>
<td>Longitudinal axis of the slide</td>
<td>$m \frac{r^2}{2}$</td>
</tr>
<tr>
<td>Right circular cylinder of radius $r$, length $l$</td>
<td>Transverse diameter</td>
<td>$m \left( \frac{r^2}{4} + \frac{l^2}{12} \right)$</td>
</tr>
<tr>
<td>Hollow circular cylinder, radii $r_1$ and $r_2$, length $l$</td>
<td>Longitudinal axis of the figure</td>
<td>$m \frac{r_1^2 + r_2^2}{2}$</td>
</tr>
<tr>
<td>Thin cylindrical shell, length $l$, mean radius $r$</td>
<td>Longitudinal axis of the figure</td>
<td>$m\pi r^2$</td>
</tr>
<tr>
<td>Hollow circular cylinder, radii $r_1$ and $r_2$, length $l$</td>
<td>Transverse diameter</td>
<td>$m \left( \frac{r_1^2 + r_2^2}{4} + \frac{l^2}{12} \right)$</td>
</tr>
<tr>
<td>Hollow circular cylinder, very thin, length $l$, mean radius $r$</td>
<td>Transverse diameter</td>
<td>$m \left( \frac{r^2}{4} + \frac{l^2}{12} \right)$</td>
</tr>
<tr>
<td>Elliptic cylinder, length $l$, transverse semiaxes $a$ and $b$</td>
<td>Longitudinal axis</td>
<td>$m \left( \frac{a^2 + b^2}{4} \right)$</td>
</tr>
<tr>
<td>Right cone, altitude $h$, radius of base $r$</td>
<td>Axis of the figure</td>
<td>$m \frac{2}{3}r^2$</td>
</tr>
<tr>
<td>Spheroid of revolution, equatorial radius $r$</td>
<td>Polar axis</td>
<td>$m \frac{2}{3}r^2$</td>
</tr>
<tr>
<td>Ellipsoid, axes $2a$, $2b$, $2c$</td>
<td>Axis $2a$</td>
<td>$m \frac{a^2 + b^2}{5}$</td>
</tr>
</tbody>
</table>
5.3.15 TABLE OF SEMI-INTEGRALS

<table>
<thead>
<tr>
<th>( f )</th>
<th>( d^{-1/2} f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( C ), a constant</td>
<td>( 2C \sqrt{\frac{\pi}{\pi}} )</td>
</tr>
<tr>
<td>( x^{-1/2} )</td>
<td>( \sqrt{\frac{\pi}{\pi}} )</td>
</tr>
<tr>
<td>( x )</td>
<td>( \frac{4x^{2/3}}{3\sqrt{\pi}} )</td>
</tr>
<tr>
<td>( x^{n}, n = 0, 1, 2, \ldots )</td>
<td>( \frac{\Gamma(p+1)}{\Gamma(p+\frac{3}{2})} x^{p+1/2} )</td>
</tr>
<tr>
<td>( x^{p}, p &gt; -1 )</td>
<td>( \sqrt{\frac{\pi}{\pi}} + \frac{(1+x)^{\tan^{-1}\sqrt{x}}}{\sqrt{\pi}} )</td>
</tr>
<tr>
<td>( \sqrt{1+x} )</td>
<td>( \frac{2}{\sqrt{\pi}} \tan^{-1}\sqrt{x} )</td>
</tr>
<tr>
<td>( \frac{1}{\sqrt{1+x}} )</td>
<td>( 2 \sinh^{-1}\sqrt{x}/\sqrt{\pi(1+x)} )</td>
</tr>
<tr>
<td>( \frac{1}{1+x} )</td>
<td>( e^x )</td>
</tr>
<tr>
<td>( e^x )</td>
<td>( e^x )</td>
</tr>
<tr>
<td>( e^x ) erf ( \sqrt{x} )</td>
<td>( \sqrt{\pi} J_1 \sqrt{x} )</td>
</tr>
<tr>
<td>( \sin(\sqrt{x}) )</td>
<td>( \sqrt{\pi} H_{-1} \sqrt{x} )</td>
</tr>
<tr>
<td>( \cos(\sqrt{x}) )</td>
<td>( \sqrt{\pi} L_{-1} \sqrt{x} )</td>
</tr>
<tr>
<td>( \sin(\sqrt{x}) )</td>
<td>( \sqrt{\pi} J_0 \sqrt{x} )</td>
</tr>
<tr>
<td>( \cos(\sqrt{x}) )</td>
<td>( \sqrt{\pi} J_0 \sqrt{x} )</td>
</tr>
<tr>
<td>( \log x )</td>
<td>( 2\sqrt{\frac{\pi}{\pi}} \log(4x) - 2 )</td>
</tr>
<tr>
<td>( \log x \sqrt{x} )</td>
<td>( \sqrt{\pi} \log \left( \frac{x}{4} \right) )</td>
</tr>
</tbody>
</table>

5.3.16 TABLES OF INTEGRALS

Many extensive compilations of integrals tables exist. No matter how extensive the integral table, it is fairly uncommon to find the exact integral desired. Usually some form of transformation will have to be made. The simplest type of transformation is substitution. Simple forms of substitutions, such as \( y = ax \) are employed, almost unconsciously, by experienced users of integral tables. Finding the right substitution is largely a matter of intuition and experience.

We adopt the following conventions in our integral tables:

- A constant of integration must be included with all indefinite integrals.
• All angles are measured in radians, and inverse trigonometric and hyperbolic functions represent principal values.

• Logarithmic expressions are to base $e = 2.71828 \ldots$, unless otherwise specified, and are to be evaluated for the absolute value of the arguments involved therein.

• When inverse trigonometric functions occur in the integrals, be sure that any replacements made for them are strictly in accordance with the rules for such functions. This causes little difficulty when the argument of the inverse trigonometric function is positive, because all angles involved are in the first quadrant. However, if the argument is negative, special care must be used. Thus, if $u > 0$ then

$$\sin^{-1} u = \cos^{-1} \sqrt{1-u^2} = \csc^{-1} \frac{1}{u} = \ldots$$

However, if $u < 0$, then

$$\sin^{-1} u = -\cos^{-1} \sqrt{1-u^2} = -\pi - \csc^{-1} \frac{1}{u} = \ldots$$

The following sections contain tables of indefinite and definite integrals.

5.4 TABLE OF INDEFINITE INTEGRALS

All integrals listed below that do not have stars next to their numbers have been verified by computer. Note that the natural logarithm function is denoted as $\log x$.

5.4.1 ELEMENTARY FORMS

1. $\int a \, dx = ax$.

2. $\int a f(x) \, dx = a \int f(x) \, dx$.

3. $\int \phi(y(x)) \, dx = \int \frac{\phi(y)}{y'} \, dy$, where $y' = \frac{dy}{dx}$.

4. $\int (u + v) \, dx = \int u \, dx + \int v \, dx$, where $u$ and $v$ are any functions of $x$.

5. $\int udv = uv - \int vdu$.

6. $\int \frac{dv}{dx} \, dx = uv - \int \frac{du}{dx} \, dx$.

7. $\int x^n \, dx = \frac{x^{n+1}}{n+1}$, except when $n = -1$.

8. $\int \frac{dx}{x} = \log x$.
9. \[ \int \frac{f'(x)}{f(x)} \, dx = \log f(x), \quad (df(x) = f'(x) \, dx). \]

10. \[ \int \frac{f'(x)}{2\sqrt{f(x)}} \, dx = \sqrt{f(x)}, \quad (df(x) = f'(x) \, dx). \]

11. \[ \int e^x \, dx = e^x. \]

12. \[ \int e^{ax} \, dx = \frac{e^{ax}}{a}. \]

13. \[ \int b^{ax} \, dx = \frac{b^{ax}}{a \log b}, \quad b > 0. \]

14. \[ \int \log x \, dx = x \log x - x. \]

15. \[ \int a^x \, dx = \frac{a^x}{\log a}, \quad a > 0. \]

16. \[ \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}. \]

17. \[ \int \frac{dx}{a^2 - x^2} = \begin{cases} \frac{1}{2a} \log \frac{a + x}{a - x}, \quad a^2 > x^2, \\ \frac{1}{a} \tanh^{-1} \frac{x}{a}, \quad a^2 > x^2. \end{cases} \]

18. \[ \int \frac{dx}{x^2 - a^2} = \begin{cases} \frac{1}{2a} \log \frac{x - a}{x + a}, \quad x^2 > a^2, \\ \frac{1}{a} \coth^{-1} \frac{x}{a}, \quad x^2 > a^2. \end{cases} \]

19. \[ \int \frac{dx}{\sqrt{a^2 - x^2}} = \begin{cases} \frac{1}{a} \sin^{-1} \frac{x}{a}, \quad a^2 > x^2, \\ -\frac{1}{a} \cos^{-1} \frac{x}{a}, \quad a^2 > x^2. \end{cases} \]

20. \[ \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log \left( x + \sqrt{x^2 \pm a^2} \right). \]

21. \[ \int \frac{dx}{x \sqrt{x^2 - a^2}} = \frac{1}{|a|} \sec^{-1} \frac{x}{a}. \]

22. \[ \int \frac{dx}{x \sqrt{a^2 \pm x^2}} = -\frac{1}{a} \log \left( \frac{a + \sqrt{a^2 \pm x^2}}{x} \right). \]

5.4.2 FORMS CONTAINING $a + bx$

23. \[ \int (a + bx)^n \, dx = \frac{(a + bx)^{n+1}}{(n+1)b}, \quad n \neq -1. \]

24. \[ \int x(a + bx)^n \, dx = \frac{1}{b^2(n+2)} \left( a + bx \right)^{n+2} - \frac{a}{b^2(n+1)} (a + bx)^{n+1}, \quad n \neq -1, n \neq -2. \]

25. \[ \int x^2(a + bx)^n \, dx = \frac{1}{b^3} \left[ (a + bx)^{n+3} - 2a \frac{(a + bx)^{n+2}}{n+2} + a^2 \frac{(a + bx)^{n+1}}{n+1} \right], \quad n \neq -1, n \neq -2, n \neq -3. \]
26. \[ \int x^n (a + bx)^n \, dx = \begin{cases} \frac{x^{n+1}(a + bx)^n}{m+n+1} + \frac{an}{m+n+1} \int x^n (a + bx)^{n-1} \, dx, \\ \frac{1}{a(n+1)} \left[ -x^{n+1}(a + bx)^{n+1} + (m+n+2) \int x^n (a + bx)^{n+1} \, dx \right], \end{cases} \]

or

\[ \frac{1}{b(m+n+1)} \left[ x^n (a + bx)^{n+1} - ma \int x^{n-1}(a + bx)^n \, dx \right]. \]

27. \[ \int \frac{dx}{a+bx} = \frac{1}{b} \log |(a + bx)|. \]

28. \[ \int \frac{dx}{(a+bx)^2} = -\frac{1}{b(a+bx)}. \]

29. \[ \int \frac{dx}{(a+bx)^3} = -\frac{2b(a+bx)^2}{(a+bx)^2}. \]

30. \[ \int \frac{x}{a+bx} \, dx = \begin{cases} \frac{1}{b^2} \left[ a + bx - a \log (a + bx) \right], \\ \frac{x}{b} - \frac{a}{b^2} \log (a + bx), \end{cases} \]

31. \[ \int \frac{x}{(a+bx)^2} \, dx = \frac{1}{b^2} \left[ \log (a + bx) + \frac{a}{a + bx} \right]. \]

32. \[ \int \frac{x}{(a+bx)^3} \, dx = \frac{1}{b^3} \left[ -\frac{1}{(n-2)(a+bx)^{n-1}} + \frac{a}{(n-1)(a + bx)^{n-2}} \right], \quad n \neq 1, \ n \neq 2. \]

33. \[ \int \frac{x^2}{a+bx} \, dx = \frac{1}{b^3} \left( \frac{1}{2} (a + bx)^2 - 2a(a + bx) + a^2 \log (a + bx) \right). \]

34. \[ \int \frac{x^2}{(a+bx)^2} \, dx = \frac{1}{b^3} \left( a + bx - 2a \log (a + bx) - \frac{a^2}{a + bx} \right). \]

35. \[ \int \frac{x^2}{(a+bx)^3} \, dx = \frac{1}{b^3} \left( \log (a + bx) + \frac{2a}{a + bx} - \frac{a^2}{2(a + bx)^2} \right). \]

36. \[ \int \frac{x^2}{(a+bx)^n} \, dx = \frac{1}{b^3} \left[ -\frac{1}{(n-3)(a+bx)^{n-3}} + \frac{2a}{(n-2)(a + bx)^{n-2}} \right], \quad n \neq 1, \ n \neq 2, \ n \neq 3. \]

37. \[ \int \frac{dx}{x(a + bx)} = -\frac{1}{a} \log \frac{a + bx}{x}. \]

38. \[ \int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a^2} \log \frac{a + bx}{x}. \]

39. \[ \int \frac{dx}{x(a+bx)^3} = \frac{1}{a^3} \left[ \frac{1}{2} \left( \frac{2a + bx}{a + bx} \right)^2 - \log \frac{a + bx}{x} \right]. \]

40. \[ \int \frac{dx}{x^2(a+b)} = -\frac{1}{ax - a^2} \log \frac{a + bx}{x}. \]

41. \[ \int \frac{dx}{x^3(a+b)} = \frac{2b - a}{2a^2} + \frac{a}{a^3} \log \frac{a + bx}{x}. \]

42. \[ \int \frac{dx}{x^2(a+bx)^2} = \frac{a + 2bx}{a^2x(a+b)} + \frac{2b}{a^3} \log \frac{a + bx}{x}. \]
5.4.3 FORMS CONTAINING $c^2 \pm x^2$ AND $x^2 - c^2$

43. $\int \frac{dx}{c^2 + x^2} = \frac{1}{c} \tan^{-1} \frac{x}{c}$.

44. $\int \frac{dx}{c^2 - x^2} = \frac{1}{2c} \log \frac{c + x}{c - x}$, $c^2 > x^2$.

45. $\int \frac{dx}{x^2 - c^2} = \frac{1}{2c} \log \frac{x - c}{x + c}$, $x^2 > c^2$.

46. $\int \frac{x}{c^2 \pm x^2} \, dx = \pm \frac{1}{2} \log(c^2 \pm x^2)$.

47. $\int \frac{x}{(c^2 \pm x^2)^n} \, dx = \mp \frac{1}{2n(c^2 \pm x^2)^{n-1}}$.

48. $\int \frac{dx}{(c^2 \pm x^2)^n} = \frac{1}{2c^2(n-1)} \left[ \frac{x}{(c^2 \pm x^2)^{n-1}} + (2n - 3) \int \frac{dx}{(c^2 \pm x^2)^{n-1}} \right]$.

49. $\int \frac{dx}{(x^2 - c^2)^n} = \frac{1}{2c^2(n-1)} \left[ -\frac{x}{(x^2 - c^2)^{n-1}} - (2n - 3) \int \frac{dx}{(x^2 - c^2)^{n-1}} \right]$.

50. $\int \frac{x}{x^2 - c^2} \, dx = \frac{1}{2} \log(x^2 - c^2)$.

51. $\int \frac{x}{(x^2 - c^2)^{n+1}} \, dx = -\frac{1}{2n(x^2 - c^2)^n}$.

5.4.4 FORMS CONTAINING $a + bx$ AND $c + dx$

$u = a + bx$, $v = c + dx$, and $k = ad - bc$. (If $k = 0$, then $v = (c/a)u$.)

52. $\int \frac{dx}{uv} = \frac{1}{k} \log \left( \frac{v}{u} \right)$.

53. $\int \frac{x}{uv} \, dx = \frac{1}{k} \left( \frac{a}{b} \log u - \frac{c}{d} \log v \right)$.

54. $\int \frac{dx}{u^2v} = \frac{1}{k} \left( \frac{1}{u} + \frac{d}{k} \log \frac{v}{u} \right)$.

55. $\int \frac{x}{u^2v} \, dx = -\frac{a}{bk} - \frac{c}{k^2} \log v$.

56. $\int \frac{x^2}{u^2v} \, dx = \frac{a^2}{bk^2} + \frac{1}{k^2} \left( \frac{c^2}{d} \log v + \frac{a(k - bc)}{b^2} \log u \right)$.

57. $\int \frac{dx}{u^m v^n} = \frac{1}{k(m-1)} \left[ \frac{-1}{u^{m-1} v^{n-1}} - b(m + n - 2) \int \frac{dx}{u^{m-1} v^{n-1}} \right]$.

58. $\int \frac{u}{v} \, dx = \frac{bx}{d} + k \frac{d^2}{d^2} \log v$.

59. $\int \frac{u^m}{v^n} \, dx = \begin{cases} -\frac{1}{k(n-1)} \left[ \frac{u^{m-1}}{v^{n-1}} + b(n - m - 2) \int \frac{u^m}{v^{n-1}} \, dx \right], \\ \text{or} \\ -\frac{1}{d(n - m - 1)} \left[ \frac{u^m}{v^{n-1}} + mk \int \frac{u^m}{v^n} \, dx \right], \\ \text{or} \\ -\frac{1}{d(n-1)} \left[ \frac{u^m}{v^{n-1}} - mb \int \frac{u^{m-1}}{v^{n-1}} \, dx \right]. \end{cases}$
5.4.5 FORMS CONTAINING $a + bx^n$

60. $\int \frac{dx}{a + bx^2} = \frac{1}{\sqrt{ab}} \tan^{-1} \frac{x\sqrt{ab}}{a}, \ ab > 0.$

61. $\int \frac{dx}{a + bx^2} = \begin{cases} \frac{1}{2\sqrt{-ab}} \log \frac{a + x\sqrt{-ab}}{a - x\sqrt{-ab}}, & ab < 0, \\ \frac{1}{\sqrt{-ab}} \tanh^{-1} \frac{x\sqrt{-ab}}{a}, & ab < 0. \end{cases}$

62. $\int \frac{dx}{a^2 + bx^2} = \frac{1}{ab} \tan^{-1} \frac{bx}{a}.

63. $\int \frac{x}{a + bx^2} \ dx = \frac{x}{2b} \log (a + bx^2).

64. $\int \frac{x^2}{a + bx^2} \ dx = \frac{x}{b} - \frac{a}{b} \int \frac{dx}{a + bx^2}.

65. $\int \frac{dx}{(a + bx^2)^2} = \frac{x}{2a(a + bx^2)} + \frac{1}{2a} \int \frac{dx}{a + bx^2}.

66. $\int \frac{dx}{a^2 - bx^2} = \frac{1}{2ab} \log \frac{a + bx}{a - bx}.

67. $\int \frac{dx}{(a + bx^2)^{m+1}} = \begin{cases} \frac{x}{2ma(a + bx^2)^m} + \frac{m - 1}{2ma} \int \frac{dx}{(a + bx^2)^m}, & m \neq 1 \\ \frac{x}{(a + bx^2)^m} + \frac{1}{2a} \int \frac{dx}{(a + bx^2)^m}, & m = 1 \end{cases}.

68. $\int \frac{xdx}{(a + bx^2)^{m+1}} = -\frac{1}{2bm(a + bx^2)^m}.

69. $\int \frac{x^2 \ dx}{(a + bx^2)^{m+1}} = \frac{x}{2mb(a + bx^2)^m} + \frac{1}{2mb} \int \frac{dx}{(a + bx^2)^m}.

70. $\int \frac{dx}{x(a + bx^2)} = \frac{1}{2a} \log \frac{x}{a + bx^2}.

71. $\int \frac{dx}{x^2(a + bx^2)} = -\frac{1}{ax} - \frac{b}{a} \int \frac{dx}{a + bx^2}.

72. $\int \frac{dx}{x(a + bx^2)^{m+1}} = \begin{cases} \frac{x}{2am(a + bx^2)^m} + \frac{1}{a} \int \frac{dx}{x(a + bx^2)^m}, & m \neq 1 \\ \frac{x}{2a^{m+1}} \left[ \frac{1}{r(a + bx^2)^r} \right] + \frac{1}{a} \int \frac{dx}{x(a + bx^2)^m}, & m = 1 \end{cases}.

73. $\int \frac{dx}{x^2(a + bx^2)^{m+1}} = \frac{1}{a} \int \frac{dx}{x^2(a + bx^2)^m} - \frac{b}{a} \int \frac{dx}{(a + bx^2)^{m+1}}.

74. $\int \frac{dx}{a + bx^3} = k \left[ \frac{1}{3a} \log \frac{(k + x)^3}{a + bx^3} + \sqrt{3} \tan^{-1} \frac{2x - k}{k \sqrt{3}} \right], \ k = \sqrt{\frac{a}{b}}.

75. $\int \frac{dx}{a + bx^3} = \frac{1}{3bk} \left[ \frac{1}{2} \log \frac{a + bx^3}{(k + x)^3} + \sqrt{3} \tan^{-1} \frac{2x - k}{k \sqrt{3}} \right], \ k = \sqrt{\frac{a}{b}}.$
76. \[ \int \frac{x^2 \, dx}{a + bx^3} = \frac{1}{3b} \log a + bx^3. \]

77. * \[ \int \frac{dx}{a + bx^3} = \begin{cases} \frac{k}{2a} \left[ \frac{1}{2} \log \frac{x^2 + 2kx + 2k^2 + \tan^{-1} \frac{2kx}{2k^2 - x^2}}{x^2 - 2kx + 2k^2} \right], & ab > 0, k = \left( \frac{a}{4b} \right)^{1/4}, \\ \frac{k}{2a} \left[ \frac{1}{2} \log \frac{x + k + \tan^{-1} x}{x - k} \right], & ab < 0, k = \left( -\frac{a}{b} \right)^{1/4}. \end{cases} \]

78. \[ \int \frac{x}{a + bx^4} \, dx = \frac{1}{2bk} \tan^{-1} \frac{x^2}{k} , \quad ab > 0, k = \sqrt{b}. \]

79. \[ \int \frac{x}{a + bx^4} \, dx = \frac{1}{4bk} \log \frac{x^2 - k}{x^2 + k} , \quad ab < 0, k = \sqrt{-b}. \]

80. \[ \int \frac{x^2}{a + bx^4} \, dx = \frac{1}{4bk} \left[ \frac{1}{2} \log \frac{x^2 - 2kx + 2k^2}{x^2 + 2kx + 2k^2} + \tan^{-1} \frac{2kx}{2k^2 - x^2} \right]. \]

81. \[ \int \frac{x^2 \, dx}{a + bx^4} = \frac{1}{4bk} \left( \frac{x - k}{x + k} + 2 \tan^{-1} \frac{x}{k} \right) , \quad ab < 0, k = \sqrt{-b}. \]

82. \[ \int \frac{x^3 \, dx}{a + bx^4} = \frac{1}{4b} \log (a + bx^4). \]

83. \[ \int \frac{dx}{x(a + bx^n)} = \frac{1}{an} \log \frac{x^n}{a + bx^n}. \]

84. \[ \int \frac{dx}{(a + bx^n)^{m+1}} = \frac{1}{a} \int \frac{dx}{(a + bx^n)^m} - \frac{b}{a} \int \frac{x^n \, dx}{(a + bx^n)^{m+1}}. \]

85. \[ \int \frac{x^n \, dx}{(a + bx^n)^{p+1}} = \frac{1}{b} \int \frac{dx}{(a + bx^n)^p} - \frac{a}{b} \int \frac{x^{m-n} \, dx}{(a + bx^n)^{p+1}}. \]

86. \[ \int \frac{dx}{x^n(a + bx^n)^{p+1}} = \frac{1}{a} \int \frac{dx}{x^n(a + bx^n)^p} - \frac{b}{a} \int \frac{x^{m-n} \, dx}{(a + bx^n)^{p+1}}. \]

87. \[ \int x^n(a + bx^n)^p \, dx = \begin{cases} \frac{1}{b(np + m + 1)} \left[ x^{m+n+1}(a + bx^n)^{p+1} - a(m - n + 1) \int x^n(a + bx^n)^p \, dx \right], & \\ \frac{1}{np + m + 1} \left[ x^{m+1}(a + bx^n)^p + anp \int x^n(a + bx^n)^{p-1} \, dx \right], & \\ \frac{1}{a(m + 1)} \left[ x^{m+1}(a + bx^n)^{p+1} - b(m + 1 + np + n) \int x^{m+n}(a + bx^n)^p \, dx \right], & \\ \frac{1}{an(p + 1)} \left[ -x^{m+1}(a + bx^n)^{p+1} + (m + 1 + np + n) \int x^n(a + bx^n)^{p+1} \, dx \right], \end{cases} \]

5.4.6 FORMS CONTAINING \( c^3 \pm x^3 \)

88. \[ \int \frac{dx}{c^3 \pm x^3} = \pm \frac{1}{6c^2} \log \left( \frac{(c \pm x)^3}{c^3 \pm x^3} \right) + \frac{1}{2c^2} \tan^{-1} \frac{2x}{c}. \]

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FORMS CONTAINING $c^4 \pm x^4$

$\int \frac{dx}{c^4 + x^4} = \frac{1}{2c^2 \sqrt{2}} \left[ \frac{1}{2} \log \left( \frac{x^2 + cx \sqrt{2} + c^2}{x^2 - cx \sqrt{2} + c^2} \right) + \tan^{-1} \frac{cx \sqrt{2}}{c^2 - x^2} \right]$. 

$\int \frac{dx}{c^4 - x^4} = \frac{1}{2c^2} \left[ \frac{1}{2} \log \left( \frac{c + x}{c - x} + \tan^{-1} \frac{x}{c} \right) \right]$. 

$\int \frac{x \, dx}{c^4 + x^4} = \frac{1}{2c^2} \tan^{-1} \frac{x^2}{c^2}$. 

$\int \frac{x \, dx}{c^4 - x^4} = \frac{1}{4c^2} \log \left( \frac{c^2 + x^2}{c^2 - x^2} \right)$. 

$\int \frac{x^2 \, dx}{c^4 + x^4} = \frac{1}{2c \sqrt{2}} \left[ \frac{1}{2} \log \left( \frac{x^2 - cx \sqrt{2} + c^2}{x^2 + cx \sqrt{2} + c^2} \right) + \tan^{-1} \frac{cx \sqrt{2}}{c^2 - x^2} \right]$. 

$\int \frac{x^2 \, dx}{c^4 - x^4} = \frac{1}{2c} \left[ \frac{1}{2} \log \left( \frac{c + x}{c - x} - \tan^{-1} \frac{x}{c} \right) \right]$. 

$\int \frac{x^3 \, dx}{c^4 \pm x^4} = \pm \frac{1}{4} \log (c^4 \pm x^4)$. 

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5.4.8 FORMS CONTAINING $a + bx + cx^2$

\[ X = a + bx + cx^2 \quad \text{and} \quad q = 4ac - b^2. \]

If $q = 0$, then $X = c \left( x + \frac{b}{2c} \right)^2$ and other formulae should be used.

108. \[ \int \frac{dx}{X} = \begin{cases} \frac{2}{\sqrt{q}} \tan^{-1} \left( \frac{2cx + b}{\sqrt{q}} \right), & q > 0, \\ \frac{-2}{\sqrt{-q}} \tanh^{-1} \left( \frac{2cx + b}{\sqrt{-q}} \right), & q < 0, \\ \frac{1}{\sqrt{-q}} \log \frac{2cx + b - \sqrt{-q}}{2cx + b + \sqrt{-q}}, & q < 0. \end{cases} \]

109. \[ \int \frac{dx}{X^2} = \frac{2cx + b}{qX} + \frac{2c}{q} \int \frac{dx}{X}. \]

110. \[ \int \frac{dx}{X^3} = \frac{2cx + b}{q} \left( \frac{1}{2X^2} + \frac{3c}{qX} \right) + \frac{6c^2}{q^2} \int \frac{dx}{X} \]

111. \[ \int \frac{dx}{X^{n+1}} = \begin{cases} \frac{(2n)!}{(n!)^2} \left( \frac{c}{q} \right)^n \left[ \frac{2cx + b}{q} \sum_{r=1}^{n} \left( \frac{q}{cX} \right)^r \right] + \int \frac{dx}{X}, & n \neq 0, \\ \frac{2cx + b}{qX} + \frac{1}{2} \int \frac{dx}{X}, & n = 0. \end{cases} \]

112. \[ \int \frac{x \log X}{X} = \frac{1}{2c} \log X - \frac{b}{2c} \int \frac{dx}{X}. \]

113. \[ \int \frac{x \log X - \frac{b}{2c}}{X} \int \frac{dx}{X}. \]

114. \[ \int \frac{x \log X}{X^n} = -\frac{b(2n-1)c}{nq} \int \frac{dx}{X^n}. \]

115. \[ \int \frac{x^2 \log X - \frac{b}{2c}}{X} \int \frac{dx}{X}. \]

116. \[ \int \frac{x^2 \log X}{X^2} = \frac{(b^2 - 2ac)x + ab}{cX} + \frac{2a}{q} \int \frac{dx}{X}. \]

117. \[ \int \frac{x^m \log X}{X^{n+1}} = \frac{x^{m-1}}{2n-m+1} \int \frac{x^{n-1}}{X^{n+1}} \int dx \]

118. \[ \int \frac{dx}{xX} = \frac{1}{2a} \log \frac{x^2}{X} - \frac{b}{2a} \int \frac{dx}{X}. \]

119. \[ \int \frac{dx}{x^2X} = \frac{b}{2a} \log \frac{X}{x} - \frac{1}{ax} + \left( \frac{b^2}{2a^2} - \frac{c}{a} \right) \int \frac{dx}{X}. \]

120. \[ \int \frac{dx}{xX^n} = \frac{1}{2a(n-1)X^{n-1}} - \frac{b}{2a} \int \frac{dx}{X^n} + \frac{1}{a} \int \frac{dx}{xX^{n-1}}. \]
\[\int\frac{dx}{x^nX^{n+1}} = -\frac{1}{(m-1)aX^{m-1}X^n} \frac{n + m - 1}{m - 1} - \frac{b}{a} \int\frac{dx}{x^{m-1}X^{n+1}} \frac{2n + m - 1}{m - 1} - \frac{c}{b} \int\frac{dx}{x^nX^{n+1}}.\]

### 5.4.9 FORMS CONTAINING $\sqrt{a + bx}$

122. \[\int \sqrt{a + bx} \, dx = \frac{2}{3b} \sqrt{(a + bx)^3}.\]

123. \[\int x\sqrt{a + bx} \, dx = -\frac{2(2a - 3bx)}{15b^2} \sqrt{(a + bx)^3}.\]

124. \[\int x^2\sqrt{a + bx} \, dx = \frac{2(8a^2 - 12abx + 15b^2x^2)}{105b^3} \sqrt{(a + bx)^3}.\]

125. \[\int x^m\sqrt{a + bx} \, dx = \begin{cases} 
\frac{2}{b(2m + 3)} \left[ x^m \sqrt{(a + bx)^3} - ma \int x^{m-1} \sqrt{a + bx} \, dx \right], \\
\frac{2}{b(2m + 1)} \sqrt{a + bx} \sum_{r=0}^{m} \frac{m!(-a)^{m-r}}{r!(m-r)!(2r+3)} (a + bx)^{r+1}.
\end{cases}\]

126. \[\int \frac{\sqrt{a + bx}}{x} \, dx = 2\sqrt{a + bx} + a \int \frac{dx}{x\sqrt{a + bx}}.\]

127. \[\int \frac{\sqrt{a + bx}}{x^m} \, dx = -\frac{1}{(m-1)a} \left[ \frac{(a + bx)^{3/2}}{x^{m-1}} + \frac{(2m - 5)b}{2} \int \frac{\sqrt{a + bx}}{x^{m-1}} \, dx \right].\]

128. \[\int \frac{dx}{\sqrt{a + bx}} = \frac{2\sqrt{a + bx}}{b}.\]

129. \[\int \frac{x \, dx}{\sqrt{a + bx}} = -\frac{2(2a - 3bx)}{3b^2} \sqrt{a + bx}.\]

130. \[\int \frac{x^2 \, dx}{\sqrt{a + bx}} = \frac{2(8a^2 - 4abx + 3b^2x^2)}{15b^3} \sqrt{a + bx}.\]

131. \[\int \frac{x^m \, dx}{\sqrt{a + bx}} = \begin{cases} 
\frac{2}{(2m+1)b} \left[ x^m \sqrt{a + bx} - ma \int \frac{x^{m-1}}{\sqrt{a + bx}} \, dx \right], \\
\frac{2(-a)^m \sqrt{a + bx}}{b^{m+1}} \sum_{r=0}^{m} \frac{(-1)^r m!(a + bx)^r}{(2r+1)r!(m-r)!a^{r+1}}.
\end{cases}\]

132. \[\int \frac{dx}{x\sqrt{a + bx}} = \begin{cases} 
\sqrt{a + bx} - a, & a < 0, \\
\sqrt{a + bx} - a, & a > 0.
\end{cases}\]

133. \[\int \frac{dx}{x^2\sqrt{a + bx}} = -\frac{\sqrt{a + bx}}{ax} - \frac{b}{2a} \int \frac{dx}{x\sqrt{a + bx}}.\]

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\[ \int \frac{dx}{x^{n} \sqrt{a + bx}} = \begin{cases} \frac{1}{n-1} \frac{(2n-3)b}{(2n-2)a} \int \frac{dx}{x^{n-1} \sqrt{a + bx}}, \\ \frac{(2n-2)!}{(n-1)!!} \left[ -\frac{\sqrt{a + bx}}{a} \sum_{r=1}^{n} \frac{r}{(r-1)!} x'(2r)! \left( -\frac{b}{4a} \right)^{n-r-1} + \left( -\frac{b}{4a} \right)^{n-1} \int \frac{dx}{x^{n-1} \sqrt{a + bx}} \right] \end{cases} \]

5.4.10 FORMS CONTAINING \( \sqrt{a + bx} \) AND \( \sqrt{c + dx} \)

\[ u = a + bx, \quad v = c + dx, \quad k = ad - bc. \]

If \( k = 0 \), then \( v = \frac{e}{a} u \), and other formulae should be used.

\[ \int \frac{dx}{\sqrt{uv}} = \begin{cases} \frac{2}{\sqrt{bd}} \tanh^{-1} \frac{\sqrt{bd}uv}{bv}, & bd > 0, \quad k < 0, \\ \frac{2}{\sqrt{bd}} \tanh^{-1} \frac{\sqrt{bd}uv}{dv}, & bd > 0, \quad k > 0, \\ \frac{1}{\sqrt{bd}} \log \left( \frac{bv + \sqrt{bd}uv}{2 \pm n} \right), & bd > 0, \\ \frac{2}{\sqrt{-bd}} \tan^{-1} \frac{bv}{\sqrt{-bd}uv}, & bd < 0, \\ \frac{1}{\sqrt{-bd}} \sin^{-1} \left( \frac{2bdx + ad + bc}{|k|} \right), & bd < 0. \end{cases} \]

\[ \int \sqrt{uv} dx = \frac{k + 2bv}{4bd} \sqrt{uv} - \frac{k^2}{8bd} \int \frac{dx}{\sqrt{uv}}. \]

\[ \int \frac{dx}{v \sqrt{u}} = \begin{cases} \frac{1}{\sqrt{kd}} \log \frac{d \sqrt{u} - \sqrt{kd}}{d \sqrt{u} + \sqrt{kd}}, & kd > 0, \\ \frac{2}{\sqrt{-kd}} \tan^{-1} \frac{d \sqrt{u}}{-\sqrt{-kd}}, & kd < 0. \end{cases} \]

\[ \int \frac{x \, dx}{\sqrt{uv}} = \frac{\sqrt{uv}}{bd} - \frac{ad + bc}{2bd} \int \frac{dx}{\sqrt{uv}}. \]

\[ \int \frac{dx}{v \sqrt{uv}} = \frac{2 \sqrt{uv}}{kv}. \]
\[ 144. \int \frac{v \, dx}{\sqrt{uv}} = \frac{\sqrt{uv}}{b} - \frac{k}{2b} \int \frac{dx}{\sqrt{uv}}. \]

\[ 145. \sqrt{\frac{v}{u}} \, dx = \frac{v}{|v|} \int \frac{v \, dx}{\sqrt{uv}}. \]

\[ 146. \int \frac{v^m \sqrt{u} \, dx}{v^m} = \frac{1}{(2m + 3)d} \left( 2v^{m+1} \sqrt{u} + k \int \frac{v^m \, dx}{\sqrt{u}} \right). \]

\[ 147. \int \frac{dx}{v^m \sqrt{u}^n} = -\frac{1}{(m-1)k} \left( \frac{n}{\sqrt{v^m}} + (m - \frac{3}{2}) b \int \frac{dx}{v^{m-1} \sqrt{u}} \right). \]

\[ 148. \int \frac{v^m}{\sqrt{u}} \, dx = \begin{cases} 
\frac{2}{b(2m+1)} \left( v^m \sqrt{u} - mk \int \frac{v^{m-1} \, dx}{\sqrt{u}} \right), \\
\text{or} \\
\frac{2(m!)^2 \sqrt{u}}{b(2m+1)!} \sum_{r=0}^{\infty} \left( \frac{-4k}{b} \right)^{m-r} (2r)! \frac{v^m}{(r!)^2} \sqrt{u}. 
\end{cases} \]

### 5.4.11 FORMS CONTAINING \( \sqrt{x^2 \pm a^2} \)

\[ 149. \int \sqrt{x^2 \pm a^2} \, dx = \frac{1}{2} \left[ x \sqrt{x^2 \pm a^2} \pm a^2 \log x + \sqrt{x^2 \pm a^2} \right]. \]

\[ 150. \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log x + \sqrt{x^2 \pm a^2}. \]

\[ 151. \int \frac{dx}{x \sqrt{x^2 - a^2}} = \frac{1}{|a|} \sec^{-1} \frac{x}{a}. \]

\[ 152. \int \frac{dx}{x \sqrt{x^2 + a^2}} = -\frac{1}{a} \log \left( \frac{a + \sqrt{x^2 + a^2}}{x} \right). \]

\[ 153. \int \frac{x^2 + a^2}{x} \, dx = \sqrt{x^2 + a^2} - a \log \left( \frac{a + \sqrt{x^2 + a^2}}{x} \right). \]

\[ 154. \int \frac{x^2 - a^2}{x} \, dx = \sqrt{x^2 - a^2} - |a| \sec^{-1} \frac{x}{a}. \]

\[ 155. \int \frac{x}{\sqrt{x^2 \pm a^2}} \, dx = \sqrt{x^2 \pm a^2}. \]

\[ 156. \int x \sqrt{x^2 \pm a^2} \, dx = \frac{1}{3} (x^2 \pm a^2)^{3/2}. \]

\[ 157. \int (x^2 \pm a^2)^{3/2} \, dx = \frac{1}{4} \left[ x \sqrt{(x^2 \pm a^2)^2} \pm \frac{3a^2 x}{2} \sqrt{x^2 \pm a^2} \right. \]

\[ + \frac{3a^4}{8} \log \left( x + \sqrt{x^2 \pm a^2} \right) \right]. \]

\[ 158. \int \frac{dx}{\sqrt{(x^2 \pm a^2)^2}} = \pm \frac{x}{a^2 \sqrt{x^2 \pm a^2}}. \]

\[ 159. \int \frac{x}{\sqrt{(x^2 \pm a^2)^2}} \, dx = - \frac{1}{\sqrt{x^2 \pm a^2}}. \]

\[ 160. \int x \sqrt{(x^2 \pm a^2)^2} \, dx = \frac{1}{5} (x^2 \pm a^2)^3. \]

\[ 161. \int x^2 \sqrt{x^2 \pm a^2} \, dx = \frac{x}{4} \sqrt{(x^2 \pm a^2)^3} + \frac{a^2}{8} \sqrt{x^2 \pm a^2} - \frac{a^4}{8} \log x + \sqrt{x^2 \pm a^2}. \]

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162. \[ \int x^3 \sqrt{x^2 + a^2} \, dx = \frac{1}{15} (3x^2 - 2a^2) \sqrt{x^2 + a^2}. \]
163. \[ \int x^3 \sqrt{x^2 - a^2} \, dx = \frac{1}{5} \sqrt{(x^2 - a^2)^3} + \frac{a^2}{3} (x^2 - a^2)^{\frac{3}{2}}. \]
164. \[ \int \frac{x^2}{\sqrt{x^2 \pm a^2}} \, dx = \frac{x}{2} \sqrt{x^2 \pm a^2} \mp \frac{a^2}{2} \log |x + \sqrt{x^2 \pm a^2}|. \]
165. \[ \int \frac{x^3}{\sqrt{x^2 \pm a^2}} \, dx = \frac{1}{3} (x^2 \pm a^2)^{\frac{3}{2}} \mp a^2 \sqrt{x^2 \pm a^2}. \]
166. \[ \int \frac{dx}{x \sqrt{x^2 \pm a^2}} \, dx = \mp \sqrt{x^2 \pm a^2} \frac{a}{a x}. \]
167. \[ \int \frac{dx}{x^3 \sqrt{x^2 + a^2}} = -\sqrt{x^2 + a^2} \frac{a}{2a x^2} + \frac{1}{2} \log \left( a + \sqrt{x^2 + a^2} \right) \frac{a}{x}. \]
168. \[ \int \frac{dx}{x^3 \sqrt{x^2 - a^2}} = -\sqrt{x^2 - a^2} \frac{a}{2a x^2} + \frac{1}{2 |a|} \sec^{-1} \frac{x}{a}. \]
169. \[ \int x^3 (x^2 \pm a^2)^{\frac{3}{2}} \, dx = \frac{x}{6} (x^2 \pm a^2)^{\frac{3}{2}} \mp \frac{a^2 x}{24} \sqrt{(x^2 \pm a^2)^3} - \frac{a^4 x}{16} \sqrt{x^2 \pm a^2} \mp \frac{a^6}{16} \log \left( x + \sqrt{x^2 \pm a^2} \right). \]
170. \[ \int x^3 (x^2 \pm a^2)^{\frac{3}{2}} \, dx = \frac{x}{7} (x^2 \pm a^2)^{\frac{3}{2}} \mp \frac{a^2}{5} (x^2 \pm a^2)^{\frac{3}{2}}. \]
171. \[ \int \frac{x^2 \pm a^2}{x^2} \, dx = -\sqrt{x^2 \pm a^2} \frac{a}{x} + \log \left( x + \sqrt{x^2 \pm a^2} \right). \]
172. \[ \int \frac{x^2 + a^2}{x^3} \, dx = -\sqrt{x^2 + a^2} \frac{a}{2x^2} \frac{1}{2} \log \left( a + \sqrt{x^2 + a^2} \right) \frac{a}{x}. \]
173. \[ \int \frac{x^2 - a^2}{x^3} \, dx = -\sqrt{x^2 - a^2} \frac{a}{2x^2} + \frac{1}{2} \sec^{-1} \frac{x}{a}. \]
174. \[ \int \frac{x^2 \pm a^2}{x^3} \, dx = \mp \left( \sqrt{x^2 \pm a^2} \right)^{\frac{1}{3}} \frac{3a^3 x}{a^3}. \]
175. \[ \int \frac{x^2 \, dx}{(x^2 \pm a^2)^{\frac{3}{2}}} = \frac{x}{\sqrt{x^2 \pm a^2}} \mp \log \left( x + \sqrt{x^2 \pm a^2} \right). \]
176. \[ \int \frac{x^3 \, dx}{(x^2 \pm a^2)^{\frac{3}{2}}} = \sqrt{x^2 \pm a^2} \pm \frac{a^2}{x^2 \pm a^2}. \]
177. \[ \int \frac{dx}{x \sqrt{x^2 + a^2}} = \frac{1}{a^2} \sqrt{x^2 + a^2} - \frac{1}{a} \log \left( a + \sqrt{x^2 + a^2} \right) \frac{a}{x}. \]
178. \[ \int \frac{dx}{x \sqrt{x^2 - a^2}} = -\frac{1}{a^2} \sqrt{x^2 - a^2} - \frac{1}{a} \sec^{-1} \frac{x}{a}. \]
179. \[ \int \frac{dx}{x^2 \sqrt{(x^2 \pm a^2)^3}} = \frac{1}{a^4} \left[ \frac{\sqrt{x^2 \pm a^2}}{x} + \frac{1}{\sqrt{x^2 \pm a^2}} \right]. \]
180. \[ \int \frac{dx}{x^3 \sqrt{(x^2 \pm a^2)^3}} = \frac{1}{2a^2 x^2} \sqrt{x^2 \pm a^2} - \frac{3}{2a} \log \left( a + \sqrt{x^2 + a^2} \right) \frac{a}{x}. \]
181. \[ \int \frac{dx}{x^3 \sqrt{(x^2 - a^2)^3}} = \frac{1}{2a^2 x^2} \sqrt{x^2 - a^2} - \frac{3}{2a^2} \sqrt{x^2 - a^2} - \frac{3}{2a} \sec^{-1} \frac{x}{a}. \]
\[ \int \frac{x^n}{\sqrt{x^2 \pm a^2}} \, dx = \frac{1}{m} \sqrt{x^2 \pm a^2} + \frac{m - 1}{m} a^2 \int \frac{x^{m-2}}{\sqrt{x^2 \pm a^2}} \, dx. \]

\[ \int \frac{x^{2m}}{\sqrt{x^2 \pm a^2}} \, dx = \frac{(2m)!}{(m!)(m!)} \left[ \sqrt{x^2 \pm a^2} \sum_{r=1}^{m} \frac{r!(r-1)!}{(2r)!} \frac{(m^2 - a^2)(2x)^{2r-1}}{(2r)!} \right]. \]

\[ \int \frac{x^{2m+1}}{\sqrt{x^2 \pm a^2}} \, dx = \sqrt{x^2 \pm a^2} \sum_{r=0}^{m} \frac{(2r)!(m!)^2}{(m+1)!}(r^2)\log |x + \sqrt{x^2 \pm a^2}|. \]

\[ \int \frac{x^m}{\sqrt{x^2 \pm a^2}} \, dx = \mp \sqrt{x^2 \pm a^2} \mp (m-2) \int \frac{dx}{(m-1)a^x \sqrt{x^2 \pm a^2}}. \]

\[ \int \frac{x^a}{\sqrt{x^2 \pm a^2}} \, dx = \sqrt{x^2 \pm a^2} \sum_{r=0}^{m} (m-1)!m!(2r)!(2m)! (2r)! \log |x + \sqrt{x^2 \pm a^2}|. \]

\[ \int \frac{x^a}{\sqrt{x^2 - a^2}} \, dx = \frac{1}{a} \log \left( a + \sqrt{x^2 - a^2} \right). \]

\[ \int \frac{x}{\sqrt{x^2 - a^2}} \, dx = - \frac{x}{a(\sqrt{x^2 - a^2})}. \]

\[ \int \frac{x}{\sqrt{a^2 - x^2}} \, dx = \frac{1}{a} \log \left( \frac{a + \sqrt{a^2 - x^2}}{x} \right). \]

\[ \int \frac{x}{\sqrt{a^2 - x^2}} \, dx = - \sqrt{a^2 - x^2}. \]

\[ \int \frac{x}{\sqrt{a^2 - x^2}} \, dx = - \frac{1}{2} \sqrt{(a^2 - x^2)^3}. \]

\[ \int \frac{dx}{\sqrt{(a^2 - x^2)^3}} = \frac{1}{a} \left( \sqrt{a^2 - x^2} + \frac{3a^2 x}{2} \sqrt{a^2 - x^2} + \frac{3a^4}{2} \sin^{-1} \frac{x}{a} \right). \]

\[ \int \frac{dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{a^2 \sqrt{a^2 - x^2}}. \]
\[
\begin{align*}
199. & \quad \int \frac{x}{\sqrt{(a^2-x^2)^3}} \, dx = \frac{1}{a^3} \log \left( \frac{a^2-x^2}{a^2} \right) \\
200. & \quad \int x \sqrt{(a^2-x^2)^3} \, dx = -\frac{1}{3} \sqrt{(a^2-x^2)^3}.
\end{align*}
\]
\[
\begin{align*}
201. & \quad \int x^2 \sqrt{a^2-x^2} \, dx = -\frac{x}{4} \sqrt{(a^2-x^2)^3} + \frac{a^2}{8} \left( x \sqrt{a^2-x^2} + a^2 \sin^{-1} \frac{x}{a} \right). \\
202. & \quad \int x^3 \sqrt{a^2-x^2} \, dx = \left( -\frac{1}{5} x^2 - \frac{2}{15} a^3 \right) \sqrt{(a^2-x^2)^3}. \\
203. & \quad \int x^2 \sqrt{(a^2-x^2)^3} \, dx = -\frac{1}{6} x \sqrt{(a^2-x^2)^3} + \frac{a^2 x}{24} \sqrt{(a^2-x^2)^3} \\
& \quad + \frac{a^4 x}{16} \sqrt{a^2-x^2} + \frac{a^6}{16} \sin^{-1} \frac{x}{a}.
\end{align*}
\]
\[
\begin{align*}
204. & \quad \int x^3 \sqrt{(a^2-x^2)^3} \, dx = \frac{1}{7} \sqrt{(a^2-x^2)^3} - \frac{a^2}{5} \sqrt{(a^2-x^2)^3}. \\
205. & \quad \int \frac{x^2}{\sqrt{a^2-x^2}} \, dx = -\frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}. \\
206. & \quad \int \frac{d x}{x^2 \sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2 x}. \\
207. & \quad \int \frac{\sqrt{a^2-x^2}}{x^2} \, dx = -\sqrt{a^2-x^2} \sin^{-1} \frac{x}{a}. \\
208. & \quad \int \frac{\sqrt{a^2-x^2}}{x^3} \, dx = -\frac{\sqrt{a^2-x^2}}{2x^2} + \frac{1}{2a} \log \left( \frac{a + \sqrt{a^2-x^2}}{x} \right). \\
209. & \quad \int \frac{\sqrt{a^2-x^2}}{x^4} \, dx = -\frac{(a^2-x^2)^3}{3a^3 x}. \\
210. & \quad \int \frac{x}{\sqrt{(a^2-x^2)^3}} \, dx = \frac{x}{\sqrt{a^2-x^2}} - \frac{x}{a} \sin^{-1} \frac{x}{a}. \\
211. & \quad \int \frac{3}{x^3} \sqrt{(a^2-x^2)^3} \, dx = -\frac{2}{3} \sqrt{(a^2-x^2)^3} - \frac{x \sqrt{a^2-x^2}}{2x^2}.
\end{align*}
\]
\[
\begin{align*}
212. & \quad \int \frac{x^3}{\sqrt{(a^2-x^2)^3}} \, dx = \frac{2}{3} \sqrt{(a^2-x^2)^3} - \frac{x \sqrt{a^2-x^2}}{2a^2 x^2} = \frac{a^2}{\sqrt{a^2-x^2}} + \sqrt{a^2-x^2}. \\
213. & \quad \int \frac{d x}{x^3 \sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{2a^2 x^2} - \frac{1}{2a} \log \left( \frac{a + \sqrt{a^2-x^2}}{x} \right). \\
214. & \quad \int \frac{d x}{x \sqrt{(a^2-x^2)^3}} = \frac{1}{a^3} \log \left( \frac{a + \sqrt{a^2-x^2}}{x} \right). \\
215. & \quad \int \frac{d x}{x^2 \sqrt{(a^2-x^2)^3}} = \frac{1}{a^3} \left( -\frac{\sqrt{a^2-x^2}}{x} + \frac{x}{\sqrt{a^2-x^2}} \right). \\
216. & \quad \int \frac{d x}{x^3 \sqrt{(a^2-x^2)^3}} = -\frac{3}{2a^2 x^2 \sqrt{a^2-x^2}} + \frac{3}{2a^2 \sqrt{a^2-x^2}} - \frac{3}{2a^2} \log \left( \frac{a + \sqrt{a^2-x^2}}{x} \right).
\end{align*}
\]
\[
\begin{align*}
217. & \quad \int \frac{x^m}{\sqrt{a^2-x^2}} \, dx = -\frac{x^{m-1} \sqrt{a^2-x^2}}{m} + \frac{(m-1)a^2}{m} \int \frac{x^{m-2}}{\sqrt{a^2-x^2}} \, dx. \\
218. & \quad \int \frac{x^{2m}}{\sqrt{a^2-x^2}} \, dx = \frac{(2m)!}{(m!)^2} \left[ -\frac{\sqrt{a^2-x^2} \sum_{r=1}^{m} \frac{r!(r-1)!}{2^{2m-2r+1}(2r)!} a^{2m-2r} x^{2r-1}}{2^{2m}} \sin^{-1} \frac{x}{a} \right] + \frac{a^{2m}}{2^{2m}} \sin^{-1} \frac{x}{a}.
\end{align*}
\]
219. \[ \int \frac{x^{2m+1}}{\sqrt{a^2 - x^2}} \, dx = -\sqrt{a^2 - x^2} \sum_{r=0}^{m} \frac{(2r)!(ml)^2}{(2m+1)!(r)!} (4a^2)^{m-r} x^{2r}. \]

220. \[ \int \frac{dx}{(m-1)a^2x^{m-1}} = -\frac{\sqrt{a^2 - x^2}}{(m-1)a^2} + \frac{(m-2)}{(m-1)a^2} \int \frac{dx}{x^{m-2}\sqrt{a^2 - x^2}}. \]

221. \[ \int \frac{dx}{x^{2m} \sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{(m-1)!(2m-2)\sqrt{a^2 - x^2}} \sum_{r=0}^{m-1} (4a^2)^{2m-2r} x^{2r+1}. \]

222. \[ \int \frac{dx}{x^{2m+1} \sqrt{a^2 - x^2}} = \frac{(2m)!}{(ml)^2} \left[ -\frac{\sqrt{a^2 - x^2}}{a^2} \sum_{r=1}^{m} \frac{1}{2(2r)!} (4a^2)^{m-r} x^{2r} \right] + \frac{1}{2^{2m}a^{2m+1}} \log \frac{a - \sqrt{a^2 - x^2}}{x}. \]

223. \[ \int \frac{dx}{(b^2 - x^2) \sqrt{a^2 - x^2}} = \begin{cases} \frac{1}{2b\sqrt{a^2 - b^2}} \log \frac{(b\sqrt{a^2 - x^2} + x\sqrt{a^2 - b^2})^2}{b^2 - x^2}, & a^2 > b^2, \\ \frac{1}{b\sqrt{a^2 - b^2}} \tan^{-1} \frac{x\sqrt{b^2 - a^2}}{b\sqrt{a^2 - x^2}}, & b^2 > a^2. \end{cases} \]

224. \[ \int \frac{dx}{(b^2 + x^2) \sqrt{a^2 - x^2}} = \frac{1}{b\sqrt{a^2 + b^2}} \tan^{-1} \frac{x\sqrt{a^2 + b^2}}{b\sqrt{a^2 - x^2}}. \]

225. \[ \int \frac{\sqrt{a^2 - x^2}}{b^2 + x^2} \, dx = \frac{\sqrt{a^2 + b^2}}{|b|} \sin^{-1} \frac{x\sqrt{a^2 + b^2}}{|a|\sqrt{x^2 + b^2}} - \sin^{-1} \frac{x}{|a|}, \quad b^2 > a^2. \]

5.4.13 FORMS CONTAINING \( \sqrt{a + bx + cx^2} \)

\( X = a + bx + cx^2 \), \( q = 4ac - b^2 \), and \( k = 4c/q \). If \( q = 0 \), then \( \sqrt{X} = \sqrt{c} |x + \frac{b}{2c}| \).

226. \[ \int \frac{dx}{\sqrt{X}} = \begin{cases} \frac{1}{\sqrt{c}} \log \frac{2\sqrt{c}X + 2cx + b}{\sqrt{q}}, & c > 0, \\ \frac{1}{\sqrt{c}} \sinh^{-1} \frac{2cx + b}{\sqrt{q}}, & c > 0, \\ -\frac{1}{\sqrt{-c}} \sin^{-1} \frac{2cx + b}{\sqrt{-q}}, & c < 0. \end{cases} \]

227. \[ \int \frac{dx}{X\sqrt{X}} = \frac{2(2cx + b)}{3q\sqrt{X}}. \]

228. \[ \int \frac{dx}{X^2\sqrt{X}} = \frac{2(2cx + b)}{3q\sqrt{X}} \left( \frac{1}{X} + 2k \right). \]

229. \[ \int \frac{dx}{X^a\sqrt{X}} = \begin{cases} \frac{2(2cx + b)\sqrt{X}}{(2n-1)q} + \frac{2k(n-1)}{2n-1} \int \frac{dx}{X^{n-1}\sqrt{X}}, \\ \frac{(2cx + b)(n!)\sqrt{X} - (2n-1)!\sqrt{X}}{q(2n)!\sqrt{X}} \sum_{r=0}^{n-1} \frac{(2r)!}{(4kX)^r(r!)^2}, \end{cases} \]

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\[ \int \sqrt[3]{X} \, dx = \frac{2(cx+b)}{4c} + \frac{1}{2k} \int \frac{dx}{\sqrt{X}} \]

\[ \int X \sqrt[3]{X} \, dx = \frac{(2cx+b)X^{1/3}}{8c} + \frac{3}{8k^2} \int \frac{dx}{\sqrt{X}} \]

\[ \int X^2 \sqrt[3]{X} \, dx = \frac{(2cx+b)X^{5/3}}{12c} + \frac{15}{8k^2} \int \frac{dx}{\sqrt{X}} \]

\[ \int X^n \sqrt[3]{X} \, dx = \begin{cases} 
\frac{(2cx+b)X^{n+1/3}}{4(n+1)c} + \frac{2n+1}{2(n+1)k} \int X^{n-1} \sqrt[3]{X} \, dx, \\
\frac{(2n+2)!}{(n+1)!} \left[ \frac{k(2cx+b)X^{n+1/3}}{c} \sum_{r=0}^{n} \frac{r!(r+1)!(4kX)^r}{(2r+2)!} + \int \frac{dx}{\sqrt{X}} \right]. 
\end{cases} \]

\[ \int \frac{x \, dx}{X} = -\frac{b}{2c} \int \frac{dx}{\sqrt{X}} \]

\[ \int \frac{x \, dx}{X \sqrt[3]{X}} = -\frac{\sqrt[3]{X}}{(2n-1)cX^n} - \frac{b}{2c} \int \frac{dx}{X^{n-1/3}} \]

\[ \int \frac{x^2 \, dx}{X \sqrt[3]{X}} = \frac{(2b^2-4ac)x + 2ab}{(2n-1)cnX^{n-1/3}} + \frac{4ac + (2n-3)b^2}{8c^2} \int \frac{dx}{X^{n-1} \sqrt{X}} \]

\[ \int \frac{x^3 \, dx}{X \sqrt[3]{X}} = \frac{(x^2 - 5bx + 5b^2X^{2/3}) - 2aX^{1/3}}{3c} + \frac{3a}{4c} - \frac{5b}{16c} \int \frac{dx}{X^{n-1} \sqrt{X}} \]

\[ \int \frac{x^n \, dx}{X \sqrt[3]{X}} = \frac{1}{nc} - \frac{x^{n-1} \sqrt[3]{X}}{(2n-1)b} \int \frac{x^{n-1} \, dx}{\sqrt[3]{X}} - \frac{(n-1)a}{nc} \int \frac{x^{n-2} \, dx}{\sqrt[3]{X}} \]

\[ \int x \sqrt[3]{X} \, dx = \frac{X^{1/3}}{3c} - \frac{b(2cx+b)}{8c^2} \sqrt[3]{X} - \frac{b}{4ck} \int \frac{dx}{\sqrt[3]{X}} \]

\[ \int xX \sqrt[3]{X} \, dx = \frac{X^{1/3} \sqrt{X}}{5c} + \frac{b}{2c} \int X \sqrt[3]{X} \, dx \]

\[ \int x^n X \sqrt[3]{X} \, dx = \frac{X^{1/3} \sqrt[3]{X}}{(2n+3)c} - \frac{b}{2c} \int X^n \sqrt[3]{X} \, dx \]

\[ \int x^2 \sqrt[3]{X} \, dx = \left( x - \frac{5b}{6c} \right) \frac{X \sqrt[3]{X}}{4c} + \frac{5b^2 - 4ac}{16c^2} \int \sqrt[3]{X} \, dx \]

\[ \int \frac{dx}{x \sqrt[3]{X}} = \begin{cases} 
\frac{1}{\sqrt{-a}} \sin^{-1} \left( \frac{bx + 2a}{\sqrt{-a}} \right), & a < 0, \\
\frac{2 \sqrt[3]{X}}{bx}, & a = 0, \\
-\frac{1}{\sqrt{a}} \log \left( \frac{2aX + bx + 2a}{x} \right), & a > 0. 
\end{cases} \]
\[ \int \frac{dx}{x^2 \sqrt{x}} = -\frac{\sqrt{x}}{ax} - \frac{b}{2a} \int \frac{dx}{x \sqrt{x}}. \]

\[ \int \frac{\sqrt{x}}{x} \, dx = \sqrt{x} + b \int \frac{dx}{\sqrt{x}} + a \int \frac{dx}{x \sqrt{x}}. \]

\[ \int \frac{\sqrt{x}}{x} \, dx = -\sqrt{x} + b \int \frac{dx}{x \sqrt{x}} + c \int \frac{dx}{\sqrt{x}}. \]

5.4.14 FORMS CONTAINING \( \sqrt{2ax - x^2} \)

\[ \int \sqrt{2ax - x^2} \, dx = \frac{1}{2} \left[ (x - a) \sqrt{2ax - x^2} + a^2 \sin^{-1} \frac{x - a}{|a|} \right]. \]

\[ \int \frac{dx}{\sqrt{2ax - x^2}} = \begin{cases} 
\cos^{-1} \left( \frac{a - x}{|a|} \right), \\
\sin^{-1} \left( \frac{x - a}{|a|} \right), 
\end{cases} \]

\[ \int \frac{x^n}{\sqrt{2ax - x^2}} \, dx = \left\{ \begin{array}{l}
\frac{-x^{n-1} \sqrt{2ax - x^2}}{n + 2} + \frac{(2n + 1)a}{n + 2} \int x^{n-1} \sqrt{2ax - x^2} \, dx, \\
\left[ \frac{x^{n+1}}{n + 2} - \sum_{r=0}^{n} \frac{(2n + 1)!(r)!a^{n-r}}{2^{n-r}(2r)!n!(n+2)!} x^{r+1} \right] + \frac{(2n + 1)a^{n+1}}{2^n n!(n+2)!} \sin^{-1} \left( \frac{x - a}{|a|} \right)\end{array} \right. \]

\[ \int \frac{dx}{x^n \sqrt{2ax - x^2}} = \begin{cases} 
\frac{-x^{n-1} \sqrt{2ax - x^2}}{n} + \frac{a(2n - 1)}{n} \int \frac{x^{n-1}}{\sqrt{2ax - x^2}} \, dx, \\
-\sqrt{2ax - x^2} \sum_{r=1}^{n} \frac{(2n)!r!(r-1)!a^{n-r}}{2^{n-r}(2r)!n!(n+2)!} x^{r-1} + \frac{(2n)!a^n}{2^n n!(n+2)!} \sin^{-1} \left( \frac{x - a}{|a|} \right). 
\end{cases} \]

\[ \int \frac{dx}{x^n \sqrt{2ax - x^2}} = \begin{cases} 
\frac{\sqrt{2ax - x^2}}{a(1-2nx^n) + (2n-1)a} \int \frac{dx}{x^{n-1} \sqrt{2ax - x^2}}, \\
-\sqrt{2ax - x^2} \sum_{r=0}^{n-1} \frac{2n-r(n-1)!n!(2r)!}{2n!(r)!a^{n-r}} x^{r+1}. 
\end{cases} \]

5.4.15 MISCELLANEOUS ALGEBRAIC FORMS

\[ \int \frac{dx}{\sqrt{2ax + x^2}} = \log \left( x + a \sqrt{2ax + x^2} \right). \]
\[259. \int \sqrt{ax^2 + c} \, dx = \begin{cases} \frac{x}{2} \sqrt{ax^2 + c} + \frac{c}{2\sqrt{-a}} \sin^{-1} \left( \frac{x}{\sqrt{-a}} \right), & a < 0, \\ \frac{x}{2} \sqrt{ax^2 + c} + \frac{c}{2\sqrt{a}} \log \left( x\sqrt{a} + \sqrt{ax^2 + c} \right), & a > 0. \end{cases}\]

\[260. \int \frac{1 + x}{1 - x} \, dx = \sin^{-1} x - \sqrt{1 - x^2}.\]

\[261. \int \frac{dx}{x\sqrt{ax^n + c}} = \begin{cases} \frac{1}{n\sqrt{c}} \log \frac{\sqrt{ax^n + c} - \sqrt{c}}{\sqrt{ax^n + c} + \sqrt{c}}, & c > 0, \\ \frac{2}{n\sqrt{c}} \log \frac{\sqrt{ax^n + c} - \sqrt{c}}{\sqrt{ax^n}}, & c > 0, \\ \frac{2}{n\sqrt{c}} \sec^{-1} \frac{\sqrt{-ax^n}}{c}, & c < 0. \end{cases}\]

\[262. \int \frac{dx}{\sqrt{ax^2 + c}} = \begin{cases} \frac{1}{\sqrt{-a}} \sin^{-1} \left( \sqrt{-a} \right), & a < 0, \\ \frac{1}{\sqrt{a}} \log \left( x\sqrt{a} + \sqrt{ax^2 + c} \right), & a > 0. \end{cases}\]

\[263. \int (ax^2 + c)^{m+1/2} \, dx = \begin{cases} \frac{x(ax^2 + c)^{m+1/2}}{2(m+1)} + \frac{(2m+1)c}{2(m+1)} \int (ax^2 + c)^{m-1/2} \, dx, \\ x\sqrt{ax^2 + c} \sum_{r=0}^{m} \frac{(2m+1)(2r)!c^{m-r}}{(2r+1)m!(m+1)!} (ax^2 + c)^r. \end{cases}\]

\[264. \int x(ax^2 + c)^{m+1/2} \, dx = \frac{(ax^2 + c)^{m+3/2}}{(2m+3)a}.\]

\[265. \int \frac{(ax^2 + c)^{m+1/2}}{x} \, dx = \begin{cases} \frac{(ax^2 + c)^{m+1/2}}{2m+1} + c \int \frac{(ax^2 + c)^{m-1/2}}{x} \, dx, \\ \sqrt{ax^2 + c} \sum_{r=0}^{m} \frac{c^{m-r}(ax^2 + c)^r}{2r+1} + c^{m+1} \int \frac{dx}{x\sqrt{ax^2 + c}}. \end{cases}\]

\[266. \int \frac{dx}{(ax^2 + c)^{m+1/2}} = \begin{cases} \frac{x}{(2m-1)c(ax^2 + c)^{m-1/2}} + \frac{2m-2}{(2m-1)c} \int \frac{dx}{(ax^2 + c)^{m-1/2}}, \\ \frac{x}{\sqrt{ax^2 + c}} \sum_{r=0}^{m-1} \frac{(2m-2r-1)(m-1)!r!(2r)!}{(2m)!} (ax^2 + c)^r. \end{cases}\]

\[267. \int \frac{dx}{x^m\sqrt{ax^2 + c}} = -\frac{\sqrt{ax^2 + c}}{(m-1)c} \sum_{r=0}^{m-1} \frac{(m-2)a}{(m-1)c} \int \frac{dx}{x^{m-2}\sqrt{ax^2 + c}}.\]
\[ 268. \int \frac{1 + x^2}{\sqrt{1 + x^4}} \, dx = \frac{1}{\sqrt{2}} \log \frac{x \sqrt{2} + \sqrt{1 + x^4}}{1 - x^2}. \]

\[ 269. \int \frac{1 - x^2}{\sqrt{1 + x^4}} \, dx = \frac{1}{\sqrt{2}} \tan^{-1} \frac{x \sqrt{2}}{\sqrt{1 + x^4}}. \]

\[ 270. \int \frac{dx}{x \sqrt{x^2 + a^2}} = -\frac{1}{na} \log \frac{a + \sqrt{x^2 + a^2}}{\sqrt{x^2}}. \]

\[ 271. \int \frac{dx}{x \sqrt{x^2 - a^2}} = -\frac{1}{na} \sin^{-1} \frac{a}{\sqrt{x^2}}. \]

\[ 272. \int \frac{dx}{\sqrt{a^2 - x^2}} = \frac{1}{3} \sin^{-1} \left( \frac{x}{a} \right)^{3/2}. \]

### 5.4.16 FORMS INVOLVING TRIGONOMETRIC FUNCTIONS

273. \[ \int \sin ax \, dx = -\frac{1}{a} \cos ax. \]

274. \[ \int \cos ax \, dx = \frac{1}{a} \sin x. \]

275. \[ \int \tan ax \, dx = -\frac{1}{a} \log \cos ax = \frac{1}{a} \log \sec ax. \]

276. \[ \int \cot ax \, dx = \frac{1}{a} \log \sin ax = -\frac{1}{a} \log \csc ax. \]

277. \[ \int \sec ax \, dx = \frac{1}{a} \log (\sec ax + \tan ax) = \frac{1}{a} \log \tan \left( \frac{\pi}{4} + \frac{ax}{2} \right). \]

278. \[ \int \csc ax \, dx = \frac{1}{a} \log (\csc ax - \cot ax) = \frac{1}{a} \log \tan \frac{ax}{2}. \]

279. \[ \int \sin^2 ax \, dx = \frac{x}{2} - \frac{1}{2a} \cos ax \sin ax = \frac{x}{2} - \frac{1}{4a} \sin 2ax. \]

280. \[ \int \sin^3 ax \, dx = -\frac{1}{3a} (\cos ax)(\sin^2 ax + 2). \]

281. \[ \int \sin^4 ax \, dx = \frac{3x}{8} - \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}. \]

282. \[ \int \sin^6 ax \, dx = -\frac{\sin^{-1} ax \cos ax}{na} + \frac{n - 1}{n} \int \sin^{n-2} ax \, dx. \]

283. \[ \int \sin^m ax \, dx = -\frac{\cos ax}{a} \sum_{j=0}^{m-1} \frac{1}{2m-2j} \left( 2r + 1 \right)! (m)! \sin^{2r+1} ax + \frac{(2m)!}{2^{2m}(m!)^2} x. \]

284. \[ \int \sin^m ax \, dx = \frac{\cos ax}{a} \sum_{r=0}^{m-1} \frac{1}{(2m+1)! (r!)^2} \sin^{2r+1} ax. \]

285. \[ \int \cos^2 ax \, dx = \frac{1}{2} x + \frac{1}{2a} \sin ax \cos ax = \frac{1}{2} x + \frac{1}{4a} \sin 2ax. \]

286. \[ \int \cos^3 ax \, dx = \frac{1}{3a} \sin ax (\cos^2 ax + 2). \]

287. \[ \int \cos^4 ax \, dx = \frac{3}{8} x + \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}. \]

288. \[ \int \cos^6 ax \, dx = \frac{1}{na} \cos^{a-1} ax \sin ax + \frac{n - 1}{n} \int \cos^{n-2} ax \, dx. \]

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\[
\int \cos^{2m+1} ax \, dx = \frac{\sin ax}{a} \sum_{r=0}^{n-1} \frac{(2m)! (r!)}{2^{m+2r} (2r+1)! (m!)^2} \cos^{2r+1} ax + \frac{(2m)!}{2^{2m} (m!)^2} x,
\]

\[
\int \cos^{2m} ax \, dx = \frac{\sin ax}{a} \sum_{r=0}^{n} \frac{2^{2m-2r} (m!)^2 (2r)!}{(2m+1)! (r!)^2} \cos^{2r} ax.
\]

\[
\int \frac{dx}{\sin^{2m} ax} = \int \csc^2 ax \, dx = -\frac{1}{a} \cot ax.
\]

\[
\int \frac{dx}{\sin^{2m} ax} = \int \csc^m ax \, dx = -\frac{1}{a(m-1)} \csc^{m-1} ax + \frac{m-2}{m-1} \int \frac{dx}{\sin^{m-2} ax}.
\]

\[
\int \frac{dx}{\sin^{2m+1} ax} = \int \csc^{2m+1} ax \, dx = -\frac{1}{a} \cos ax \sum_{r=0}^{n-1} \frac{(2m)! (r!)}{2^{2m-2r} (2r+1)! (m!)^2} \cos^{2r+1} ax + \frac{(2m)!}{a 2^{2m} (m!)^2} \log \tan \frac{ax}{2}.
\]

\[
\int \frac{dx}{\cos^{2m} ax} = \int \sec^2 ax \, dx = \frac{1}{a} \tan ax.
\]

\[
\int \frac{dx}{\cos^{2m+1} ax} = \int \sec^m ax \, dx = \frac{1}{a(m-1)} \sin ax \csc^{m-1} ax + \frac{m-2}{m-1} \int \frac{dx}{\cos^{m-2} ax}.
\]

\[
\int \frac{dx}{\cos^{2m+1} ax} = \int \sec^{2m} ax \, dx = \frac{1}{a} \sin ax \sum_{r=0}^{n-1} \frac{2^{2m-2r-1} (m-1)! (r!)}{2^{2m} (m!)^2} \cos^{2r+1} ax.
\]

\[
\int \frac{dx}{\cos^{2m+2} ax} = \int \sec^{2m+1} ax \, dx = \frac{1}{a} \sin ax \sum_{r=0}^{n-1} \frac{2^{2m-2r-1} (m-1)! (2r+1)!}{2^{2m} (m!)^2} \cos^{2r+2} ax + \frac{(2m)!}{a 2^{2m} (m!)^2} \log (\sec ax + \tan ax).
\]

\[
\int (\sin mx)(\sin nx) \, dx = \frac{\sin (m-n) x}{2(m-n)} - \frac{\sin (m+n) x}{2(m+n)}, \quad m^2 \neq n^2.
\]

\[
\int (\cos mx)(\cos nx) \, dx = \frac{\cos (m-n) x}{2(m-n)} + \frac{\cos (m+n) x}{2(m+n)}, \quad m^2 \neq n^2.
\]

\[
\int (\sin ax)(\cos ax) \, dx = \frac{1}{2a} \sin^2 ax.
\]

\[
\int (\sin mx)(\cos nx) \, dx = -\frac{\cos (m-n) x}{2(m-n)} - \frac{\cos (m+n) x}{2(m+n)}, \quad m^2 \neq n^2.
\]

\[
\int (\sin^2 ax)(\cos^2 ax) \, dx = -\frac{1}{32a} \sin 4ax + \frac{x}{8}.
\]

\[
\int (\sin ax)(\cos^n ax) \, dx = -\frac{\cos^{n+1} ax}{(m+1)a}.
\]

\[
\int (\cos^n ax)(\sin^m ax) \, dx = \begin{cases} 
\frac{\cos^{m-1} ax \sin^{m+1} ax}{(m+n) a} + \frac{m-1}{m+n} \int (\cos^{m-2} ax)(\sin^m ax) \, dx, \\
\text{or} \\
\frac{\cos^{n+1} ax \sin^{m-1} ax}{(m+n) a} + \frac{n-1}{m+n} \int (\cos^m ax)(\sin^{m-2} ax) \, dx.
\end{cases}
\]

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\[
307. \quad \int \frac{\cos^n ax}{\sin^m ax} \, dx = \begin{cases} 
- \frac{\cos^{n+1} ax}{a(n-1) \sin^{n-1} ax} - \frac{m-n+2}{n-1} \int \frac{\cos^n ax}{\sin^m ax} \, dx, \\
\text{or} \\
\frac{\cos^{m-1} ax}{a(m-n) \sin^{m-1} ax} + \frac{m-1}{m-n} \int \frac{\cos^n ax}{\sin^m ax} \, dx,
\end{cases}
\]

\[
308. \quad \int \frac{\sin^n ax}{\cos^m ax} \, dx = \begin{cases} 
- \frac{\sin^{n+1} ax}{a(n-1) \cos^{n-1} ax} - \frac{m-n+2}{n-1} \int \frac{\sin^n ax}{\cos^m ax} \, dx, \\
\text{or} \\
\frac{\sin^{m-1} ax}{a(m-n) \cos^{m-1} ax} + \frac{m-1}{m-n} \int \frac{\sin^n ax}{\cos^m ax} \, dx,
\end{cases}
\]

\[
309. \quad \int \frac{\sin ax}{\cos^n ax} \, dx = \frac{1}{a \cos ax} = \sec ax
\]

\[
310. \quad \int \frac{\sin^2 ax}{\cos ax} \, dx = -\frac{1}{a} \sin ax + \frac{1}{a} \log \tan \left( \frac{\pi}{4} + \frac{ax}{2} \right).
\]

\[
311. \quad \int \frac{\cos ax}{\sin^n ax} \, dx = -\csc ax = -\frac{1}{a \sin ax}
\]

\[
312. \quad \int \frac{dx}{(\sin ax)(\cos ax)} = \frac{1}{a} \log \tan ax.
\]

\[
313. \quad \int \frac{dx}{(\sin ax)(\cos^2 ax)} = \frac{1}{a} \left( \sec ax + \log \tan \frac{ax}{2} \right).
\]

\[
314. \quad \int \frac{dx}{(\sin ax)(\cos^3 ax)} = \frac{1}{a(n-1) \cos^{n-1} ax} + \int \frac{dx}{(\sin ax)(\cos^{n-2} ax)}.
\]

\[
315. \quad \int \frac{dx}{(\sin^2 ax)(\cos ax)} = -\frac{1}{a} \csc ax + \frac{1}{a} \log \tan \left( \frac{\pi}{4} + \frac{ax}{2} \right).
\]

\[
316. \quad \int \frac{dx}{(\sin^2 ax)(\cos^2 ax)} = -\frac{2}{a} \cot 2ax.
\]

\[
317. \quad \int \frac{dx}{\sin^n ax \cos^m ax} = \begin{cases} 
\int \frac{1}{a(n-1) \sin^{m-1} ax \cos^{n-1} ax} + \frac{m+n-2}{m-1} \int \frac{dx}{\sin^m ax \cos^n ax}, \\
\text{or} \\
\int \frac{1}{a(n-1) \sin^{m-1} ax \cos^{n-1} ax} + \frac{m+n-2}{n-1} \int \frac{dx}{\sin^m ax \cos^n ax},
\end{cases}
\]

\[
318. \quad \int \sin (a + bx) \, dx = -\frac{1}{b} \cos (a + bx).
\]

\[
319. \quad \int \cos (a + bx) \, dx = \frac{1}{b} \sin (a + bx).
\]

\[
320. \quad \int \frac{dx}{1 + \sin ax} = \pm \frac{1}{a} \tan \left( \frac{\pi}{4} + \frac{ax}{2} \right).
\]

\[
321. \quad \int \frac{dx}{1 + \cos ax} = \frac{1}{a} \tan \frac{ax}{2}.
\]

\[
322. \quad \int \frac{dx}{1 - \cos ax} = -\frac{1}{a} \cot \frac{ax}{2}.
\]

\[
323. \quad \int \frac{dx}{a + b \sin x} = \begin{cases} 
\frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left( \frac{a \tan \frac{x}{2} + b}{\sqrt{a^2 - b^2}} \right), \\
\text{or} \\
\frac{1}{\sqrt{b^2 - a^2}} \log \left( \frac{a \tan \frac{x}{2} + b \pm \sqrt{b^2 - a^2}}{a \tan \frac{x}{2} + b + \sqrt{b^2 - a^2}} \right),
\end{cases}
\]

©1996 CRC Press LLC
\[ \int \frac{dx}{a + b \cos x} = \begin{cases} \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left( \frac{\sqrt{a^2 - b^2} \tan \frac{x}{2}}{a + b} \right), \\
\frac{1}{\sqrt{b^2 - a^2}} \log \left( \frac{\sqrt{b^2 - a^2} \tan \frac{x}{2} + a + b}{\sqrt{b^2 - a^2} \tan \frac{x}{2} - a - b} \right), \end{cases} \]

\[ \int \frac{dx}{a + b \sin x + c \cos x} = \begin{cases} \frac{1}{\sqrt{b^2 + c^2 - a^2}} \log \left( \frac{b - \sqrt{b^2 + c^2 - a^2} + (a - c) \tan \frac{x}{2}}{b + \sqrt{b^2 + c^2 - a^2} + (a - c) \tan \frac{x}{2}} \right), & a \neq c, \ a^2 < b^2 + c^2, \\
\frac{2}{\sqrt{a^2 - b^2 - c^2}} \tan^{-1} \left( \frac{b + (a - c) \tan \frac{x}{2}}{\sqrt{a^2 - b^2 - c^2}} \right), & a^2 > b^2 + c^2, \\
\frac{1}{a} \left[ a - (b + c) \sin x - (b - c) \sin x \right] \frac{1}{a - (b + c) \sin x + (b - c) \sin x}, & a^2 = b^2 + c^2. \end{cases} \]

\[ \int \frac{dx}{a + b \cos x + c^2 \sin^2 x} = \frac{1}{2b} \log \left( \frac{a + b \tan x}{a} \right) \tan^{-1} \left( \frac{a}{b} \right). \]

\[ \int \frac{dx}{a \cos^2 x + b \sin^2 x} = \frac{1}{a} \tan^{-1} \left( \frac{a \tan x}{b} \right). \]

\[ \int \frac{dx}{a \cos x + b \sin x} = \int \frac{dx}{a + b \tan x} = \frac{1}{c(a^2 + b^2)} [acx + b \log (a \cos c x + b \sin c x)] \]

\[ \int \frac{dx}{b + a \cot c x} = \frac{1}{c(a^2 + b^2)} [bcx - a \log (a \cos c x + b \sin c x)] \]

\[ \int \frac{dx}{a \cos^2 x + 2b \cos x \sin x + c \sin^2 x} = \begin{cases} \frac{c \tan x + b - \sqrt{b^2 - ac}}{2\sqrt{b^2 - ac}} \tan^{-1} \left( \frac{c \tan x + b + \sqrt{b^2 - ac}}{\sqrt{b^2 - ac}} \right), & b^2 > ac, \\
\frac{1}{\sqrt{ac - b^2}} \tan^{-1} \left( \frac{\sqrt{ac - b^2}}{c \tan x + b} \right), & b^2 < ac, \end{cases} \]

\[ \int \frac{dx}{\sin ax} = \pm x + \frac{1}{a} \tan \left( \frac{\pi}{4} + \frac{ax}{2} \right). \]

\[ \frac{dx}{(\sin x)(1 + \sin ax)} = \frac{1}{a} \tan \left( \frac{\pi}{4} + \frac{ax}{2} \right) + \frac{1}{a} \log \tan \frac{ax}{2}. \]

\[ \int \frac{dx}{(1 + \sin ax)^2} = -\frac{1}{2a} \tan \left( \frac{\pi}{4} + \frac{ax}{2} \right) - \frac{1}{6a} \tan^3 \left( \frac{\pi}{4} - \frac{ax}{2} \right). \]
\[
\int \frac{dx}{(1 - \sin ax)^2} = \frac{1}{2a} \cot \left( \frac{\pi}{4} - \frac{ax}{2} \right) + \frac{1}{6a} \cot^3 \left( \frac{\pi}{4} - \frac{ax}{2} \right).
\]
\[
\int \frac{\sin ax}{(1 + \sin ax)^2} \, dx = -\frac{1}{2a} \tan \left( \frac{\pi}{4} - \frac{ax}{2} \right) + \frac{1}{6a} \tan^3 \left( \frac{\pi}{4} - \frac{ax}{2} \right).
\]
\[
\int \frac{\sin ax}{(1 - \sin ax)^2} \, dx = -\frac{1}{2a} \cot \left( \frac{\pi}{4} - \frac{ax}{2} \right) + \frac{1}{6a} \cot^3 \left( \frac{\pi}{4} - \frac{ax}{2} \right).
\]
\[
\int \frac{dx}{a^2 + b^2 \sin^2 cx} = \frac{1}{a \sqrt{a^2 + b^2}} \tan^{-1} \left( \frac{\sqrt{a^2 + b^2} \tan cx}{a} \right).
\]
\[
\int \frac{dx}{a^2 - b^2 \sin^2 cx} = \frac{1}{a \sqrt{a^2 - b^2}} \tan^{-1} \left( \frac{\sqrt{a^2 - b^2} \tan cx}{a} \right), \quad a^2 > b^2.
\]
\[
\int \frac{\cos ax}{1 + \cos ax} \, dx = x - \frac{1}{a} \tan \frac{ax}{2}.
\]
\[
\int \frac{\cos ax}{1 - \cos ax} \, dx = -x - \frac{1}{a} \cot \frac{ax}{2}.
\]
\[
\int \frac{dx}{\cos ax)(1 + \cos ax)} = \frac{1}{a} \log \tan \left( \frac{\pi}{4} + \frac{ax}{2} \right) - \frac{1}{a} \tan \frac{ax}{2}.
\]
\[
\int \frac{dx}{\cos ax)(1 - \cos ax)} = \frac{1}{a} \log \tan \left( \frac{\pi}{4} + \frac{ax}{2} \right) - \frac{1}{a} \cot \frac{ax}{2}.
\]
\[
\int \frac{dx}{(1 + \cos ax)^2} = \frac{1}{2a} \tan \frac{ax}{2} + \frac{1}{6a} \tan^3 \frac{ax}{2}.
\]
\[
\int \frac{dx}{(1 - \cos ax)^2} = -\frac{1}{2a} \cot \frac{ax}{2} - \frac{1}{6a} \cot^3 \frac{ax}{2}.
\]
\[
\int \frac{\cos ax}{(1 + \cos ax)^2} \, dx = \frac{1}{2a} \tan \frac{ax}{2} - \frac{1}{6a} \tan^3 \frac{ax}{2}.
\]
\[
\int \frac{\cos ax}{(1 - \cos ax)^2} \, dx = \frac{1}{2a} \cot \frac{ax}{2} - \frac{1}{6a} \cot^3 \frac{ax}{2}.
\]
\[
\int \frac{a + b \cos x}{a + b \cos x} \, dx = \frac{x}{a} + \frac{a}{b} \int \frac{dx}{a + b \cos x}.
\]
\[
\int \frac{dx}{\cos ax(a + b \cos x)} = \frac{1}{a} \log \tan \left( \frac{x}{2} + \frac{\pi}{4} \right) - \frac{b}{a} \int \frac{dx}{a + b \cos x}.
\]
\[
\int \frac{dx}{a(a + b \cos x)} = \frac{1}{a^2 - b^2} \log \tan \frac{ax}{2}.
\]
\[
\int \frac{dx}{a(a + b \cos x)^2} = \frac{b}{a} \int \frac{dx}{a + b \cos x}.
\]
\[
\int\frac{dx}{a^2 + b^2 - 2ab \cos cx} = \frac{2}{c(a^2 - b^2)} \tan^{-1} \left( \frac{a + b}{a - b} \tan \frac{cx}{2} \right).
\]

\[
\int\frac{dx}{a^2 + b^2 \cos^2 cx} = \frac{1}{ac\sqrt{a^2 + b^2}} \tan^{-1} \frac{a \tan cx}{\sqrt{a^2 + b^2}}.
\]

\[
\int\frac{dx}{a^2 - b^2 \cos^2 cx} = \begin{cases} 
\frac{1}{ac\sqrt{a^2 - b^2}} \tan^{-1} \frac{a \tan cx}{\sqrt{a^2 - b^2}} & a^2 > b^2, \\
\text{or} & \\
\frac{1}{2ac\sqrt{b^2 - a^2}} \log \frac{a \tan cx - \sqrt{b^2 - a^2}}{a \tan cx + \sqrt{b^2 - a^2}} & b^2 > a^2.
\end{cases}
\]

\[
\int \frac{\sin ax}{1 \pm \cos ax} \, dx = \mp \frac{1}{a} \log (1 \pm \cos ax).
\]

\[
\int \frac{\cos ax}{1 \pm \sin ax} \, dx = \pm \frac{1}{a} \log (1 \pm \sin ax).
\]

\[
\int \frac{dx}{(\sin ax)(1 \pm \cos ax)} = \pm \frac{1}{2a(1 \pm \cos ax)} + \frac{1}{2a} \log \frac{a \tan \frac{ax}{2}}{2}.
\]

\[
\int \frac{dx}{(\cos ax)(1 \pm \sin ax)} = \mp \frac{1}{2a(1 \pm \sin ax)} + \frac{1}{2a} \log \frac{a \tan \left( \frac{ax}{2} + \frac{\pi}{4} \right)}{2}.
\]

\[
\int \frac{\sin ax}{(\cos ax)(1 \pm \cos ax)} \, dx = \frac{1}{a} \log (\sec ax \pm 1).
\]

\[
\int \frac{\cos ax}{(\sin ax)(1 \pm \sin ax)} \, dx = -\frac{1}{a} \log (\csc ax \pm 1).
\]

\[
\int \frac{\sin ax}{(\cos ax)(1 \pm \sin ax)} \, dx = \frac{1}{a(1 \pm \cos ax)} \pm \frac{1}{2a} \log \left( \frac{ax}{2} \pm \frac{\pi}{8} \right).
\]

\[
\int \frac{dx}{\sin ax \pm \cos ax} = \frac{1}{a\sqrt{2}} \log \left( \frac{ax}{2} \pm \frac{\pi}{8} \right).
\]

\[
\int \frac{1}{1 + \cos ax \pm \sin ax} = \pm \frac{1}{a} \log \left( 1 \pm \tan \frac{ax}{2} \right).
\]

\[
\int \frac{dx}{a^2 \cos^2 cx - b^2 \sin^2 cx} = \frac{1}{2abc} \log \frac{b \tan cx + a}{b \tan cx - a}.
\]

\[
\int x \sin ax \, dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax.
\]

\[
\int x^2 \sin ax \, dx = \frac{2x}{a^2} \sin ax + \frac{2 - a^2 x^2}{a^3} \cos ax.
\]

\[
\int x^3 \sin ax \, dx = \frac{3a^2 x^2 - 6}{a^4} \sin ax + \frac{6x - a^2 x^3}{a^4} \cos ax.
\]

\[
\int x^n \sin ax \, dx = \begin{cases} 
-\frac{1}{a} x^n \cos ax + \frac{m}{a} \int x^{m-1} \cos ax \, dx, \\
\text{or} & \\
\cos ax \sum_{r=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^{r+1} \frac{m!}{(m-2r)!} a^{2r+1} + \sin ax \sum_{r=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^{r+1} \frac{m!}{(m-2r-1)!} a^{2r+2}.
\end{cases}
\]
375. \[ \int x \cos ax \, dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax. \]
376. \[ \int x^2 \cos ax \, dx = \frac{2x}{a^2} \cos ax + \frac{a^2 x^2 - 2}{a^3} \sin ax. \]
377. \[ \int x^3 \cos ax \, dx = \frac{3a^2 x^2 - 6}{a^4} \cos ax + \frac{a^2 x^3 - 6x}{a^5} \sin ax. \]
378. \[ \int x^n \cos ax \, dx = \frac{x^n \sin ax - m}{a} \int x^{n-1} \sin ax \, dx, \]
or
\[ \sin ax \sum_{r=0}^{\left[ \frac{n}{2} \right]} (-1)^r \frac{m!}{(m-2r)!} \frac{x^{m-2r}}{a^{2r+1}} + \cos ax \sum_{r=0}^{\left[ \frac{n}{2} \right] - 1} (-1)^r \frac{m!}{(m-2r-1)!} \frac{x^{m-2r-1}}{a^{2r+2}}. \]
379. \[ \int \frac{\sin ax}{x} \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{(ax)^{2n+1}}{(2n+1)(2n+1)!}. \]
380. \[ \int \frac{\cos ax}{x} \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{(ax)^{2n}}{(2n)(2n)!}. \]
381. \[ \int x \sin^2 ax \, dx = \frac{x^2}{4} - \frac{x}{4a} \sin 2ax - \frac{1}{8a^2} \cos 2ax. \]
382. \[ \int x^2 \sin^2 ax \, dx = \frac{x^3}{6} - \left( \frac{x^2}{4a} - \frac{1}{8a^3} \right) \sin 2ax - \frac{x}{4a^2} \cos 2ax. \]
383. \[ \int x \sin^3 ax \, dx = \frac{x}{12a} \cos 3ax - \frac{1}{36a^2} \sin 3ax - \frac{3x}{4a} \cos ax + \frac{3}{4a^2} \sin ax. \]
384. \[ \int x \cos^2 ax \, dx = \frac{x^2}{4} + \frac{x}{4a} \sin 2ax + \frac{1}{8a^2} \cos 2ax. \]
385. \[ \int x^2 \cos^2 ax \, dx = \frac{x^3}{6} + \left( \frac{x^2}{4a} - \frac{1}{8a^3} \right) \sin 2ax + \frac{x}{4a^2} \cos 2ax. \]
386. \[ \int x \cos^3 ax \, dx = \frac{x}{12a} \sin 3ax + \frac{1}{36a^2} \cos 3ax + \frac{3x}{4a} \sin ax + \frac{3}{4a^2} \cos ax. \]
387. \[ \int \frac{\sin ax}{x^n} \, dx = \frac{\sin ax}{(1 - m)x^{m-1}} + \frac{a}{m - 1} \int \frac{\cos ax}{x^{m-1}} \, dx. \]
388. \[ \int \frac{\cos ax}{x^n} \, dx = \frac{\cos ax}{(1 - m)x^{m-1}} + \frac{a}{1 - m} \int \frac{\sin ax}{x^{m-1}} \, dx. \]
389. \[ \int \frac{x}{1 \pm \sin ax} \, dx = \mp \frac{x \cos ax}{a(1 \pm \sin ax)} + \frac{1}{a^2} \log (1 \pm \sin ax). \]
390. \[ \int \frac{x}{1 + \cos ax} \, dx = \frac{x}{a} \tan \frac{ax}{2} + \frac{2}{a^2} \log \cos \frac{ax}{2}. \]
391. \[ \int \frac{x}{1 - \cos ax} \, dx = - \frac{x}{a} \cot \frac{ax}{2} + \frac{2}{a^2} \log \sin \frac{ax}{2}. \]
392. \[ \int \frac{x + \sin x}{1 + \cos x} \, dx = x \tan \frac{x}{2}. \]
393. \[ \int \frac{x - \sin x}{1 - \cos x} \, dx = - x \cot \frac{x}{2}. \]
394. \[ \int \frac{\sqrt{1 - \cos ax}}{a} \, dx = - \frac{2 \sin ax}{a \sqrt{1 - \cos ax}} = - \frac{2\sqrt{2}}{a} \cos \frac{ax}{2}. \]
395. \[ \int \frac{\sqrt{1 + \cos ax}}{ax} \, dx = \frac{2 \sin ax}{a \sqrt{1 + \cos ax}} = \frac{2\sqrt{2}}{a} \sin \frac{ax}{2} \]

For the following six integrals, each \( k \) represents an integer.

396. \[ \int \frac{\sqrt{1 + \sin x}}{\sin x} \, dx = \begin{cases} 2 \left( \sin \frac{x}{2} - \cos \frac{x}{2} \right), & (8k - 1) \frac{\pi}{2} < x \leq (8k + 3) \frac{\pi}{2}, \\ -2 \left( \sin \frac{x}{2} - \cos \frac{x}{2} \right), & (8k + 3) \frac{\pi}{2} < x \leq (8k - 1) \frac{\pi}{2}. \end{cases} \]

397. \[ \int \frac{\sqrt{1 - \sin x}}{\sin x} \, dx = \begin{cases} 2 \left( \sin \frac{x}{2} + \cos \frac{x}{2} \right), & (8k - 3) \frac{\pi}{2} < x \leq (8k + 1) \frac{\pi}{2}, \\ -2 \left( \sin \frac{x}{2} + \cos \frac{x}{2} \right), & (8k + 1) \frac{\pi}{2} < x \leq (8k - 3) \frac{\pi}{2}. \end{cases} \]

398. \[ \int \frac{dx}{\sqrt{1 - \cos x}} = \begin{cases} \sqrt{2} \log \tan \frac{x}{4}, & 4k\pi < x \leq (4k + 2)\pi, \\ -\sqrt{2} \log \tan \frac{x}{4} & (4k + 2)\pi < x \leq 4k\pi. \end{cases} \]

399. \[ \int \frac{dx}{\sqrt{1 + \cos x}} = \begin{cases} \sqrt{2} \log \tan \left( \frac{x}{4} + \frac{\pi}{4} \right), & (4k - 1)\pi < x \leq (4k + 1)\pi, \\ -\sqrt{2} \log \tan \left( \frac{x}{4} + \frac{\pi}{4} \right) & (4k + 1)\pi < x \leq (4k - 1)\pi. \end{cases} \]

400. \[ \int \frac{dx}{\sqrt{1 - \sin x}} = \begin{cases} \sqrt{2} \log \tan \left( \frac{x}{4} - \frac{\pi}{8} \right), & (8k + 1) \frac{\pi}{2} < x \leq (8k + 5) \frac{\pi}{2}, \\ -\sqrt{2} \log \tan \left( \frac{x}{4} - \frac{\pi}{8} \right) & (8k + 5) \frac{\pi}{2} < x \leq (8k + 1) \frac{\pi}{2}. \end{cases} \]

401. \[ \int \frac{dx}{\sqrt{1 + \sin x}} = \begin{cases} \sqrt{2} \log \tan \left( \frac{x}{4} + \frac{\pi}{8} \right), & (8k - 1) \frac{\pi}{2} < x \leq (8k + 3) \frac{\pi}{2}, \\ -\sqrt{2} \log \tan \left( \frac{x}{4} + \frac{\pi}{8} \right) & (8k + 3) \frac{\pi}{2} < x \leq (8k - 1) \frac{\pi}{2}. \end{cases} \]

402. \[ \int \tan^2 ax \, dx = \frac{1}{a} \tan ax - x. \]

403. \[ \int \tan^3 ax \, dx = \frac{1}{2a} \tan^2 ax + \frac{1}{a} \log \cos ax. \]

404. \[ \int \tan^4 ax \, dx = \frac{1}{3a} \tan^3 ax - \frac{1}{a} \tan ax + x. \]

405. \[ \int \tan^n ax \, dx = \frac{1}{a(n - 1)} \tan^{n-1} ax - \int \tan^{n-2} ax \, dx. \]

406. \[ \int \cot^2 ax \, dx = -\frac{1}{a} \cot ax - x. \]

407. \[ \int \cot^3 ax \, dx = -\frac{1}{2a} \cot^2 ax - \frac{1}{a} \log \sin ax. \]

408. \[ \int \cot^4 ax \, dx = -\frac{1}{3a} \cot^3 ax + \frac{1}{a} \cot ax + x. \]

409. \[ \int \cot^n ax \, dx = -\frac{1}{a(n - 1)} \cot^{n-1} ax - \int \cot^{n-2} ax \, dx. \]

410. \[ \int \frac{x}{\sin^2 ax} \, dx = \int x \csc^2 ax \, dx = -\frac{x \cot ax}{a} + \frac{1}{a^2} \log \sin ax. \]

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411.  \[
\int \frac{x}{\sin^n ax} \, dx = \int x \csc^n ax \, dx = -\frac{x \cos ax}{a(n-1) \sin^{n-1} ax} + \frac{1}{a^2(n-1)(n-2) \sin^{n-2} ax} + \frac{n-2}{n-1} \int \frac{x}{\sin^{n-2} ax} \, dx
\]

412.  \[
\int \frac{x}{\cos^n ax} \, dx = \int x \sec^n ax \, dx = \frac{x \tan ax}{a} + \frac{1}{a^2} \log \cos ax
\]

413.  \[
\int \frac{x}{\cos^n ax} \, dx = \int x \sec^n ax \, dx = \frac{x \sin ax}{a(n-1) \cos^{n-1} ax} + \frac{1}{a^2(n-1)(n-2) \cos^{n-2} ax} + \frac{n-2}{n-1} \int \frac{x}{\cos^{n-2} ax} \, dx
\]

414.  \[
\int \frac{\sin ax}{\sqrt{1+ b^2 \sin^2 ax}} \, dx = -\frac{1}{ab} \sin^{-1} \frac{b \cos ax}{\sqrt{1+b^2}}.
\]

415.  \[
\int \frac{\sin ax}{\sqrt{1- b^2 \sin^2 ax}} \, dx = -\frac{1}{ab} \log \left( b \cos ax + \sqrt{1- b^2 \sin^2 ax} \right).
\]

416.  \[
\int (\sin ax) \sqrt{1 + b^2 \sin^2 ax} \, dx = -\frac{\cos ax}{2a} \sqrt{1 + b^2 \sin^2 ax} - \frac{1+b^2}{2ab} \sin^{-1} \frac{b \cos ax}{\sqrt{1+b^2}}.
\]

417.  \[
\int (\sin ax) \sqrt{1- b^2 \sin^2 ax} \, dx = -\frac{\cos ax}{2a} \sqrt{1- b^2 \sin^2 ax} - \frac{1-b^2}{2ab} \log \left( b \cos ax + \sqrt{1- b^2 \sin^2 ax} \right).
\]

418.  \[
\int \frac{\cos ax}{\sqrt{1+ b^2 \sin^2 ax}} \, dx = \frac{1}{ab} \log \left( b \sin ax + \sqrt{1+b^2 \sin^2 ax} \right).
\]

419.  \[
\int \frac{\cos ax}{\sqrt{1- b^2 \sin^2 ax}} \, dx = \frac{1}{ab} \sin^{-1} \left( b \sin ax \right).
\]

420.  \[
\int (\cos ax) \sqrt{1+ b^2 \sin^2 ax} \, dx = \frac{\sin ax}{2a} \sqrt{1+ b^2 \sin^2 ax} + \frac{1}{2ab} \log \left( b \sin ax + \sqrt{1+b^2 \sin^2 ax} \right).
\]

421.  \[
\int (\cos ax) \sqrt{1- b^2 \sin^2 ax} \, dx = \frac{\sin ax}{2a} \sqrt{1- b^2 \sin^2 ax} + \frac{1}{2ab} \sin^{-1} \left( b \sin ax \right).
\]

For the following integral, \(k\) represents an integer and \(a > |b|\)

422.  \[
\int \frac{dx}{\sqrt{a + b \tan cx}} = \begin{cases} 
\frac{1}{c \sqrt{a-b}} \sin^{-1} \left( \sqrt{\frac{a-b}{a}} \sin cx \right), & (2k-1) \frac{\pi}{2} < x \leq (2k+1) \frac{\pi}{2}, \\
\text{or} \\
\frac{-1}{c \sqrt{a-b}} \sin^{-1} \left( \sqrt{\frac{a-b}{a}} \sin cx \right), & (2k+1) \frac{\pi}{2} < x \leq (2k-1) \frac{\pi}{2}.
\end{cases}
\]

423.  \[
\int \cos^n x \, dx = \frac{1}{2^{n-1}} \sum_{k=0}^{n-1} \binom{n}{k} \sin \left[ (n-2k+1) \frac{x}{2} \right] + \frac{1}{2^n} \binom{n}{\frac{x}{2}}, \quad n \text{ is an even integer}.
\]

424.  \[
\int \cos^n x \, dx = \frac{1}{2^{n-1}} \sum_{k=0}^{n-1} \binom{n}{k} \sin \left[ (n-2k) \frac{x}{2} \right], \quad n \text{ is an odd integer}.
\]

425.  \[
\int \sin^n x \, dx = \frac{1}{2^{n-1}} \sum_{k=0}^{n-1} \binom{n}{k} \sin \left[ (n-2k)(\frac{x}{2} - \frac{\pi}{2}) \right] + \frac{1}{2^n} \binom{n}{\frac{x}{2}}, \quad n \text{ is an even integer}.
\]
\[ \int \sin^n x \, dx = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} \sin \left( \frac{(n-2k)(\frac{x}{2})}{2k-n} \right), \quad n \text{ is an odd integer.} \]

### 5.4.17 FORMS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

\[ \int \sin^{-1} ax \, dx = x \sin^{-1} ax + \frac{\sqrt{1-a^2x^2}}{a} \]

\[ \int \cos^{-1} ax \, dx = x \cos^{-1} ax - \frac{\sqrt{1-a^2x^2}}{a} \]

\[ \int \tan^{-1} ax \, dx = x \tan^{-1} ax - \frac{1}{2a} \log (1+a^2x^2). \]

\[ \int \cot^{-1} ax \, dx = x \cot^{-1} ax + \frac{1}{2a} \log (1+a^2x^2). \]

\[ \int \sec^{-1} ax \, dx = x \sec^{-1} ax - \frac{1}{a} \log \left( ax + \sqrt{a^2x^2-1} \right). \]

\[ \int \csc^{-1} ax \, dx = x \csc^{-1} ax + \frac{1}{a} \log \left( ax + \sqrt{a^2x^2-1} \right). \]

\[ \int \left( \sin^{-1} \frac{x}{a} \right) \, dx = x \sin^{-1} \frac{x}{a} + \sqrt{a^2-x^2}, \quad a > 0. \]

\[ \int \left( \cos^{-1} \frac{x}{a} \right) \, dx = x \cos^{-1} \frac{x}{a} - \sqrt{a^2-x^2}, \quad a > 0. \]

\[ \int \left( \tan^{-1} \frac{x}{a} \right) \, dx = x \tan^{-1} \frac{x}{a} - \frac{a}{2} \log (a^2+x^2). \]

\[ \int \left( \cot^{-1} \frac{x}{a} \right) \, dx = x \cot^{-1} \frac{x}{a} + \frac{a}{2} \log (a^2+x^2). \]

\[ \int x \sin^{-1} (ax) \, dx = \frac{1}{4a^2} \left( (2a^2x^2-1) \sin^{-1} (ax) + ax \sqrt{1-a^2x^2} \right). \]

\[ \int x \cos^{-1} (ax) \, dx = \frac{1}{4a^2} \left( (2a^2x^2-1) \cos^{-1} (ax) - ax \sqrt{1-a^2x^2} \right). \]

\[ \int x^n \sin^{-1} (ax) \, dx = \frac{x^{n+1}}{n+1} \sin^{-1} (ax) - \frac{a}{n+1} \int \frac{x^{n+1}}{\sqrt{1-a^2x^2}} \, dx, \quad n \neq -1. \]

\[ \int x^n \cos^{-1} ax \, dx = \frac{x^{n+1}}{n+1} \cos^{-1} (ax) + \frac{a}{n+1} \int \frac{x^{n+1}}{\sqrt{1-a^2x^2}} \, dx, \quad n \neq -1. \]

\[ \int x \tan^{-1} (ax) \, dx = \frac{1+a^2x^2}{2a^2} \tan^{-1} (ax) - \frac{x}{2a}. \]

\[ \int x^n \tan^{-1} (ax) \, dx = \frac{x^{n+1}}{n+1} \tan^{-1} (ax) - \frac{a}{n+1} \int \frac{x^{n+1}}{1+a^2x^2} \, dx. \]

\[ \int x \cot^{-1} (ax) \, dx = \frac{1+a^2x^2}{2a^2} \cot^{-1} (ax) + \frac{x}{2a}. \]

\[ \int x^n \cot^{-1} (ax) \, dx = \frac{x^{n+1}}{n+1} \cot^{-1} (ax) + \frac{a}{n+1} \int \frac{x^{n+1}}{1+a^2x^2} \, dx. \]

\[ \int \frac{\sin^{-1} (ax)}{x^2} \, dx = a \log \left( \frac{1 - \sqrt{1-a^2x^2}}{x} \right) - \sin^{-1} (ax) \frac{1}{x}. \]
\[ \int \frac{\cos^{-1}(ax)}{x^2} \, dx = -\frac{1}{x} \cos^{-1}(ax) + a \log \frac{1 + \sqrt{1 - a^2x^2}}{x}. \]
\[ \int \frac{\tan^{-1}(ax)}{x^2} \, dx = -\frac{1}{x} \tan^{-1}(ax) - \frac{a}{2} \log \frac{1 + a^2x^2}{x^2}. \]
\[ \int \frac{\cot^{-1}(ax)}{x^2} \, dx = -\frac{1}{x} \cot^{-1}(ax) - \frac{a}{2} \log \frac{x^2}{1 + a^2x^2}. \]
\[ \int (\sin^{-1}(ax))^2 \, dx = x(\sin^{-1}(ax))^2 - 2x + \frac{2\sqrt{1 - a^2x^2}}{a} \sin^{-1}(ax). \]
\[ \int (\cos^{-1}(ax))^2 \, dx = x(\cos^{-1}(ax))^2 - 2x - \frac{2\sqrt{1 - a^2x^2}}{a} \cos^{-1}(ax). \]
\[ \int (\sin^{-1}(ax))^n \, dx = \frac{x(\sin^{-1}(ax))^{n+1}}{n+1} - \frac{n\sqrt{1 - a^2x^2}}{n+1} (\sin^{-1}(ax))^{n-1} - n(n-1) \int (\sin^{-1}(ax))^{n-2} \, dx, \]
\[ \text{or} \]
\[ \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^r \frac{n!}{(n-2r)!} x(\sin^{-1}(ax))^{n-2r} = \sum_{r=0}^{\frac{n-1}{2}} (-1)^r \frac{n!\sqrt{1 - a^2x^2}}{(n-2r-1)!a} (\sin^{-1}(ax))^{n-2r-1}. \]
\[ \int (\cos^{-1}(ax))^n \, dx = \frac{x(\cos^{-1}(ax))^{n+1}}{n+1} - \frac{n\sqrt{1 - a^2x^2}}{n+1} (\cos^{-1}(ax))^{n-1} - n(n-1) \int (\cos^{-1}(ax))^{n-2} \, dx, \]
\[ \text{or} \]
\[ \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^r \frac{n!}{(n-2r)!} x(\cos^{-1}(ax))^{n-2r} = \sum_{r=0}^{\frac{n-1}{2}} (-1)^r \frac{n!\sqrt{1 - a^2x^2}}{(n-2r-1)!a} (\cos^{-1}(ax))^{n-2r-1}. \]
\[ \int \frac{\sin^{-1}ax}{\sqrt{1 - a^2x^2}} \, dx = \frac{1}{2a} (\sin^{-1}ax)^2. \]
\[ \int \frac{x^n \sin^{-1}ax}{\sqrt{1 - a^2x^2}} \, dx = -\frac{x^{n-1}}{na^2} \sqrt{1 - a^2x^2} \sin^{-1}ax + \frac{x^n}{n^2a} + \frac{n-1}{na^2} \int \frac{x^{n-2} \sin^{-1}ax}{\sqrt{1 - a^2x^2}} \, dx. \]
\[ \int \frac{\cos^{-1}ax}{\sqrt{1 - a^2x^2}} \, dx = -\frac{1}{2a} (\cos^{-1}ax)^2. \]
\[ \int \frac{x^n \cos^{-1}ax}{\sqrt{1 - a^2x^2}} \, dx = -\frac{x^{n-1}}{na^2} \sqrt{1 - a^2x^2} \cos^{-1}ax - \frac{x^n}{n^2a} + \frac{n-1}{na^2} \int \frac{x^{n-2} \cos^{-1}ax}{\sqrt{1 - a^2x^2}} \, dx. \]
\[ \int \frac{\tan^{-1}ax}{1 + a^2x^2} \, dx = \frac{1}{2a} (\tan^{-1}ax)^2. \]
\[ \int \frac{\cot^{-1}ax}{1 + a^2x^2} \, dx = \frac{1}{2a} (\cot^{-1}ax)^2. \]
\[ \int x \sec^{-1}ax \, dx = \frac{x^2}{2} \sec^{-1}ax - \frac{1}{2a^2} \sqrt{a^2x^2 - 1}. \]
\[ \int x^n \sec^{-1}ax \, dx = \frac{x^{n+1}}{n+1} \sec^{-1}ax - \frac{1}{n+1} \int \frac{x^n}{\sqrt{a^2x^2 - 1}} \, dx. \]
461. \[ \int \frac{\sec^{-1} ax}{x^2} \, dx = -\frac{\sec^{-1} ax}{x} + \frac{\sqrt{a^2 x^2 - 1}}{x}. \]

462. \[ \int x \csc^{-1} ax \, dx = \frac{x^2}{2} \csc^{-1} ax + \frac{1}{2a^2} \sqrt{a^2 x^2 - 1}. \]

463. \[ \int x^n \csc^{-1} ax \, dx = \frac{x^{n+1}}{n+1} \csc^{-1} ax + \frac{1}{n+1} \int \frac{x^n}{\sqrt{a^2 x^2 - 1}} \, dx. \]

464. \[ \int \frac{\csc^{-1} ax}{x^2} \, dx = -\frac{\csc^{-1} ax}{x} - \frac{\sqrt{a^2 x^2 - 1}}{x}. \]

### 5.4.18 LOGARITHMIC FORMS

465. \[ \int \log x \, dx = x \log x - x. \]

466. \[ \int x \log x \, dx = \frac{x^2}{2} \log x - \frac{x^2}{4}. \]

467. \[ \int x^2 \log x \, dx = \frac{x^3}{3} \log x - \frac{x^3}{9}. \]

468. \[ \int x^n \log x \, dx = \frac{x^{n+1}}{n+1} \log x - \frac{x^{n+1}}{(n+1)^2}. \]

469. \[ \int (\log x)^2 \, dx = x(\log x)^2 - 2x \log x + 2x. \]

470. \[ \int (\log x)^n \, dx = \left\{ \begin{array}{ll}
& x(\log x)^n - n \int (\log x)^{n-1} \, dx, \quad n \neq 1,
\text{or} & (\log x)^n = \frac{1}{n+1} (\log x)^{n+1}.
\end{array} \right. \]

471. \[ \int (\log x)^n \, dx = \frac{1}{n+1} (\log x)^{n+1}. \]

472. \[ \int \frac{dx}{\log x} = \log(\log x) + \log x + \frac{(\log x)^2}{2 - 2!} + \frac{(\log x)^3}{3 - 3!} + \ldots \]

473. \[ \int \frac{dx}{x \log x} = \log(\log x). \]

474. \[ \int \frac{dx}{x(\log x)^n} = \frac{1}{1 - n)(\log x)^{n-1}. \]

475. \[ \int \frac{x^m \, dx}{(\log x)^n} = \frac{x^{m+1}}{(1-n)(\log x)^{n-1}} + \frac{m+1}{m+1} \int \frac{x^m \, dx}{(\log x)^{n-1}}. \]

476. \[ \int x^n (\log x)^m \, dx = \left\{ \begin{array}{ll}
& \frac{x^{n+1}}{m+1} (\log x)^m - \frac{n}{m+1} \int x^n (\log x)^{m-1} \, dx, \\
\text{or} & (\log x)^m \sum_{r=0}^{n-1} \frac{(-\log x)^r}{r!(m+1)^{n-r}}.
\end{array} \right. \]

477. \[ \int x^p \cos(b \log x) \, dx = \frac{x^{p+1}}{(p+1)^2 + b^2} \left[ b \sin(b \log x) + (p+1) \cos(b \log x) \right]. \]

478. \[ \int x^p \sin(b \log x) \, dx = \frac{x^{p+1}}{(p+1)^2 + b^2} \left[ (p+1) \sin(b \log x) - b \cos(b \log x) \right]. \]
479. \[ \int \log (ax + b) \, dx = \frac{ax + b}{a} \log (ax + b) - x. \]

480. \[ \int \frac{\log (ax + b)}{x^2} \, dx = \frac{a}{b} \log x - \frac{ax + b}{bx} \log (ax + b). \]

481. \[ \int x^n \log (ax + b) \, dx = \frac{1}{m + 1} \left[ x^{m+1} - \left( \frac{b}{a} \right)^{m+1} \right] \log (ax + b) \]
\[ - \frac{1}{m + 1} \left( \frac{b}{a} \right)^{m+1} \sum_{r=1}^{m+1} \frac{1}{r} \left( \frac{-ax}{b} \right)^r. \]

482. \[ \int \frac{\log (ax + b)}{x^m} \, dx = -\frac{1}{m-1} \log (ax + b) + \frac{1}{m-1} \left( \frac{a}{b} \right)^{m-1} \log \frac{ax + b}{x} \]
\[ + \frac{1}{m-1} \left( \frac{a}{b} \right)^{m-1} \sum_{r=1}^{m-1} \frac{1}{r} \left( \frac{-b}{ax} \right)^r, \quad m > 2. \]

483. \[ \int \frac{x + a}{x - a} \, dx = (x + a) \log (x + a) - (x - a) \log (x - a). \]

484. \[ \int x^n \log \frac{x + a}{x - a} \, dx = \frac{x^{m+1} - (-a)^{m+1}}{m + 1} \log (x + a) - \frac{x^{m+1} - a^{m+1}}{m + 1} \log (x - a) \]
\[ + \frac{2a^{m+1}}{m + 1} \sum_{r=1}^{m+1} \frac{1}{m - 2r + 2} \left( \frac{x}{a} \right)^{m-2r+2}. \]

485. \[ \int \frac{1}{x^2} \log \frac{x + a}{x - a} \, dx = -\frac{1}{x} \log \frac{x - a}{x + a} - \frac{1}{a} \log \frac{x^2 - a^2}{x^2}. \]

For the following two integrals, \( X = a + bx + cx^2. \)

486. * \[ \int \log X \, dx = \begin{cases} (x + \frac{b}{2c}) \log X - 2x + \frac{\sqrt{4ac - b^2}}{c} \tan^{-1} \frac{2cx + b}{\sqrt{4ac - b^2}}, & b^2 - 4ac < 0, \\ (x + \frac{b}{2c}) \log X - 2x + \frac{\sqrt{b^2 - 4ac}}{2c} \tanh^{-1} \frac{2cx + b}{\sqrt{b^2 - 4ac}}, & b^2 - 4ac > 0. \end{cases} \]

487. * \[ \int x^n \log X \, dx = \frac{x^{n+1}}{n + 1} \log X - \frac{2c}{n + 1} \int \frac{x^{n+2}}{x} \, dx - \frac{b}{n + 1} \int \frac{x^{n+1}}{X} \, dx. \]

488. \[ \int \log (x^2 + a^2) \, dx = x \log (x^2 + a^2) - 2x + 2a \tan^{-1} \frac{x}{a}. \]

489. \[ \int \log (x^2 - a^2) \, dx = x \log (x^2 - a^2) - 2x + a \log \frac{x + a}{x - a}. \]

490. \[ \int x \log (x^2 + a^2) \, dx = \frac{1}{2} (x^2 + a^2) \log (x^2 + a^2) - \frac{1}{2} x^2. \]

491. \[ \int \log (x + \sqrt{x^2 + a^2}) \, dx = x \log (x + \sqrt{x^2 + a^2}) - \sqrt{x^2 + a^2}. \]

492. \[ \int x \log (x + \sqrt{x^2 + a^2}) \, dx = \frac{x^2 + a^2}{2} \log (x + \sqrt{x^2 + a^2}) - \frac{x \sqrt{x^2 + a^2}}{4}. \]

493. \[ \int x^n \log (x + \sqrt{x^2 + a^2}) \, dx = \frac{x^{m+1}}{m + 1} \log (x + \sqrt{x^2 + a^2}) \]
\[ - \frac{1}{m + 1} \int \frac{x^{m+1}}{\sqrt{x^2 + a^2}} \, dx. \]
\[ \int \frac{\log (x + \sqrt{x^2 + a^2})}{x^2} \, dx = -\frac{\log (x + \sqrt{x^2 + a^2})}{x} - \frac{1}{a} \log \frac{a + \sqrt{x^2 + a^2}}{x}. \]

\[ \int \frac{\log (x + \sqrt{x^2 - a^2})}{x^2} \, dx = -\frac{\log (x + \sqrt{x^2 - a^2})}{x} + \frac{1}{|a|} \sec^{-1} \frac{x}{a}. \]

\[ \int x^n \log (x^2 - a^2) \, dx = \frac{1}{n+1} \left[ x^{n+1} \log (x^2 - a^2) - a^{n+1} \log (x - a) - (-a)^{n+1} \log (x + a) - 2 \sum_{r=0}^{\infty} \frac{a^{2r} x^{n-2r+1}}{n-2r+1} \right]. \]

### 5.4.19 EXPONENTIAL FORMS

\[ \int e^x \, dx = e^x. \]

\[ \int e^{-x} \, dx = -e^{-x}. \]

\[ \int e^{ax} \, dx = \frac{e^{ax}}{a}. \]

\[ \int x e^{ax} \, dx = \frac{e^{ax}}{a^2} (ax - 1). \]

\[ \int x^n e^{ax} \, dx = \left\{ \begin{array}{ll}
\frac{x^n e^{ax}}{a} - \frac{m}{a} \int x^{m-1} e^{ax} \, dx, & \text{or} \\
\sum_{r=0}^{m} (-1)^r \frac{m! x^{m-r}}{(m-r)! a^{r+1}}, & \text{or} 
\end{array} \right. \]

\[ \int \frac{e^{ax}}{x^m} \, dx = \log x + \frac{ax^2}{2!} + \frac{a^2 x^3}{3!} + \ldots. \]

\[ \int \frac{e^{ax}}{x^m} \, dx = -\frac{e^{ax}}{x} + \frac{a}{m-1} \int \frac{e^{ax}}{x^{m-1}} \, dx. \]

\[ \int e^{ax} \log x \, dx = \frac{e^{ax} \log x}{a} - \frac{1}{a} \int e^{ax} \, dx. \]

\[ \int \frac{dx}{1 + e^x} = x - \log (1 + e^x) = \log \frac{e^x}{1 + e^x}. \]

\[ \int \frac{dx}{a + be^{ax}} = \frac{x}{a} - \frac{1}{ap} \log (a + be^{ax}). \]

\[ \int \frac{dx}{ae^{ax} + be^{-ax}} = \frac{1}{m \sqrt{ab}} \tan^{-1} \left( \frac{e^{ax}}{\sqrt{b}} \right), \quad a > 0, \ b > 0. \]

\[ \int \frac{dx}{ae^{ax} - be^{-ax}} = \left\{ \begin{array}{ll}
\frac{1}{2m \sqrt{ab}} \log \frac{\sqrt{a} e^{ax} - \sqrt{b}}{\sqrt{a} e^{ax} + \sqrt{b}}, & a > 0, \ b > 0, \\
\frac{-1}{m \sqrt{ab}} \tanh^{-1} \left( \frac{\sqrt{a}}{\sqrt{b}} e^{ax} \right), & a > 0, \ b > 0.
\end{array} \right. \]

\[ \int (a^x - a^{-x}) \, dx = \frac{a^x + a^{-x}}{\log a}. \]

\[ \int \frac{e^{ax}}{b + ce^{ax}} \, dx = \frac{1}{ac} \log (b + ce^{ax}). \]
\[ \int \frac{xe^{ax}}{(1 + ax)^2} \, dx = \frac{e^{ax}}{a^2(1 + ax)}. \]

\[ \int xe^{-x^2} \, dx = -\frac{1}{2}e^{-x^2}. \]

\[ \int e^{ax} \sin (bx) \, dx = \frac{e^{ax} [a \sin (bx) - b \cos (bx)]}{a^2 + b^2}. \]

\[ \int e^{ax} \sin (bx) \sin (cx) \, dx = \frac{e^{ax} [(b - c) \sin (b - c)x + a \cos (b - c)x]}{2 \left[a^2 + (b - c)^2\right]} - \frac{e^{ax} [(b + c) \sin (b + c)x + a \cos (b + c)x]}{2 \left[a^2 + (b + c)^2\right]}. \]

\[ \int e^{ax} \sin (bx) \cos (cx) \, dx = \frac{e^{ax} [a \sin (b - c)x - (b - c) \cos (b - c)x]}{2 \left[a^2 + (b - c)^2\right]} + \frac{e^{ax} [a \sin (b + c)x - (b + c) \cos (b + c)x]}{2 \left[a^2 + (b + c)^2\right]} \frac{2a}{2a + 4b^2}. \]

\[ \int e^{ax} \cos (bx) \, dx = \frac{e^{ax}}{a^2 + b^2} [a \cos (bx) + b \sin (bx)]. \]

\[ \int e^{ax} \cos (bx) \cos (cx) \, dx = \frac{e^{ax} [(b - c) \sin (b - c)x + a \cos (b - c)x]}{2 \left[a^2 + (b - c)^2\right]} + \frac{e^{ax} [(b + c) \sin (b + c)x + a \cos (b + c)x]}{2 \left[a^2 + (b + c)^2\right]}. \]

\[ \int e^{ax} \cos (bx) \cos (cx) \, dx = \frac{e^{ax} \cos c}{2a} + \frac{e^{ax} [a \cos 2bx + c + 2b \sin 2bx + c]}{2 \left[a^2 + 4b^2\right]}. \]

\[ \int e^{ax} \sin (bx) \cos (cx) \, dx = \frac{e^{ax} \sin c}{2a} + \frac{e^{ax} [a \sin 2bx + c - 2b \cos 2bx + c]}{2 \left[a^2 + 4b^2\right]}. \]

\[ \int e^{ax} \sin^n (bx) \, dx = \frac{1}{a^2 + nb^2} \left[a \sin (bx) - nb \cos (bx)\right] e^{ax} \sin^{n-1} (bx) + n(n - 1)b^2 \int e^{ax} \sin^{n-2} (bx) \, dx. \]

\[ \int e^{ax} \cos^n (bx) \, dx = \frac{1}{a^2 + nb^2} \left[a \cos (bx) + nb \sin (bx)\right] e^{ax} \cos^{n-1} (bx) + n(n - 1)b^2 \int e^{ax} \cos^{n-2} (bx) \, dx. \]

\[ \int x^m e^x \sin x \, dx = \frac{1}{2} x^m e^x (\sin x - \cos x) - \frac{m}{2} \int x^{m-1} e^x \sin x \, dx + \frac{m}{2} \int x^{m-1} e^x \cos x \, dx. \]

\[ \int x^m e^x \sin bx \, dx = \frac{x^m e^x a \sin (bx) - b \cos (bx)}{a^2 + b^2} - \frac{m}{a^2 + b^2} \int x^{m-1} e^x (a \sin (bx) - b \cos (bx)) \, dx. \]

\[ \int x^m e^x \cos x \, dx = \frac{1}{2} x^m e^x (\sin x + \cos x) - \frac{m}{2} \int x^{m-1} e^x \sin x \, dx - \frac{m}{2} \int x^{m-1} e^x \cos x \, dx. \]
527. \( \int x^m e^{ax} \cos bx \, dx = \frac{x^m e^{ax} a \cos (bx) + b \sin (bx)}{a^2 + b^2} \)
\( \quad - \frac{m}{a^2 + b^2} \int x^{m-1} e^{ax} (a \cos (bx) + b \sin (bx)) \, dx. \)

528. \( \int e^{ax} \cos^n x \, dx = \)
\[ \frac{e^{ax} (\cos^{n-1} x) (\sin^n x) [a \cos x + (m + n) \sin x]}{(m + n)^2 + a^2} \]
\( \quad \frac{na}{(m + n)^2 + a^2} \int e^{ax} (\cos^{m-1} x) (\sin^{n-1} x) \, dx \)
\( \quad + \frac{(m + n)^2 + a^2}{(m + n)(n + 1)(m + n - 1) \sin^n x} \int e^{ax} (\cos^{m-1} x) (\sin^{n-1} x) \, dx, \)
\( \quad \text{or} \)
\[ \frac{e^{ax} (\cos^{m-1} x) (\sin^{n-1} x) [a \sin x - (m + n) \cos x]}{(m + n)^2 + a^2} \]
\( \quad \frac{na}{(m + n)^2 + a^2} \int e^{ax} (\cos^{m-1} x) (\sin^{n-1} x) \, dx \)
\( \quad + \frac{(m + n)^2 + a^2}{(m + n)(n + 1)(m + n - 1) \cos^n x} \int e^{ax} (\cos^{m-1} x) (\sin^{n-1} x) \, dx, \)
\( \quad \text{or} \)
\[ e^{ax} (\cos^{m-1} x) (\sin^{n-1} x) \left[ a \sin x \cos x + m \sin^2 x - n \cos^2 x \right] \]
\( \quad \frac{(m + n)^2 + a^2}{(m + n)(n + 1)(m + n - 1) \sin^n x} \int e^{ax} (\cos^{m-1} x) (\sin^{n-1} x) \, dx. \)

529. \( \int x e^{ax} \sin (bx) \, dx = \frac{x e^{ax}}{a^2 + b^2} [a \sin (bx) - b \cos (bx)] \)
\( \quad \frac{e^{ax}}{(a^2 + b^2)^2} \left[ (a^2 - b^2) \sin bx - 2ab \cos (bx) \right]. \)

530. \( \int x e^{ax} \cos (bx) \, dx = \frac{x e^{ax}}{a^2 + b^2} [a \cos (bx) + b \sin (bx)] \)
\( \quad \frac{e^{ax}}{(a^2 + b^2)^2} \left[ (a^2 - b^2) \cos bx + 2ab \sin (bx) \right]. \)

531. \( \int \frac{e^{ax}}{\sin^n x} \, dx = - \frac{e^{ax} [a \sin x + (n - 2) \cos x]}{(n - 1)(n - 2) \sin^{n-1} x} \)
\( \quad + \frac{a^2 + (n - 2)^2}{(n - 1)(n - 2)} \int \frac{e^{ax}}{\sin^{n-2} x} \, dx. \)

532. \( \int \frac{e^{ax}}{\cos^n x} \, dx = \frac{e^{ax} [a \cos x - (n - 2) \sin x]}{(n - 1)(n - 2) \cos^{n-1} x} \)
\( \quad + \frac{a^2 + (n - 2)^2}{(n - 1)(n - 2)} \int \frac{e^{ax}}{\cos^{n-2} x} \, dx. \)

533. \( \int e^{ax} \tan^n x \, dx = e^{ax} \tan^{n-1} x \frac{1}{n - 1} - \frac{a}{n - 1} \int e^{ax} \tan^{n-1} x \, dx - \int e^{ax} \tan^{n-2} x \, dx. \)
5.4.20 HYPERBOLIC FORMS

534. \( \int \sinh x \, dx = \cosh x \).

535. \( \int \cosh x \, dx = \sinh x \).

536. \( \int \tanh x \, dx = \log \cosh x \).

537. \( \int \coth x \, dx = \log \sinh x \).

538. \( \int \sech x \, dx = \tanh^{-1} (\sinh x) \).

539. \( \int \csch x \, dx = \log \tanh \left( \frac{x}{2} \right) \).

540. \( \int x \sinh x \, dx = x \cosh x - \sinh x \).

541. \( \int x^n \sinh x \, dx = x^n \cosh x - n \int x^{n-1} (\cosh x) \, dx \).

542. \( \int x \cosh x \, dx = x \sinh x - \cosh x \).

543. \( \int x^n \cosh x \, dx = x^n \sinh x - n \int x^{n-1} (\sinh x) \, dx \).

544. \( \int \sech x \tanh x \, dx = -\sech x \).

545. \( \int \csch x \coth x \, dx = -\csch x \).

546. \( \int \sinh^2 x \, dx = \frac{\sinh 2x}{4} - \frac{x}{2} \).

547. \( \int \sinh^m x \cosh^n x \, dx = \begin{cases} \frac{1}{m+n} \sinh^{m+1} x \cosh^{n-1} x + \frac{n-1}{m+n} \int \sinh^m x \cosh^{n-2} x \, dx, & m+n \neq 0, \\ \text{or} \\ \frac{1}{m+n} \sinh^{m-1} x \cosh^{n+1} x - \frac{m-1}{m+n} \int \sinh^{m-2} x \cosh^n x \, dx, & m+n \neq 0. \end{cases} \)

548. \( \int \frac{dx}{(\sinh^m x)(\cosh^n x)} = \begin{cases} \frac{1}{(m-1)(\sinh^{m-1} x)(\cosh^{n-1} x)} - \frac{m+n-2}{m-1} \int \frac{dx}{(\sinh^{m-2} x)(\cosh^n x)} \, dx, & m \neq 1, \\ \text{or} \\ \frac{1}{(n-1)(\sinh^{m-1} x)(\cosh^{n-1} x)} + \frac{m+n-2}{n-1} \int \frac{dx}{(\sinh^m x)(\cosh^{n-2} x)} \, dx, & n \neq 1. \end{cases} \)

549. \( \int \tanh^2 x \, dx = x - \tanh x \).

550. \( \int \tanh^n x \, dx = -\frac{\tanh^{n-1} x}{n-1} + \int (\tanh^{n-2} x) \, dx, \quad n \neq 1. \)

551. \( \int \sech^2 x \, dx = \tanh x \).

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\[552. \int \cosh^2 x \, dx = \frac{\sinh 2x}{4} + \frac{x}{2}\]
\[553. \int \coth^2 x \, dx = x - \coth x.\]
\[554. \int \coth^n x \, dx = -\frac{\coth^{n-1} x}{n-1} + \int \coth^{n-2} x \, dx, \quad n \neq 1.\]
\[555. \int \csch^2 x \, dx = -\coth x.\]
\[556. \int (\sinh mx)(\sinh nx) \, dx = \frac{\sinh (m+n)x}{2(m+n)} - \frac{\sinh (m-n)x}{2(m-n)}, \quad m^2 \neq n^2.\]
\[557. \int (\cosh mx)(\cosh nx) \, dx = \frac{\sinh (m+n)x}{2(m+n)} + \frac{\sinh (m-n)x}{2(m-n)}, \quad m^2 \neq n^2.\]
\[558. \int (\sinh mx)(\cosh nx) \, dx = \frac{\cosh (m+n)x}{2(m+n)} + \frac{\cosh (m-n)x}{2(m-n)}, \quad m^2 \neq n^2.\]
\[559. \int \left( \frac{\sinh^{-1} x}{a} \right) \, dx = x \sinh^{-1} \left( \frac{x}{a} \right) - \sqrt{x^2 + a^2}, \quad a > 0.\]
\[560. \int x \left( \frac{\sinh^{-1} x}{a} \right) \, dx = \left( \frac{x^2}{2} + \frac{a^2}{4} \right) \sinh^{-1} \left( \frac{x}{a} \right) - \frac{x}{4} \sqrt{x^2 + a^2}, \quad a > 0.\]
\[561. \int x^n \sinh^{-1} x \, dx = \frac{x^{n+1}}{n+1} \sinh^{-1} x - \frac{1}{n+1} \int \frac{x^{n+1}}{\sqrt{1+x^2}} \, dx, \quad n \neq -1.\]
\[562. \int \cosh^{-1} \left( \frac{x}{a} \right) \, dx = \left\{ \begin{array}{ll}
z \cosh^{-1} \left( \frac{z}{a} \right) - \sqrt{z^2 - a^2}, & \cosh^{-1} \left( \frac{z}{a} \right) > 0, \\
z \cosh^{-1} \left( \frac{z}{a} \right) + \sqrt{z^2 - a^2}, & \cosh^{-1} \left( \frac{z}{a} \right) < 0, \quad a > 0.
\end{array} \right.\]
\[563. \int x \left( \cosh^{-1} \left( \frac{x}{a} \right) \right) \, dx = \left( \frac{x^2}{2} - \frac{a^2}{4} \right) \cosh^{-1} \left( \frac{x}{a} \right) - \frac{x}{4} \sqrt{x^2 - a^2}.\]
\[564. \int x^n \cosh^{-1} x \, dx = \frac{x^{n+1}}{n+1} \cosh^{-1} x - \frac{1}{n+1} \int \frac{x^{n+1}}{\sqrt{x^2 - 1}} \, dx, \quad n \neq -1.\]
\[565. \int \left( \tanh^{-1} \left( \frac{x}{a} \right) \right) \, dx = x \tanh^{-1} \left( \frac{x}{a} \right) + \frac{a}{2} \log (a^2 - x^2), \quad |\frac{x}{a}| < 1.\]
\[566. \int \left( \coth^{-1} \left( \frac{x}{a} \right) \right) \, dx = x \coth^{-1} \left( \frac{x}{a} \right) + \frac{a}{2} \log (x^2 - a^2), \quad |\frac{x}{a}| > 1.\]
\[567. \int x \left( \tanh^{-1} \left( \frac{x}{a} \right) \right) \, dx = \frac{x^2 - a^2}{2} \tanh^{-1} \left( \frac{ax}{2} \right) + \frac{a}{2}, \quad |\frac{x}{a}| < 1.\]
\[568. \int x^n \tanh^{-1} x \, dx = \frac{x^{n+1}}{n+1} \tanh^{-1} x - \frac{1}{n+1} \int \frac{x^{n+1}}{1-x^2} \, dx, \quad n \neq -1.\]
\[569. \int x \left( \coth^{-1} \left( \frac{x}{a} \right) \right) \, dx = \frac{x^2 - a^2}{2} \coth^{-1} \left( \frac{ax}{2} \right) + \frac{a}{2}, \quad |\frac{x}{a}| > 1.\]
\[570. \int x^n \coth^{-1} x \, dx = \frac{x^{n+1}}{n+1} \coth^{-1} x + \frac{1}{n+1} \int \frac{x^{n+1}}{x^2 - 1} \, dx, \quad n \neq -1.\]
\[571. \int \text{sech}^{-1} x \, dx = x \text{sech}^{-1} x + \sin^{-1} x.\]
\[572. \int x \text{sech}^{-1} x \, dx = \frac{x^2}{2} \text{sech}^{-1} x - \frac{1}{2} \sqrt{1-x^2}.\]
\[573. \int x^n \text{sech}^{-1} x \, dx = \frac{x^{n+1}}{n+1} \text{sech}^{-1} x + \frac{1}{n+1} \int \frac{x^n}{\sqrt{1-x^2}} \, dx, \quad n \neq -1.\]
\[ \int \text{csch}^{-1} x \, dx = x \text{csch}^{-1} x + \frac{x}{|x|} \sinh^{-1} x. \]

\[ \int x \text{csch}^{-1} x \, dx = \frac{x^2}{2} \text{csch}^{-1} x + \frac{x}{2 |x|} \sqrt{1 + x^2}. \]

\[ \int x^n \text{csch}^{-1} x \, dx = \frac{x^{n+1}}{n+1} \text{csch}^{-1} x + \frac{1}{n+1 |x|} \int \frac{x^n}{\sqrt{1+x^2}} \, dx, \quad n \neq -1. \]

### 5.4.21 BESSEL FUNCTIONS

\( Z_p(x) \) represents any of the Bessel functions \( \{ J_p(x), Y_p(x), K_p(x), I_p(x) \} \).

\[ \int x^{p+1} Z_p(x) \, dx = x^{p+1} Z_{p+1}(x). \]

\[ \int x^{-p} Z_p(x) \, dx = -x^{-p+1} Z_{p-1}(x). \]

\[ \int x \left[ Z_p(ax) \right]^2 \, dx = \frac{x^2}{2} \left[ \left( Z_p(ax) \right)^2 - Z_{p-1}(ax) Z_{p+1}(ax) \right]. \]

\[ \int Z_1(x) \, dx = -Z_0(x). \]

\[ \int x Z_0(x) \, dx = x Z_1(x). \]

### 5.5 TABLE OF DEFINITE INTEGRALS

All integrals listed below that do not have stars next to their numbers have been automatically verified by computer.

\[ \int_0^\infty x^{n-1} e^{-x} \, dx = \Gamma(n), \quad n \text{ is a positive integer}. \]

\[ \int_0^\infty x^n p^x \, dx = \frac{n!}{(\log p)^{n+1}}, \quad p > 0, \quad n \text{ is a non-negative integer}. \]

\[ \int_0^\infty x^{n-1} e^{-(a+1)x} \, dx = \frac{\Gamma(n)}{(a+1)^n}, \quad n > 0, \quad a > -1. \]

\[ \int_0^1 x^m \left( \log \left( \frac{1}{x} \right) \right)^n \, dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}, \quad m > -1, \quad n > -1. \]

\[ \int_0^1 x^{m-1} (1-x)^{n-1} \, dx = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} \, dx = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}, \quad n > 0, \quad m > 0. \]

\[ \int_a^b (x-a)^m (b-x)^n \, dx = \frac{(b-a)^{n+m+1} \Gamma(m+1) \Gamma(n+1)}{\Gamma(m+n+2)}, \quad m > -1, \quad n > -1, \quad b > a. \]

\[ \int_1^\infty \frac{dx}{x^m} = \frac{1}{m-1}, \quad m > 1. \]

\[ \int_0^\infty \frac{dx}{(1+x)x^p} = \pi \cot \pi p, \quad 0 < p < 1. \]
\[ \int_0^\infty \frac{dx}{(1-x)x^p} = -\pi \cot p\pi, \quad 0 < p < 1. \]

\[ \int_0^1 \frac{x^p}{(1-x)^p} dx = p\pi \cosec p\pi, \quad |p| < 1. \]

\[ \int_0^1 \frac{x^p}{(1-x)^{p+1}} dx = -\pi \cosec p\pi, \quad -1 < p < 0. \]

\[ \int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}, \quad 0 < p < 1. \]

\[ \int_0^\infty \frac{x^{n-1}}{1+x^n} dx = \frac{\pi}{n \sin \frac{n\pi}{n}}, \quad 0 < m < n. \]

\[ \int_0^\infty \frac{x^a}{(m+x^n)^{n/2}} dx = \frac{m^{(a+1-b)/n} \Gamma\left(\frac{a+1}{m}\right) \Gamma\left(c - \frac{a+1}{m}\right)}{b \Gamma(c)}, \quad a > -1, \ b > 0, \ m > 0, \ c > \frac{a+1}{m}. \]

\[ \int_0^\infty \frac{dx}{(1+x)^{1/2}} = \pi. \]

\[ \int_0^\infty \frac{a}{a^2 + x^2} dx = \begin{cases} \frac{\pi}{2}, & a > 0, \\ 0, & a = 0, \\ -\frac{\pi}{2}, & a < 0. \end{cases} \]

\[ \int_0^a (a^2 - x^2)^{n/2} dx = \int_0^a \frac{1}{2} (a^2 - x^2)^{n/2} dx = \frac{n!!}{(n+1)!!} \frac{\pi a^{n+1}}{2}, \quad a > 0, \ n \text{ is an odd integer.} \]

\[ \int_0^a x^m (a^2 - x^3)^{n/2} dx = \frac{1}{2} a^{m+3/2} \frac{\Gamma\left(\frac{m+4}{2}\right) \Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{m+4+2n}{2}\right)}, \quad a > 0, \ m > -1, \ n > -2. \]

\[ \int_0^{\pi/2} \sin^n x \ dx = \int_0^{\pi/2} \cos^n x \ dx = \begin{cases} \frac{\sqrt{n} \Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{n+2}{2}\right)}{2 \Gamma\left(\frac{n+4}{2}\right)}, & n > -1, \\ \frac{(n-1)!! \pi}{2}, & n \neq 0, \ n \text{ is an even integer,} \\ \frac{(n-1)!! \pi}{n!!}, & n \neq 1, \ n \text{ is an odd integer.} \end{cases} \]

\[ \int_0^\infty \frac{\sin ax}{x} \ dx = \begin{cases} \frac{\pi}{2}, & a > 0, \\ 0, & a = 0, \\ -\frac{\pi}{2}, & a < 0. \end{cases} \]

\[ \int_0^\infty \frac{\cos x}{x} \ dx = \infty. \]

\[ \int_0^\infty \frac{\tan x}{x} \ dx = \frac{\pi}{2}. \]

\[ \int_0^\infty \frac{\tan ax}{x} \ dx = \frac{\pi}{2}, \quad a > 0. \]
\[ \int_0^\pi \sin(nx) \cdot \sin(mx) \, dx = \int_0^\pi \cos(nx) \cdot \cos(mx) \, dx = 0, \quad n \neq m, \quad n \text{ is an integer, } m \text{ is an integer.} \]

\[ \int_0^{\pi/n} \sin(nx) \cdot \cos(nk) \, dx = \int_0^{\pi/n} \cos(nx) \cdot \sin(nk) \, dx = 0, \quad n \text{ is an integer.} \]

\[ \int_0^\infty \frac{\sin(ax) \cdot \cos(bx)}{x^2} \, dx = \begin{cases} 
\frac{2a}{a^2 - b^2} & \text{if } a - b \text{ is an odd integer,} \\
0 & \text{if } a - b \text{ is an even integer.}
\end{cases} \]

\[ \int_0^\infty \frac{\sin(ax) \cdot \sin(bx)}{x^2} \, dx = \begin{cases} 
\frac{\pi a}{2} & \text{if } 0 < a < b, \\
0 & \text{if } 0 < b < a.
\end{cases} \]

\[ \int_0^\pi \sin(mx) \, dx = \int_0^\pi \cos(mx) \, dx = \frac{\pi}{2}, \quad m \text{ is an integer.} \]

\[ \int_0^\infty \frac{\sin^2 px}{x^2} \, dx = \frac{\pi}{2} |p| \quad \text{for } 0 < p < 1. \]

\[ \int_0^\infty \frac{\cos^2 px}{x^2} \, dx = \frac{\pi}{2} |p| \quad \text{for } 0 < p < 1. \]

\[ \int_0^\infty \frac{1 - \cos px}{x^2} \, dx = \frac{\pi}{2} |p| \quad \text{for } 0 < p < 1. \]

\[ \int_0^\infty \frac{\sin px \cos qx}{x} \, dx = \begin{cases} 
\frac{\pi}{2} & \text{if } p > q > 0, \\
0 & \text{if } p > 0. \end{cases} \]

\[ \int_0^\infty \frac{\cos px}{x^2 + a^2} \, dx = \frac{\pi}{2|a|} e^{-|ma|}. \]

\[ \int_0^\infty \frac{\sin^2 x}{x^2} \, dx = \frac{1}{2} \sqrt{\frac{\pi}{2}}. \]

\[ \int_0^\infty \frac{\sin (ax^n)}{x} \, dx = \frac{1}{n a^{1/n}} \Gamma \left( \frac{1}{n} \right) \sin \frac{\pi}{2n}, \quad n > 1. \]

\[ \int_0^\infty \frac{\cos (ax^n)}{x} \, dx = \frac{1}{n a^{1/n}} \Gamma \left( \frac{1}{n} \right) \cos \frac{\pi}{2n}, \quad n > 1. \]

\[ \int_0^\infty \frac{\sin x}{\sqrt{x}} \, dx = \int_0^\infty \frac{\cos x}{\sqrt{x}} \, dx = \sqrt{\frac{\pi}{2}}. \]

\[ \int_0^\infty \frac{\sin^3 x}{x} \, dx = \frac{\pi}{4}. \]
\[ \int_0^{\infty} \frac{\sin^3 x}{x^2} \, dx = \frac{3}{4} \log 3. \]

\[ \int_0^{\infty} \frac{\sin^3 x}{x^3} \, dx = \frac{3\pi}{8}. \]

\[ \int_0^{\infty} \frac{\sin^4 x}{x^4} \, dx = \frac{\pi}{3}. \]

\[ \int_0^{\infty} \frac{dx}{1 + a \cos x} = \frac{\cos^{-1} a}{\sqrt{1 - a^2}}, \quad |a| < 1. \]

\[ \int_0^{\infty} \frac{dx}{a + b \cos x} = \frac{\pi}{\sqrt{a^2 - b^2}}, \quad a > b \geq 0. \]

\[ \int_0^{\infty} \frac{1 + a \cos x \, dx}{a \cos x} = \frac{2\pi}{\sqrt{1 - a^2}}, \quad |a| < 1. \]

\[ \int_0^{\infty} \frac{\cos ax - \cos bx}{x} \, dx = \log \left| \frac{b}{a} \right|. \]

\[ \int_0^{\infty} \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \frac{\pi}{2|ab|}, \quad a > 0, \ b > 0. \]

\[ \int_0^{\infty} \frac{dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^2} = \frac{\pi(a^2 + b^2)}{4a^3b^3}, \quad a > 0, \ b > 0. \]

\[ \int_0^{\infty} \sin^{n-1} x \cos^{m-1} x \, dx = \frac{1}{2} B \left( \frac{n}{2}, \frac{m}{2} \right), \quad m \text{ is a positive integer, } n \text{ is a positive integer}. \]

\[ \int_0^{\infty} \sin^{2n+1} x \, dx = \frac{1}{2} \left( \frac{2n+1}{2n+1} \right)!!, \quad n \text{ is a positive integer}. \]

\[ \int_0^{\infty} \sin^{2n} x \, dx = \frac{1}{2} \left( \frac{2n}{2n} \right)!! \frac{\pi}{2}, \quad n \text{ is a positive integer}. \]

\[ \int_0^{\infty} \frac{x}{\sin x} \, dx = 2 \left( \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots \right). \]

\[ \int_0^{\infty} \frac{dx}{1 + \tan^m x} = \frac{\pi}{4}, \quad m \text{ is a non-negative integer}. \]

\[ \int_0^{\infty} \frac{\cos x \, dx}{\sqrt{\cos x}} = \frac{(2\pi)^{1/2}}{(\Gamma(1/4))^2}. \]

\[ \int_0^{\infty} \frac{\tan^h x \, dx}{\sin x} = \frac{\pi}{2 \cos \left( \frac{h\pi}{2} \right)}, \quad 0 < h < 1. \]

\[ \int_0^{\infty} \frac{\tan^{-1} ax - \tan^{-1} bx}{x} \, dx = \frac{\pi}{2} \log \left| \frac{a}{b} \right|, \quad a > 0, \ b > 0. \]

\[ \int_0^{\infty} e^{-ax} \, dx = \frac{1}{a}, \quad a > 0. \]

\[ \int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} \, dx = \log \left| \frac{b}{a} \right|, \quad a > 0, \ b > 0. \]

\[ \int_0^{\infty} x^n e^{-ax} \, dx = \begin{cases} \frac{\Gamma(n + 1)}{a^{n+1}}, & a > 0, \ n > -1, \\ \frac{n!}{a^{n+1}}, & a > 0, \ n \text{ is a positive integer}. \end{cases} \]

\[ \int_0^{\infty} x^n e^{-axp} \, dx = \frac{\Gamma((n + 1)/p)}{pa^{(n+1)/p}}, \quad a > 0, \ p > 0, \ n > -1. \]
\[ \int_0^\infty e^{-ax^2} \, dx = \frac{1}{2a} \sqrt{\pi}, \quad a > 0. \]

\[ \int_0^b e^{-ax^2} \, dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \text{erf}(b\sqrt{a}), \quad a > 0. \]

\[ \int_b^\infty e^{-ax^2} \, dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \text{erfc}(b\sqrt{a}), \quad a > 0. \]

\[ \int_0^\infty xe^{-ax^2} \, dx = \frac{1}{2}, \quad a > 0. \]

\[ \int_0^\infty x^2 e^{-ax^2} \, dx = \frac{\sqrt{\pi}}{4}. \]

\[ \int_0^\infty x^n e^{-ax^2} \, dx = \frac{(n-1)!}{2(2a)^{n/2}} \sqrt{\frac{\pi}{a}}, \quad a > 0, \quad n > 0. \]

\[ \int_0^\infty x^{2n+1} e^{-ax^2} \, dx = \frac{n!}{2a^{n+1}}, \quad a > 0, \quad n > -1. \]

\[ \int_0^1 x^n e^{-ax} \, dx = \frac{m!}{a^{m+1}} \left[ 1 - e^{-a} \sum_{i=0}^m \frac{a^i}{i!} \right]. \]

\[ \int_0^\infty e^{-x^2 + bx^2} \, dx = \frac{1}{2a} \sqrt{\frac{\pi}{b}}, \quad a > 0, \quad b > 0. \]

\[ \int_0^\infty \sqrt{x} e^{-ax} \, dx = \frac{1}{2a} \sqrt{\frac{\pi}{a}}, \quad a > 0. \]

\[ \int_0^\infty e^{-ax} \, dx = \frac{\sqrt{\pi}}{\sqrt{a}}, \quad a > 0. \]

\[ \int_0^\infty e^{-ax} \cos mx \, dx = \frac{a}{a^2 + m^2}, \quad a > 0. \]

\[ \int_0^\infty e^{-ax} \cos (bx + c) \, dx = \frac{a \cos c - b \sin c}{a^2 + b^2}, \quad a > 0. \]

\[ \int_0^\infty e^{-ax} \sin mx \, dx = \frac{m}{a^2 + m^2}, \quad a > 0. \]

\[ \int_0^\infty e^{-ax} \sin (bx + c) \, dx = \frac{b \cos c + a \sin c}{a^2 + b^2}, \quad a > 0. \]

\[ \int_0^\infty xe^{-ax} \sin bx \, dx = \frac{2ab}{(a^2 + b^2)^2}, \quad a > 0. \]

\[ \int_0^\infty xe^{-ax} \cos bx \, dx = \frac{a^2 - b^2}{(a^2 + b^2)^2}, \quad a > 0. \]

\[ \int_0^\infty x^n e^{-ax} \sin bx \, dx = \frac{n! \left[(a+ib)^{n+1} - (a-ib)^{n+1}\right]}{2a(a^2+b^2)^{n+1}}, \quad a > 0. \]

\[ \int_0^\infty x^n e^{-ax} \cos bx \, dx = \frac{n! \left[(a-ib)^{n+1} + (a+ib)^{n+1}\right]}{2a(a^2+b^2)^{n+1}}, \quad a > 0, \quad n > -1. \]

\[ \int_0^\infty e^{-ax} \sin x \, dx = \cot^{-1} a, \quad a > 0. \]

\[ \int_0^\infty e^{-ax} \cos bx \, dx = \sqrt{\frac{\pi}{2a}} \exp\left(-\frac{b^2}{4a}\right), \quad ab > 0. \]
\[ \int_0^\infty e^{-x \cos \phi} x b^{-1} \sin(x \sin \phi) \, dx = \Gamma(b) \sin(b \phi), \quad b > 0, \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2}. \]

\[ \int_0^\infty e^{-x \cos \phi} x b^{-1} \cos(x \sin \phi) \, dx = \Gamma(b) \cos(b \phi), \quad b > 0, \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2}. \]

\[ \int_0^\infty x^{b-1} \cos x \, dx = \Gamma(b) \cos \left(\frac{b \pi}{2}\right), \quad 0 < b < 1. \]

\[ \int_0^\infty x^{b-1} \sin x \, dx = \Gamma(b) \sin \left(\frac{b \pi}{2}\right), \quad 0 < b < 1. \]

\[ \int_0^1 (\log x)^n \, dx = (-1)^n n!, \quad n < -1. \]

\[ \int_0^1 \sqrt{\log \frac{1}{x}} \, dx = \frac{\sqrt{\pi}}{2}. \]

\[ \int_0^1 \left(\log \frac{1}{x}\right)^n \, dx = n!. \]

\[ \int_0^1 x \log (1 - x) \, dx = -\frac{3}{4}. \]

\[ \int_0^1 x \log (1 + x) \, dx = \frac{1}{4}. \]

\[ \int_0^1 x^n (\log x)^n \, dx = \frac{(-1)^n \Gamma(n + 1)}{(m + 1)^{n+1}}, \quad m > -1, \quad n \text{ is a positive integer.} \]

\[ \int_0^1 \log \frac{x}{1 + x} \, dx = -\frac{\pi^2}{12}. \]

\[ \int_0^1 \log \frac{x}{1 - x} \, dx = -\frac{\pi^2}{6}. \]

\[ \int_0^1 \log \left(\frac{1 + x}{x}\right) \, dx = \frac{\pi^2}{12}. \]

\[ \int_0^1 \log \left(\frac{1 - x}{x}\right) \, dx = -\frac{\pi^2}{6}. \]

\[ \int_0^1 (\log x) \log (1 + x) \, dx = 2 - 2 \log 2 - \frac{\pi^2}{12}. \]

\[ \int_0^1 (\log x) \log (1 - x) \, dx = 2 - \frac{\pi^2}{6}. \]

\[ \int_0^1 \log \frac{x}{1 - x^2} \, dx = -\frac{\pi}{8}. \]

\[ \int_0^1 \log \left(\frac{1 + x}{1 - x}\right) \, dx = \frac{\pi^2}{4}. \]

\[ \int_0^1 \log \frac{x}{\sqrt{1 - x}} \, dx = -\pi \log 2. \]

\[ \int_0^1 x^m \left(\log \frac{1}{x}\right)^n \, dx = \frac{\Gamma(n + 1)}{(m + 1)^{n+1}}, \quad m > -1, \quad n > -1. \]

\[ \int_0^1 x^p - x^q \, dx = \log \left(\frac{p + 1}{q + 1}\right), \quad p > -1, \quad q > -1. \]

\[ \int_0^1 \frac{dx}{\sqrt{\log (-\log x)}} = \sqrt{\pi}. \]
687. \[ \int_0^\infty \log \left( \frac{e^x + 1}{e^x - 1} \right) \, dx = \frac{\pi^2}{4}. \]

688. \[ \int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \cos x \, dx = -\frac{\pi}{2} \log 2. \]

689. \[ \int_0^{\pi/2} \log \sec x \, dx = \int_0^{\pi/2} \log \cosec x \, dx = \frac{\pi}{2} \log 2. \]

690. \[ \int_0^\pi x \log \sin x \, dx = -\frac{\pi^2}{2} \log 2. \]

691. \[ \int_0^{\pi/2} (\sin x) \log \sin x \, dx = \log 2 - 1. \]

692. \[ \int_0^\infty \log \tan x \, dx = 0. \]

693. \[ \int_0^\infty \log (a \pm b \cos x) \, dx = \pi \log \left( \frac{a + \sqrt{a^2 - b^2}}{2} \right), \quad a \geq b. \]

694. \[ \int_0^\infty \log (a^2 - 2ab \cos x + b^2) \, dx = \begin{cases} 2\pi \log a, & a \geq b > 0, \\ or \\ 2\pi \log b, & b \geq a > 0. \end{cases} \]

695. \[ \int_0^\infty \frac{\sin ax}{\sinh bx} \, dx = \frac{\pi}{2} \tanh \frac{a\pi}{2|b|}. \]

696. \[ \int_0^\infty \frac{\cos ax}{\cosh bx} \, dx = \frac{\pi}{2} \sech \frac{a\pi}{2b}. \]

697. \[ \int_0^\infty \frac{dx}{\cosh ax} = \frac{\pi}{2 |a|}. \]

698. \[ \int_0^\infty \frac{x}{\sinh ax} \, dx = \frac{\pi^2}{4a^2}, \quad a \geq 0. \]

699. \[ \int_0^\infty e^{-ax} \cosh (bx) \, dx = \frac{a}{a^2 - b^2}, \quad |b| < a. \]

700. \[ \int_0^\infty e^{-ax} \sinh (bx) \, dx = \frac{b}{a^2 - b^2}, \quad |b| < a. \]

701. \[ \int_0^\infty e^{bx} \csc \frac{a\pi}{b} \, dx = \frac{\pi}{2a} - \frac{1}{2a^2}, \quad b \geq 0. \]

702. \[ \int_0^\infty \frac{\sinh ax}{e^{bx} + 1} \, dx = \frac{1}{2a} - \frac{\pi}{2b} \cot \frac{a\pi}{b}, \quad b \geq 0. \]

703. \[ \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{\pi}{2} \left[ 1 + \left( \frac{1}{2} \right)^2 k^2 + \left( \frac{1 \cdot 3}{2 \cdot 4} \right)^2 k^4 + \left( \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \right)^2 k^6 + \ldots \right], \quad k^2 < 1. \]

704. \[ \int_0^{\pi/2} \frac{dx}{(1 - k^2 \sin^2 x)^{3/2}} = \frac{\pi}{2} \left[ 1 + \left( \frac{1}{2} \right)^2 k^2 + \left( \frac{1 \cdot 3}{2 \cdot 4} \right)^2 k^4 + \left( \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \right)^2 k^6 + \ldots \right], \quad k^2 < 1. \]

705. \[ \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 x} \, dx = \frac{\pi}{2} \left[ 1 - \left( \frac{1}{2} \right)^2 k^2 - \left( \frac{1 \cdot 3}{2 \cdot 4} \right)^2 k^4 \right. \]

\[ \left. - \left( \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \right)^2 k^6 - \ldots \right], \quad k^2 < 1. \]
\[ \int_0^\infty e^{-x} \log x \, dx = -\gamma. \]
\[ \int_0^\infty e^{-x^2} \log x \, dx = -\frac{\sqrt{\pi}}{4} (\gamma + 2 \log 2). \]
\[ \int_0^\infty \left( \frac{1}{1-e^{-x}} - \frac{1}{x} \right) e^{-x} \, dx = \gamma. \]
\[ \int_0^\infty \frac{1}{x} \left( \frac{1}{1-e^{-x}} - \frac{1}{x} \right) \, dx = \gamma. \]

### 5.6 ORDINARY DIFFERENTIAL EQUATIONS

#### 5.6.1 LINEAR DIFFERENTIAL EQUATIONS

Any linear differential equation can be written in the form

\[ b_n(x)y^{(n)} + b_{n-1}(x)y^{(n-1)} + \cdots + b_1(x)y' + b_0(x)y = R(x) \quad (5.6.1) \]

or \( p(D)y = R(x) \), where \( D \) is the differentiation operator \( (Dy = dy/dx) \), \( p(D) \) is a polynomial in \( D \) with coefficients \( \{b_i\} \) depending on \( x \), and \( R(x) \) is an arbitrary function. In this notation, a power of \( D \) denotes repeated differentiation, that is, \( D^ny = d^ny/dx^n \). For such an equation, the general solution has the form

\[ y(x) = y_h(x) + y_p(x) \quad (5.6.2) \]

where \( y_h(x) \) is the homogeneous solution and \( y_p(x) \) is the particular solution. These functions satisfy \( p(D)y_h = 0 \) and \( p(D)y_p = R(x) \).

**Vector representation**

Equation (5.6.1) can be written in the form \( \frac{dy}{dx} = A(x)y + r(x) \) where

\[
\begin{bmatrix}
y \\
y' \\
y'' \\
\vdots \\
y^{(n-1)}
\end{bmatrix}
= \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 1 & \cdots \\
-\frac{b_0}{b_n} & -\frac{b_1}{b_n} & -\frac{b_2}{b_n} & \cdots & -\frac{b_{n-1}}{b_n}
\end{bmatrix}
\begin{bmatrix}
y \\
y' \\
y'' \\
\vdots \\
y^{(n-1)}
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
\frac{r_n}{b_n}
\end{bmatrix}.
\]

**Homogeneous solution**

For the special case of a linear differential equation with constant coefficients (i.e., the \( \{b_i\} \) in Equation (5.6.1) are constants), the procedure for finding the homogeneous solution is as follows:
1. Factor the polynomial $p(D)$ into real and complex linear factors, just as if $D$ were a variable instead of an operator.

2. For each nonrepeated linear factor of the form $(D - a)$, where $a$ is real, write a term of the form $ce^{ax}$, where $c$ is an arbitrary constant.

3. For each repeated real linear factor of the form $(D - a)^m$, write $m$ terms of the form

$$c_1 e^{ax} + c_2 xe^{ax} + c_3 x^2 e^{ax} + \cdots + c_m x^{m-1} e^{ax} \quad (5.6.3)$$

where the $c_i$'s are arbitrary constants.

4. For each nonrepeated conjugate complex pair of factors of the form $(D - a + ib)(D - a - ib)$, write two terms of the form

$$c_1 e^{ax} \cos bx + c_2 e^{ax} \sin bx. \quad (5.6.4)$$

5. For each repeated conjugate complex pair of factors of the form $(D - a + ib)^m(D - a - ib)^m$, write $2m$ terms of the form

$$c_1 e^{ax} \cos bx + c_2 e^{ax} \sin bx + c_3 xe^{ax} \cos bx + c_4 xe^{ax} \sin bx + \cdots + c_{2m-1} x^{m-1} e^{ax} \cos bx + c_{2m} x^{m-1} e^{ax} \sin bx. \quad (5.6.5)$$

6. The sum of all the terms thus written is the homogeneous solution.

Example

For the linear equation

$$y^{(7)} - 14y^{(6)} + 80y^{(5)} - 242y^{(4)} + 419y^{(3)} - 416y'' + 220y' - 48y = 0,$$

$p(D)$ factors as $p(D) = (D - 1)^3(D - 2)^2(D - 3)(D - 4)$. The roots are thus $\{1, 1, 1, 2, 2, 3, 4\}$. Hence, the homogeneous solution has the form

$$y_h(x) = \left(c_0 + c_1 x + c_2 x^2\right) e^x + (c_3 + c_4 x) e^{2x} + c_5 e^{3x} + c_6 e^{4x}$$

where $\{c_0, \ldots, c_6\}$ are arbitrary constants.
**Particular solutions**

The following are solutions for some specific ordinary differential equations. In these tables we assume that \( P(x) \) is a polynomial of degree \( n \) and \( \{a, b, p, q, r, s\} \) are constants. In all of these tables, when using “\( \cos \)” instead of “\( \sin \)” in \( R(x) \), use the given result, but replace “\( \sin \)” by “\( \cos \)”, and replace “\( \cos \)” by “−\( \sin \).

<table>
<thead>
<tr>
<th>If ( R(x) ) is</th>
<th>A particular solution to ( y' - ay = R(x) ) is</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( e^{rx} )</td>
<td>( e^{rx}/(r - a) ).</td>
</tr>
<tr>
<td>2. ( \sin sx )</td>
<td>(-\frac{a \sin sx + \cos sx}{a^2 + s^2} = -(a^2 + s^2)^{-1/2} \sin(sx + \tan^{-1} \frac{s}{a}) ).</td>
</tr>
<tr>
<td>3. ( P(x) )</td>
<td>(-\frac{1}{a} \left[ P(x) + \frac{P'(x)}{a} + \frac{P''(x)}{a^2} + \cdots + \frac{P^{(n)}(x)}{a^n} \right] ).</td>
</tr>
<tr>
<td>4. ( e^{rx} \sin sx )</td>
<td>Replace ( a ) by ( a - r ) in formula 2 and multiply by ( e^{rx} ).</td>
</tr>
<tr>
<td>5. ( P(x)e^{rx} )</td>
<td>Replace ( a ) by ( a - r ) in formula 3 and multiply by ( e^{rx} ).</td>
</tr>
</tbody>
</table>
| 6. \( P(x) \sin sx \) | \(-\sin sx \left[ \frac{s}{a^2 + s^2} P(x) + \frac{s^2 - r^2}{(a^2 + s^2)^2} P'(x) \right. \\
| & \left. + \cdots + \frac{s^k - r^k}{(a^2 + s^2)^k} \frac{P^{(k)}(x)}{a^k s^k} \right] \)
| & \(-\cos sx \left[ \frac{s}{a^2 + s^2} P(x) + \frac{2as}{(a^2 + s^2)^2} P'(x) \right. \\
| & \left. + \cdots + \frac{(k-1)s^{k-1} - r^k}{(a^2 + s^2)^k} \frac{P^{(k)}(x)}{a^k s^k} \right] \). |
| 7. \( P(x)e^{rx} \sin sx \) | Replace \( a \) by \( a - r \) in formula 6 and multiply by \( e^{rx} \). |
| 8. \( e^{sx} \) | \( xe^{ax} \). |
| 9. \( e^{ax} \sin sx \) | \(-e^{ax} \cos sx/s \). |
| 10. \( P(x)e^{ax} \) | \( e^{ax} \int x \cdot P(z) \, dz \). |
| 11. \( P(x)e^{ax} \sin sx \) | \(-e^{ax} \sin sx \left[ \frac{P(x)}{s} - \frac{P''(x)}{s^3} + \frac{P^{(4)}(x)}{s^5} + \cdots \right] \)
<p>| &amp; (-\frac{e^{ax} \cos sx}{s} \left[ P(x) - \frac{P''(x)}{s^3} + \frac{P^{(4)}(x)}{s^5} + \cdots \right] ). |</p>
<table>
<thead>
<tr>
<th>If $R(x)$ is</th>
<th>A particular solution to $y'' - 2ay' + a^2y = R(x)$ is</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^{rx}$</td>
<td>$e^{rx}/(r - a)^2$.</td>
</tr>
<tr>
<td>sin $sx$</td>
<td>$\frac{(a^2 - s^2) \sin sx + 2as \cos sx}{(a^2 + s^2)^2}$ $\frac{1}{a^2 + s^2} \sin \left(sx + \tan^{-1} \frac{2as}{a^2 - s^2}\right)$.</td>
</tr>
<tr>
<td>$P(x)$</td>
<td>$\frac{1}{a^2} \left[P(x) + \frac{2P'(x)}{a} + \frac{3P''(x)}{a^2} + \cdots + \frac{(n+1)P^{(n)}(x)}{a^n}\right]$.</td>
</tr>
<tr>
<td>$e^{rx}\sin sx$</td>
<td>Replace $a$ by $a - r$ in formula 13 and multiply by $e^{rx}$.</td>
</tr>
<tr>
<td>$P(x)e^{rx}$</td>
<td>Replace $a$ by $a - r$ in formula 14 and multiply by $e^{rx}$.</td>
</tr>
<tr>
<td>$P(x)\sin sx$</td>
<td>$\sin sx \left[\frac{a^2 - s^2}{(a^2 + s^2)^2} P(x) + 2 \frac{a^2 - 3as^2}{(a^2 + s^2)^2} P'(x)$ $+ \cdots + (k - 1) \frac{a^2 - (k-1)s^2 + \cdots + (k-1)s^2 - s^2}{(a^2 + s^2)^2} P^{(k-2)}(x) + \cdots \right] + \cos sx \left[\frac{2as}{(a^2 + s^2)^2} P(x) + 2 \frac{3as - s^2}{(a^2 + s^2)^2} P'(x)$ $+ \cdots + (k - 1) \frac{(k-1)s^2 - 3as + 3s^2 - s^2}{(a^2 + s^2)^2} P^{(k-2)}(x) + \cdots \right]$.</td>
</tr>
<tr>
<td>$P(x)e^{rx}\sin sx$</td>
<td>Replace $a$ by $a - r$ in formula 17 and multiply by $e^{rx}$.</td>
</tr>
<tr>
<td>$e^{ax}$</td>
<td>$x^2 e^{ax}/2$.</td>
</tr>
<tr>
<td>$e^{ax}\sin sx$</td>
<td>$-e^{ax} \sin sx/s^2$.</td>
</tr>
<tr>
<td>$P(x)e^{ax}$</td>
<td>$e^{ax} \int \int P(z) , dz , dy$.</td>
</tr>
<tr>
<td>$P(x)e^{ax}\sin sx$</td>
<td>$\frac{-e^{ax} \sin sx}{s^2} \left[P(x) - \frac{3P'(x)}{s} + \frac{5P''(x)}{s^2} + \cdots \right]$ $- \frac{e^{ax} \cos sx}{s^2} \left[2P(x) - \frac{4P'(x)}{s} + \frac{6P''(x)}{s^2} + \cdots \right]$.</td>
</tr>
<tr>
<td>If ( R(x) ) is</td>
<td>A particular solution to ( y'' + qy = R(x) ) is</td>
</tr>
<tr>
<td>-----------------</td>
<td>------------------------------------------------</td>
</tr>
<tr>
<td>23. ( e^{rx} )</td>
<td>( e^{rx}/(r^2 + q). )</td>
</tr>
<tr>
<td>24. ( \sin sx )</td>
<td>( \sin sx/(q - s^2). )</td>
</tr>
<tr>
<td>25. ( P(x) )</td>
<td>( \text{Re} \left{ \frac{1}{q} \left[ P(x) - \frac{P''(x)}{q} + \frac{P''(x)}{q^2} + \cdots + (-1)^k \frac{P^{(2k)}(x)}{q^k} + \cdots \right] \right}. )</td>
</tr>
<tr>
<td>26. ( e^{r x} \sin sx )</td>
<td>( \frac{e^{r x}}{(r^2 - q^2 + (2x)^2)} \sin \left[ sx - \tan^{-1} \frac{2x}{r^2 - q^2 + (2x)^2} \right]. )</td>
</tr>
<tr>
<td>27. ( P(x)e^{rx} )</td>
<td>( \frac{e^{r x}}{q + r^2} \left[ P(x) - \frac{2x}{q + r^2} P'(x) + \frac{3s^2 - q}{(q + r^2)^2} P''(x) + \cdots + (-1)^k \frac{k^{-3} - q^{-3}}{(q + r^2)^{k+1}} P^{(k)}(x) + \cdots \right]. )</td>
</tr>
<tr>
<td>28. ( P(x) \sin sx )</td>
<td>( \frac{\sin sx}{q - x} \left[ P(x) - \frac{3s^2 + q}{(q - x)^2} P''(x) + \cdots + (-1)^k \frac{(2k+1)x^{2k+2} + (2k+1)x^{2k-2} + \cdots}{(q - x)^{2k+1}} P^{(2k)}(x) + \cdots \right] )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>If ( R(x) ) is</th>
<th>A particular solution to ( y'' + b^2 y = R(x) ) is</th>
</tr>
</thead>
<tbody>
<tr>
<td>29. ( \sin bx )</td>
<td>(-x \cos bx/2b. )</td>
</tr>
<tr>
<td>30. ( P(x) \sin bx )</td>
<td>( \frac{\sin bx}{(2b)^2} \left[ P(x) - \frac{P''(x)}{(2b)^2} + \frac{P^{(4)}(x)}{(2b)^4} + \cdots \right] )</td>
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<td></td>
<td>( -\cos bx \int \left[ P(x) - \frac{P''(x)}{(2b)^2} + \cdots \right] dx. )</td>
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</table>
If $R(x)$ is a particular solution to $y'' + py' + qy = R(x)$ is

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<table>
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<tbody>
<tr>
<td>31. $e^{rx}$</td>
<td>$e^{rx}/(r^2 + pr + q)$.</td>
</tr>
<tr>
<td>32. $\sin sx$</td>
<td>$(\frac{(q-r)x - pr\cos sx}{(q-r)^2 + (pr)^2}) = \frac{1}{\sqrt{(q-r)^2 + (pr)^2}} \sin \left( sx - \tan^{-1} \frac{pr}{q-r} \right)$.</td>
</tr>
<tr>
<td>33. $P(x)$</td>
<td>$\frac{1}{q} \left[ P(x) - \frac{p}{q} P'(x) + \frac{p^2-q}{q^2} P''(x) - \frac{p^2-2pq}{q^3} P'''(x) + \ldots + \left( (-1)^{n} \frac{p^r-\left(C_{r}^{n}\right)}{q^r} q^{n} + \frac{\left(C_{r}^{n}\right)}{q^r} q^{n} \right) \right]$.</td>
</tr>
<tr>
<td>34. $e^{sx} \sin sx$</td>
<td>Replace $p$ by $p + 2r$, and $q$ by $q + pr + r^2$ in formula 32 and multiply by $e^{rx}$.</td>
</tr>
<tr>
<td>35. $P(x)e^{sx}$</td>
<td>Replace $p$ by $p + 2r$, and $q$ by $q + pr + r^2$ in formula 33 and multiply by $e^{rx}$.</td>
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</tbody>
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<tbody>
<tr>
<td>36. $e^{rx}$</td>
<td>$e^{rx}/(r-a)^n$.</td>
</tr>
<tr>
<td>37. $\sin sx$</td>
<td>$\frac{(-1)^{n}}{n} \left[ \left( a^n - \binom{n}{2} a^{n-2} s^2 + \binom{n}{4} a^{n-4} s^4 - \ldots \right) \sin sx + \left( \binom{n}{3} a^{n-1} s^3 + \ldots \right) \cos sx \right]$.</td>
</tr>
<tr>
<td>38. $P(x)$</td>
<td>$\frac{(-1)^{n}}{n} \left[ P(x) + \binom{n}{1} \frac{P'(x)}{a} + \binom{n}{2} \frac{P''(x)}{a^2} + \binom{n}{3} \frac{P'''(x)}{a^3} + \ldots \right]$.</td>
</tr>
<tr>
<td>39. $e^{sx} \sin sx$</td>
<td>Replace $a$ by $a - r$ in formula 37 and multiply by $e^{rx}$.</td>
</tr>
<tr>
<td>40. $P(x)e^{sx}$</td>
<td>Replace $a$ by $a - r$ in formula 38 and multiply by $e^{rx}$.</td>
</tr>
</tbody>
</table>

**Second order linear constant coefficient equation**

Consider $ay'' + by' + cy = 0$, where $a$, $b$, and $c$ are real constants. Let $m_1$ and $m_2$ be the roots of $am^2 + bm + c = 0$. There are three forms of the solution:

1. If $m_1$ and $m_2$ are real and distinct, then $y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
2. If $m_1$ and $m_2$ are real and equal, then $y(x) = c_1 e^{m_1 x} + c_2 xe^{m_1 x}$
3. If $m_1 = p + iq$ and $m_2 = p - iq$ (with $p = -b/2$ and $q = \sqrt{4ac - b^2}/2$), then $y(x) = e^{px} (c_1 \cos qx + c_2 \sin qx)$
Consider $ay'' + by' + cy = R(x)$, where $a$, $b$, and $c$ are real constants. Let $m_1$ and $m_2$ be as above.

1. If $m_1$ and $m_2$ are real and distinct, then $y(x) = C_1 e^{m_1 x} + C_2 e^{m_2 x} + e^{m_1 x} \int x e^{-m_1 z} R(z) \, dz + e^{m_2 x} \int x e^{-m_2 z} R(z) \, dz$. 

2. If $m_1$ and $m_2$ are real and equal, then $y(x) = C_1 e^{m_1 x} + C_2 x e^{m_1 x} + x e^{m_1 x} \int e^{-m_1 z} R(z) \, dz - e^{m_1 x} \int x e^{-m_1 z} R(z) \, dz$. 

3. If $m_1 = p + iq$ and $m_2 = p - iq$, then $y(x) = e^{px} (c_1 \cos qx + c_2 \sin qx) + e^{px} \sin qx / q \int e^{-pz} R(z) \cos qz \, dz - e^{px} \cos qx / q \int e^{-pz} R(z) \sin qx \, dz$. 

Damping: none, under, over, and critical

Consider the linear ordinary differential equation (ODE) $x'' + \mu x' + x = 0$. If the damping coefficient $\mu$ is positive, then all solutions decay to $x = 0$. If $\mu = 0$, the system is undamped and the solution oscillates without decaying. The value of $\mu$ such that the roots of the characteristic equation $\lambda^2 + \mu \lambda + 1 = 0$ are real and equal is the critical damping coefficient. If $\mu$ is less than (greater than) the critical damping coefficient, then the system is under (over) damped.

In the following figure all curves have the same initial values: $x(0) = 2$ and $x'(0) = -2.5$. Reading down, at the left-most depression, are the curves

- $x'' + 3x' + x = 0$ Overdamped
- $x'' + 2x' + x$ Critically damped
- $x'' + 0.2x' + x = 0$ Underdamped
- $x'' + x = 0$ Undamped
## 5.6.2 SOLUTION TECHNIQUES

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<th>Differential equation</th>
<th>Solution or solution technique</th>
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<td><strong>Autonomous equation</strong></td>
<td>$f(y^{(n)}, y^{(n-1)}, \ldots, y', y) = 0$</td>
</tr>
<tr>
<td><strong>Bernoulli’s equation</strong></td>
<td>$y' + f(x)y = g(x)y^n$</td>
</tr>
<tr>
<td><strong>Clairaut’s equation</strong></td>
<td>$f(xy' - y) = g(y')$</td>
</tr>
<tr>
<td><strong>Constant coefficient equation</strong></td>
<td>$a_0y^{(n)} + a_1y^{(n-1)} + \ldots + a_ny = 0$</td>
</tr>
<tr>
<td><strong>Dependent variable missing</strong></td>
<td>$f(y^{(n)}, y^{(n-1)}, \ldots, y'', y') = 0$</td>
</tr>
<tr>
<td><strong>Euler’s equation</strong></td>
<td>$a_0xny^{(n)} + a_1x^{n-1}y^{(n-1)} + \ldots + a_ny = 0$</td>
</tr>
<tr>
<td><strong>Exact equation</strong></td>
<td>$M(x, y)dx + N(x, y)dy = 0$</td>
</tr>
<tr>
<td><strong>Homogeneous equation</strong></td>
<td>$y' = f \left( \frac{y}{x} \right)$</td>
</tr>
<tr>
<td><strong>Linear first order equation</strong></td>
<td>$y' + f(x)y = g(x)$</td>
</tr>
<tr>
<td><strong>Reducible to homogeneous</strong></td>
<td>$(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0$</td>
</tr>
<tr>
<td><strong>Reducible to separable</strong></td>
<td>$(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0$</td>
</tr>
<tr>
<td><strong>Separation of variables</strong></td>
<td>$y' = f(x)g(y)$</td>
</tr>
</tbody>
</table>

## 5.6.3 INTEGRATING FACTORS

An integrating factor is a multiplicative term that makes a differential equation become exact. If the differential equation $M(x, y)dx + N(x, y) \neq 0$ is not exact (i.e., $M_y \neq N_x$) then it may always be made exact if you can find the integrating factor.
1. If \( \frac{1}{M} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = f(x) \), a function of \( x \) alone, then \( u = \exp \left( \int f(x) \, dx \right) \) is an integrating factor.

2. If \( \frac{1}{N} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = g(y) \), a function of \( y \) alone, then \( u = \exp \left( \int g(y) \, dy \right) \) is an integrating factor.

**Example**

The equation \( y \, dx + dy = 0 \) has \( \{ M = y/x, \, N = 1 \} \) and \( f(x) = 1/x \). Hence \( u = \exp \left( \int \frac{1}{x} \, dz \right) = \exp(\log x) = x \) is an integrating factor. Multiplying the original equation by \( u(x) \) results in \( y \, dx + x \, dy = 0 \) or \( d(xy) = 0 \).

**5.6.4 VARIATION OF PARAMETERS**

If the equation \( L[y] = y'' + P(x)y' + Q(x)y = R(x) \) has the homogeneous solutions \( y_1(x) \) and \( y_2(x) \) (i.e., \( L[y_i] = 0 \)), then the solution to the original equation is given by

\[
y(x) = -y_1(x) \int \frac{y_1(x)R(x)}{W(y_1, y_2)} \, dx + y_2(x) \int \frac{y_1(x)R(z)}{W(y_1, y_2)} \, dx,
\]

where \( W(y_1, y_2) = y'_1y_2 - y_1y'_2 \) is the Wronskian.

**Example**

The homogeneous solutions to \( y'' + y = \csc x \) are clearly \( y_1(x) = \sin x \) and \( y_2(x) = \cos x \). Here, \( W(y_1, y_2) = -1 \). Hence, \( y(x) = \sin x \log(\sin x) - x \cos x \).

**5.6.5 GREEN’S FUNCTIONS**

Let \( L[y] = f(x) \) be a linear differential equation for \( y(x) \) with the linear homogeneous boundary conditions \( \{ B_i[y] = 0 \} \), for \( i = 1, 2, \ldots, n \). If there is a Green’s function \( G(x; z) \) that satisfies

\[
L[G(x; z)] = \delta(x - z),
\]

\[
B_i[G(x; z)] = 0,
\]

where \( \delta \) is Dirac’s delta function, then the solution of the original system can be written as \( y(x) = \int G(x; z) f(z) \, dz \), integrated over an appropriate region.

**Example**

To solve \( y'' = f(x) \) with \( y(0) = 0 \) and \( y(L) = 0 \), the appropriate Green’s function is

\[
G(x; z) = \begin{cases} 
\frac{x(z - L)}{z(x - L)} & \text{for } 0 \leq x \leq z, \\
\frac{L}{z(x - L)} & \text{for } z \leq x \leq L.
\end{cases}
\]
Hence, the solution is
\[ y(x) = \int_0^L G(x; z) f(z) \, dz = \int_0^x \frac{z(x-L)}{L} f(z) \, dz + \int_x^L \frac{x(z-L)}{L} f(z) \, dz. \]

### 5.6.6 LIST OF GREEN’S FUNCTIONS

For the following, the Green’s function is \( G(x, \xi) \) when \( x \leq \xi \) and \( G(\xi, x) \) when \( x \geq \xi \).

1. For the equation \( \frac{d^2 y}{dx^2} = f(x) \) with
   
   (a) \( y(0) = y(1) = 0 \), \( G(x, \xi) = -(1-\xi)x \),
   
   (b) \( y(0) = 0, y'(1) = 0 \), \( G(x, \xi) = -x \),
   
   (c) \( y(0) = -y(1), y'(0) = -y'(1) \), \( G(x, \xi) = -(\frac{1}{4}(x-\xi) - \frac{1}{4}) \), and
   
   (d) \( y(-1) = y(1) = 0 \), \( G(x, \xi) = -(\frac{1}{4}(x-\xi - x\xi + 1)) \).

2. For the equation \( \frac{d^2 y}{dx^2} - y = f(x) \) with \( y \) finite in \((-\infty, \infty)\),
   \( G(x, \xi) = -\frac{1}{2} e^{x-\xi} \).

3. For the equation \( \frac{d^2 y}{dx^2} + k^2 y = f(x) \) with
   
   (a) \( y(0) = y(1) = 0 \), \( G(x, \xi) = -\frac{\sin kx \sin k(1-\xi)}{k \sin k} \),
   
   (b) \( y(-1) = y(1), y'(-1) = y'(1) \), and \( G(x, \xi) = \frac{\cos k(x-\xi + 1)}{2k \sin k} \).

4. For the equation \( \frac{d^2 y}{dx^2} - k^2 y = f(x) \) with
   
   (a) \( y(0) = y(1) = 0 \), \( G(x, \xi) = -\frac{\sinh kx \sinh k(1-\xi)}{k \sinh k} \),
   
   (b) \( y(-1) = y(1), y'(-1) = y'(1) \), and \( G(x, \xi) = -\frac{\cosh k(x-\xi + 1)}{2k \sinh k} \).

5. For the equation \( \frac{d}{dx} \left( x \frac{dy}{dx} \right) = f(x) \), with \( y(0) \) finite and \( y(1) = 0 \),
   \( G(x, \xi) = \ln \xi \).

6. For the equation \( \frac{d}{dx} \left( x \frac{dy}{dx} \right) - \frac{m^2}{x} y = f(x) \), with \( y(0) \) finite and \( y(1) = 0 \),
   \( G(x, \xi) = -\frac{1}{2m} \left[ \left( \frac{\xi}{x} \right)^m - (x\xi)^m \right], (m = 1, 2, \ldots) \).

7. For the equation \( \frac{d}{dx} \left( (1-x^2) \frac{dy}{dx} \right) - \frac{m^2}{1-x^2} y = f(x) \), with \( y(-1) \) and \( y(1) \)
   finite, \( G(x, \xi) = -\frac{1}{2m} \left( \frac{1+1}{1+1} \right)^{m/2}, (m = 1, 2, \ldots) \).

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8. For the equation \( \frac{d^4y}{dx^4} = f(x) \), with \( y(0) = y'(0) = y(1) = y'(1) = 0 \),
\[ G(x, \xi) = \frac{x^2(\xi-1)^2}{6}(2x\xi + x - 3\xi). \]

5.6.7 TRANSFORM TECHNIQUES

Transforms can sometimes be used to solve linear differential equations. Laplace transforms (page 539) are appropriate for initial value problems, while Fourier transforms (page 530) are appropriate for boundary value problems.

Example

Consider the linear second order equation \( y'' + y = p(x) \), with the initial conditions \( y(0) = 0 \) and \( y'(0) = 0 \). Multiplying this equation by \( e^{-sx} \), and integrating with respect to \( x \) from 0 to \( \infty \), results in

\[ \int_0^\infty e^{-sx}y''(x) \, dx + \int_0^\infty e^{-sx}y(x) \, dx = \int_0^\infty e^{-sx}p(x) \, dx. \]

Integrating by parts, and recognizing that \( Y(s) = \mathcal{L}\{y(x)\} = \int_0^\infty e^{-sx}y(x) \, dx \) is the Laplace transform of \( y \), this simplifies to

\[ (s^2 + 1)Y(s) = \int_0^\infty e^{-sx}p(x) \, dx = \mathcal{L}\{p(x)\}. \]

If \( p(x) \equiv 1 \), then \( \mathcal{L}\{p(x)\} = s^{-1} \). The table of Laplace transforms shows that the \( y(x) \) corresponding to \( Y(s) = 1/[s(1 + s^2)] \) is \( y(x) = \mathcal{L}^{-1}[Y(s)] = 1 - \cos x \).

5.6.8 NAMED ORDINARY DIFFERENTIAL EQUATIONS

1. Airy equation: \( y'' = xy \)
   Solution: \( y = c_1 \text{Ai}(x) + c_2 \text{Bi}(x) \)

2. Bernoulli equation: \( y' = a(x)y^n + b(x)y \)

3. Bessel equation: \( x^2y'' + xy' + (\lambda^2x^2 - n^2)y = 0 \)
   Solution: \( y = c_1J_n(\lambda x) + c_2Y_n(\lambda x) \)

4. Bessel equation (transformed): \( x^2y'' + (2p + 1)xy' + (\lambda^2x^{2p} + \beta^2)y = 0 \)
   Solution: \( y = x^{-p} \left[ c_1J_{\nu}(\frac{\lambda}{r}x^p) + c_2Y_{\nu}(\frac{\lambda}{r}x^p) \right] q = \sqrt{\beta^2 - \lambda^2} \)

5. Bôcher equation: \( y'' + \frac{1}{2} \left[ \frac{m_1}{x-a_1} + \cdots + \frac{m_{n+1}}{x-a_{n+1}} \right] y' \]
   \[ + \frac{1}{4} \left[ \frac{A_0 + A_1x + \cdots + A_{n-1}x^{n-1}}{(x-a_1)^{2n-2} \cdots (x-a_{n-1})^{2n-2}} \right] y = 0 \]

6. Duffing’s equation: \( y'' + y + \epsilon y^3 = 0 \)

7. Emden–Fowler equation: \( (x^p y')' \pm x^a y^n = 0 \)

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8. Hypergeometric equation: 
\[ y'' + \left( \frac{1-a-a'}{x-a} + \frac{1-b-b'}{x-b} + \frac{1-c-c'}{x-c} \right) y' - \left( \frac{a}{(x-a)(b-c)} + \frac{b}{(x-b)(c-a)} + \frac{c}{(x-c)(a-b)} \right) y = 0 \]

Solution: 
\[ y = P \left\{ \frac{a}{\alpha}, \frac{b}{\beta}, \frac{c}{\gamma}, \frac{x}{\xi}, \frac{x'}{\eta}, \xi' \right\} \] (Riemann’s P function)

9. Legendre equation: 
\[ (1 - x^2)y'' - 2xy' + n(n + 1)y = 0 \]
Solution: 
\[ y = c_1 P_n(x) + c_2 Q_n(x) \]

10. Mathieu equation: 
\[ y'' + (a - 2q \cos 2x) y = 0 \]

11. Painlevé transcendent (first equation): 
\[ y'' = 6y^2 + x \]

12. Parabolic cylinder equation: 
\[ y'' + (ax^2 + bx + c)y = 0 \]

13. Riccati equation: 
\[ y' = a(x)y^2 + b(x)y + c(x) \]

5.6.9 LIAPUNOV’S DIRECT METHOD

If, as \( x(t) \) evolves, the function \( V \) depends on \( x \) so that \( V(x) > 0 \) and \( \frac{dV}{dt} < 0 \), then the system is asymptotically stable: \( V[x(t)] \to 0 \) as \( x \to \infty \). For example, for the nonlinear system of differential equations with \( a > 0 \)

\[ \begin{align*}
\dot{x}_1 &= -ax_1 - x_1x_2^2, \\
\dot{x}_2 &= -ax_2 + x_1^2x_2,
\end{align*} \]

define \( V(x) = x_1^2 + x_2^2 \). Since \( \dot{V} = -2a(x_1^2 + x_2^2) = -2aV, V(t) = V_0e^{-2at} \). Hence \( x_1(t) \) and \( x_2(t) \) both decay to 0.

5.6.10 LIE GROUPS

An algorithm for integrating second order ordinary differential equations is given by:

1. Determine the admitted Lie algebra \( L_r \), where \( r \) is the dimension.
2. If \( r > 2 \), determine a subalgebra \( L_2 \subset L_r \). If \( r < 2 \), then Lie groups are not useful for the given equation.
3. From the commutator and pseudoscalar product, change the basis to obtain one of the four cases in the table below.
4. Introduce canonical variables specified by the change of basis into the original differential equation. Integrate this new equation.
5. Rewrite the solution in terms of the original variables.

The invertible transformation \( \{ \bar{x} = \phi(x, y, a), \bar{y} = \psi(x, y, a) \} \) forms a one parameter group if \( \phi(\bar{x}, \bar{y}, b) = \phi(x, y, a + b) \) and \( \psi(\bar{x}, \bar{y}, b) = \psi(x, y, a + b) \). For small \( a \), these transformations become

\[ \begin{align*}
\bar{x} &= x + a\xi(x, y) + O(a^2) \\
\bar{y} &= y + a\eta(x, y) + O(a^2).
\end{align*} \] (5.6.8)
The infinitesimal generator is \( X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} \). If \( D = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y} + \ldots \), then the derivatives of the new variables are

\[
\begin{align*}
\dot{y} &= \frac{d\dot{y}}{d\xi} = \frac{D\psi}{D\phi} = \frac{\psi_x + y'\psi_y}{\phi_x + y'\phi_y} = P(x, y, y', a) = y' + a\zeta_1 + O(a^2), \\
\ddot{y} &= \frac{d\ddot{y}}{d\xi} = \frac{DP}{D\phi} = \frac{P_x + y'P_y + y''P_y}{\phi_x + y'\phi_y} = y'' + a\zeta_2 + O(a^2), \quad (5.6.9)
\end{align*}
\]

where

\[
\begin{align*}
\zeta_1 &= D(\eta) - y'D(\xi) = \eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y, \quad \text{and} \\
\zeta_2 &= D(\eta) - y''D(\xi) = \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 \\
&\quad - y'^3\xi_{yy} + (\eta_y - 2\xi_x) - 3y'\xi_y y''. \quad (5.6.10)
\end{align*}
\]

The prolongations of \( X \) are \( X^{(1)} = X + \zeta_1 \frac{\partial}{\partial y} \) and \( X^{(2)} = X^{(1)} + \zeta_2 \frac{\partial}{\partial y} \). For a given differential equation, the different infinitesimal generators will generate an \( r \)-dimensional Lie group \( (L_r) \).

For the equation \( F(x, y, y', y'') = 0 \) to be invariant under the action of the above group, \( X^{(2)}F |_{F=0} = 0 \). When \( F = y'' - f(x, y, y') \), this determining equation becomes

\[
\begin{align*}
\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - y'^3\xi_y y + (\eta_y - 2\xi_x - 3y'\xi_y) f - \left[ \eta_x + (\eta_y - \xi_x)y' - y'^2 \right] f_y \\
- \xi f_x - \eta f_y &= 0. \quad (5.6.11)
\end{align*}
\]

Given the two generators \( X_1 = \xi_1 \frac{\partial}{\partial x} + \eta_1 \frac{\partial}{\partial y} \) and \( X_2 = \xi_2 \frac{\partial}{\partial x} + \eta_2 \frac{\partial}{\partial y} \), the pseudoscalar product is \( X_1 \triangledown X_2 = \xi_1\eta_2 - \xi_2\eta_1 \), and the commutator is \([X_1, X_2] = X_1X_2 - X_2X_1 \). By a suitable choice of basis, any two-dimensional Lie algebra can be reduced to one of four types:

<table>
<thead>
<tr>
<th>No.</th>
<th>Commutator</th>
<th>Pseudoscalar</th>
<th>Typified by</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>( [X_1, X_2] = 0 )</td>
<td>( X_1 \triangledown X_2 \neq 0 )</td>
<td>( {X_1 = \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial y}} )</td>
</tr>
<tr>
<td>II</td>
<td>( [X_1, X_2] = 0 )</td>
<td>( X_1 \triangledown X_2 = 0 )</td>
<td>( {X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y}} )</td>
</tr>
<tr>
<td>III</td>
<td>( [X_1, X_2] = X_1 )</td>
<td>( X_1 \triangledown X_2 \neq 0 )</td>
<td>( {X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y} + y \frac{\partial}{\partial y}} )</td>
</tr>
<tr>
<td>IV</td>
<td>( [X_1, X_2] = X_1 )</td>
<td>( X_1 \triangledown X_2 = 0 )</td>
<td>( {X_1 = \frac{\partial}{\partial y}, \quad X_2 = y \frac{\partial}{\partial y}} )</td>
</tr>
</tbody>
</table>

### 5.6.11 TYPES OF CRITICAL POINTS

An ODE may have several types of critical points; these include improper node, deficient improper node, proper node, saddle, center, and focus. See Figure 5.6.1.
FIGURE 5.6.1
Types of critical points. Clockwise from upper left: center, improper node, deficient improper node, spiral, star, saddle.

5.6.12 STOCHASTIC DIFFERENTIAL EQUATIONS

A stochastic differential equation has the form

\[ dX(t) = a(X(t)) \, dt + b(X(t)) \, dB(t) \]  \hfill (5.6.12)

where \( B(t) \) is a random Brownian motion. Brownian motion has a Gaussian probability distribution and independent increments. The probability density function \( f_{X(t)} \) for \( X(t) \) satisfies the forward Kolmogorov equation

\[ \frac{\partial}{\partial t} f_{X(t)}(x) = \frac{1}{2} \frac{d^2}{dx^2} \left[ b^2(x) f_{X(t)}(x) \right] - \frac{d}{dx} \left[ a(x) f_{X(t)}(x) \right]. \]  \hfill (5.6.13)

The conditional expectation \( u(t, x) = \mathbb{E} [\phi(X(t)) \mid X(0) = x] \) satisfies

\[ \frac{\partial}{\partial t} u(t, x) = \frac{1}{2} b^2(x) \frac{\partial^2}{\partial x^2} u(t, x) + a(x) \frac{\partial}{\partial x} u(t, x) \text{ with } u(0, x) = \phi(x). \]  \hfill (5.6.14)
5.7 PARTIAL DIFFERENTIAL EQUATIONS

5.7.1 CLASSIFICATIONS OF PDES

Consider second order partial differential equations, with two independent variables, of the form

\[ A(x, y) \frac{\partial^2 u}{\partial x^2} + B(x, y) \frac{\partial^2 u}{\partial x \partial y} + C(x, y) \frac{\partial^2 u}{\partial y^2} = \Psi\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, x, y\right). \]  

(5.7.1)

If \[ B^2 - 4AC > 0 \]

then Equation (5.7.1) is hyperbolic at that point. If an equation is of the same type at all points, then the equation is simply of that type.

5.7.2 NAMED PARTIAL DIFFERENTIAL EQUATIONS

1. Biharmonic equation: \[ \nabla^4 u = 0 \]
2. Burgers’ equation: \[ u_t + uu_x = \nu u_{xx} \]
3. Diffusion (or heat) equation: \[ \nabla (c(x, t) \nabla u) = u_t \]
4. Hamilton–Jacobi equation: \[ V_t + H(t, x, V_x, \ldots, V_x, V_x) = 0 \]
5. Helmholtz equation: \[ \nabla^2 u + k^2 u = 0 \]
6. Korteweg de Vries equation: \[ u_t + u_{xxx} - 6u u_x = 0 \]
7. Laplace’s equation: \[ \nabla^2 u = 0 \]
8. Navier–Stokes equations: \[ u_t + (u \cdot \nabla) u = -\nabla p + \nu \nabla^2 u \]
9. Poisson equation: \[ \nabla^2 u = -4\pi \rho(x) \]
10. Schrödinger equation: \[ -\hbar^2 \nabla^2 u + V(x) u = i\hbar u_t \]
11. Sine–Gordon equation: \[ u_{xx} - u_{yy} \pm \sin u = 0 \]
12. Tricomi equation: \[ u_{yy} = y u_{xx} \]
13. Wave equation: \[ c^2 \nabla^2 u = u_{tt} \]
14. Telegraph equation: \[ u_{xx} = au_{tt} + bu_t + cu \]

5.7.3 WELL-POSEDNESS OF PDES

Partial differential equations involving \( u(x) \) usually have the following types of boundary conditions:
1. Dirichlet conditions: \( u = 0 \) on the boundary
2. Neumann conditions: \( \frac{\partial u}{\partial n} = 0 \) on the boundary
3. Cauchy conditions: \( u \) and \( \frac{\partial u}{\partial n} \) specified on the boundary

A well-posed differential equation meets these conditions:

1. The solution exists.
2. The solution is unique.
3. The solution is stable (i.e., the solution depends continuously on the boundary conditions and initial conditions).

<table>
<thead>
<tr>
<th>Type of boundary conditions</th>
<th>Type of equation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Elliptic</td>
</tr>
<tr>
<td>Dirichlet</td>
<td></td>
</tr>
<tr>
<td>Open (spacelike) surface</td>
<td>Undetermined</td>
</tr>
<tr>
<td>Closed surface</td>
<td>Unique, stable solution</td>
</tr>
<tr>
<td>Neumann</td>
<td></td>
</tr>
<tr>
<td>Open (spacelike) surface</td>
<td>Undetermined</td>
</tr>
<tr>
<td>Closed surface</td>
<td>Overdetermined</td>
</tr>
<tr>
<td>Cauchy</td>
<td></td>
</tr>
<tr>
<td>Open (spacelike) surface</td>
<td>Not physical results</td>
</tr>
<tr>
<td>Closed surface</td>
<td>Overdetermined</td>
</tr>
</tbody>
</table>

### 5.7.4 GREEN’S FUNCTIONS

In the following, \( \mathbf{r} = (x, y, z) \), \( \mathbf{r}_0 = (x_0, y_0, z_0) \), \( R^2 = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \), \( P^2 = (x-x_0)^2 + (y-y_0)^2 \).

1. For the potential equation \( \nabla^2 G + k^2 G = -4\pi \delta(\mathbf{r} - \mathbf{r}_0) \), with the radiation condition (outgoing waves only), the solution is

\[
G = \begin{cases} 
\frac{2\pi i}{k} e^{ik|x-x_0|} & \text{in one dimension,} \\
\frac{i\pi}{k} H_0^{(1)}(kP) & \text{in two dimensions, and} \\
\frac{e^{ikR}}{R} & \text{in three dimensions,}
\end{cases}
\]

where \( H_0^{(1)}(\cdot) \) is a Hankel function.

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2. For the \( N \)-dimensional diffusion equation

\[
\nabla^2 G - a^2 \frac{\partial G}{\partial t} = -4\pi \delta(r - r_0)\delta(t - t_0),
\]

with the initial condition \( G = 0 \) for \( t < t_0 \), and the boundary condition \( G = 0 \) at \( r = \infty \), the solution is

\[
G = \frac{4\pi}{a^2} \left( \frac{a}{2\sqrt{\pi(t - t_0)}} \right)^N \exp \left( -\frac{a^2 |r - r_0|^2}{4(t - t_0)} \right).
\]

3. For the wave equation

\[
\nabla^2 G - \frac{1}{c^2} \frac{\partial^2 G}{\partial t^2} = -4\pi \delta(r - r_0)\delta(t - t_0),
\]

with the initial conditions \( G = G_t = 0 \) for \( t < t_0 \), and the boundary condition \( G = 0 \) at \( r = \infty \), the solution is

\[
G = \begin{cases} 
2c\pi H \left[ (t - t_0) - \frac{|x - x_0|}{c} \right] & \text{in one space dimension,} \\
\frac{2c}{\sqrt{c^2(t - t_0)^2 - p^2}} H \left[ (t - t_0) - \frac{p}{c} \right] & \text{in two space dimensions, and} \\
\frac{1}{\pi} \delta \left[ \frac{x}{c} - (t - t_0) \right] & \text{in three space dimensions.}
\end{cases}
\]

where \( H(\cdot) \) is the Heaviside function.

### 5.7.5 QUASI-LINEAR EQUATIONS

Consider the quasi-linear differential equation for \( u(x) = u(x_1, x_2, \ldots, x_N) \),

\[
a_1(x, u)u_{x_1} + a_2(x, u)u_{x_2} + \cdots + a_N(x, u)u_{x_N} = b(x, u).
\]

Defining \( \frac{\partial x_k}{\partial s} = a_k(x, u) \), for \( k = 1, 2, \ldots, N \), the original equation becomes \( \frac{du}{dt} = b(x, u) \). To solve the original system, the ordinary differential equations for \( u(s, t) \) and the \( \{x_k(s, t)\} \) must be solved. Their initial conditions can often be parameterized as (with \( t = (t_1, \ldots, t_{N-1}) \))

\[
\begin{align*}
    u(s = 0, t) & = v(t), \\
x_1(s = 0, t) & = h_1(t), \\
x_2(s = 0, t) & = h_2(t), \\
    \vdots \\
x_N(s = 0, t) & = h_N(t),
\end{align*}
\]

from which the solution follows. This results in an implicit solution.
Example
For the equation \( u_x + x^2 u_y = -yu \) with \( u = f(y) \) when \( x = 0 \), the corresponding equations are
\[
\frac{\partial x}{\partial s} = 1, \quad \frac{\partial y}{\partial s} = x^2, \quad \frac{du}{ds} = -yu.
\]
The original initial data can be written parametrically as \( x(s = 0, t) = 0, y(s = 0, t_1) = t_1 \), and \( u(s = 0, t_1) = f(t_1) \). Solving for \( x \) results in \( u(s, t_1) = \frac{t_1^4}{16} + t_1 \). Finally, the equation for \( u \) is integrated to obtain \( u(s, t_1) = f(t_1) \exp\left(\frac{-s^4}{16} - st_1\right) \). These solutions constitute an implicit solution of the original system.

In this case, it is possible to eliminate the \( s \) and \( t_1 \) variables analytically to obtain the explicit solution: \( u(x, y) = f\left(y - \frac{x^3}{3}\right) \exp\left(\frac{x^4}{4} - xy\right) \).

5.7.6 EXACT SOLUTIONS OF LAPLACE’S EQUATION

1. If \( \nabla^2 u = 0 \) in a circle of radius \( R \) and \( u(R, \theta) = f(\theta) \), for \( 0 \leq \theta < 2\pi \), then
   \[
   u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr \cos(\theta - \phi)} f(\phi) \, d\phi.
   \]
2. If \( \nabla^2 u = 0 \) in a sphere of radius one and \( u(1, \theta, \phi) = f(\theta, \phi) \), then
   \[
   u(r, \theta, \phi) = \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} f(\sigma \gamma, \sigma \omega) (1 - 2r \cos \gamma + r^2) \sin \Theta d\Theta d\Phi,
   \]
   where \( \cos \gamma = \cos \theta \cos \sigma \gamma + \sin \theta \sin \sigma \gamma \cos(\phi - \Phi) \).
3. If \( \nabla^2 u = 0 \) in the half plane \( y \geq 0 \), and \( u(x, 0) = f(x) \), then
   \[
   u(x, y) = \frac{1}{\pi} \int_{-\infty}^\infty \frac{f(t) y}{(x - t)^2 + y^2} \, dt.
   \]
4. If \( \nabla^2 u = 0 \) in the half space \( z \geq 0 \), and \( u(x, y, 0) = f(x, y) \), then
   \[
   u(x, y, z) = \frac{z}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{f(\zeta, \eta)}{[(x - \zeta)^2 + (y - \eta)^2 + z^2]^{3/2}} \, d\zeta \, d\eta.
   \]

5.7.7 SOLUTIONS TO THE WAVE EQUATION

Consider the wave equation \( \frac{\partial^2 u}{\partial t^2} = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \cdots + \frac{\partial^2 u}{\partial x^n} \), with \( x = (x_1, \ldots, x_n) \) and the initial data \( u(0, \mathbf{x}) = f(\mathbf{x}) \) and \( u_t(0, \mathbf{x}) = g(\mathbf{x}) \). When \( n \) is odd (and \( n \geq 3 \)), the solution is
\[
\begin{aligned}
u(t, \mathbf{x}) &= \frac{1}{1 \cdot 3 \cdots (n - 2)} \left\{ \frac{\partial}{\partial t} \left( \frac{\partial}{\partial t} \right)^{(n-3)/2} t^{n-2} \omega[f; \mathbf{x}, t] + \left( \frac{\partial}{\partial t} \right)^{(n-3)/2} t^{n-2} \omega[g; \mathbf{x}, t] \right\},
\end{aligned}
\]
where \( \omega[h; x, t] \) is the average of the function \( h(x) \) over the surface of an \( n \)-dimensional sphere of radius \( t \) centered at \( x \); that is, \( \omega[h; x, t] = \frac{1}{\sigma_{n-1}(t)} \int h(0, \xi) \, d\Omega \), where \( |\xi - x|^2 = t^2 \), \( \sigma_{n-1}(t) \) is the surface area of the \( n \)-dimensional sphere of radius \( t \), and \( d\Omega \) is an element of area.

When \( n \) is even, the solution is given by

\[
\begin{align*}
\frac{\partial}{\partial t} (\frac{\partial}{\partial t})^{(n-2)/2} \frac{1}{2} & \int_{t-\epsilon}^{t+\epsilon} \omega[h; x, \rho] \frac{\rho^{n-1}}{\sqrt{t^2 - \rho^2}} \, d\rho \\
\end{align*}
\]

where \( \omega[h; x, t] \) is defined as above. Since this expression is integrated over \( \rho \), the values of \( f \) and \( g \) must be known everywhere in the interior of the \( n \)-dimensional sphere.

Using \( u_n \) for the solution in \( n \) dimensions, the above simplify to

\[
\begin{align*}
\frac{1}{2} \left[ f(x - t) + f(x + t) \right] + \frac{1}{2} \int_{x-t}^{x+t} g(\xi) \, d\xi, \\
\frac{1}{2} \left[ f(x + ct) - f(ct - x) \right] + \frac{1}{2} \int_{ct-x}^{ct+x} g(\xi) \, d\xi,
\end{align*}
\]

where \( R(t) \) is the region \( \{(\zeta_1, \zeta_2) \mid \zeta_1^2 + \zeta_2^2 \leq t^2\} \) and

\[
\omega[h; x, t] = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi h(x_1 + t \sin \theta \cos \phi, x_2 + t \sin \theta \sin \phi, x_3 + t \cos \phi) \times \sin \theta \, d\theta \, d\phi.
\]

- The solution of the wave equation

\[
\begin{align*}
v_{tt} &= c^2 v_{xx}, \\
v(0, t) &= 0, \quad \text{for } 0 < t < \infty, \\
v(x, 0) &= f(x), \quad \text{for } 0 \leq x < \infty, \\
v_t(x, 0) &= g(x), \quad \text{for } 0 \leq x < \infty,
\end{align*}
\]

is

\[
\begin{align*}
v(x, t) &= \begin{cases} \frac{1}{2} \left[ f(x + ct) + f(x - ct) \right] + \frac{1}{2} \int_{ct-x}^{ct-x} g(\xi) \, d\xi, & \text{for } x \geq ct, \\
\frac{1}{2} \left[ f(x + ct) - f(ct - x) \right] + \frac{1}{2} \int_{ct-x}^{ct-x} g(\xi) \, d\xi, & \text{for } x < ct.
\end{cases}
\end{align*}
\]

- The solution of the inhomogeneous wave equation

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} = F(t, x, y, z),
\end{align*}
\]

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with the initial conditions \( u(0, x, y, z) = 0 \) and \( u_t(0, x, y, z) = 0 \), is

\[
u(t, x, y, z) = \frac{1}{4\pi} \iiint_{\rho \leq t} \frac{F(t - \rho, \eta, \xi)}{\rho} \, d\eta \, d\xi,
\]

with \( \rho = \sqrt{(x - \xi)^2 + (y - \eta)^2 + (z - \xi)^2} \).

### 5.7.8 SEPARATION OF VARIABLES

A solution of a linear PDE is attempted in the form \( u(x) = u(x_1, x_2, \ldots, x_n) = X_1(x_1)X_2(x_2)\ldots X_n(x_n) \). Logical reasoning may determine the \( \{X_i\} \).

- For example, the diffusion or heat equation in a circle is

\[
\frac{\partial u}{\partial t} = \nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}
\]

for \( u(t, r, \theta) \). If \( u(t, r, \theta) = T(t)R(r)\Theta(\theta) \), then

\[
\frac{1}{T} \frac{dT}{dt} \left( \frac{dR}{dr} \right) + \frac{1}{R} \frac{d^2 R}{d\theta^2} - \frac{1}{T} \frac{dT}{dt} = 0.
\]

Logical reasoning leads to

\[
\frac{1}{T} \frac{dT}{dt} = -\lambda, \quad \frac{1}{R} \frac{d^2 R}{d\theta^2} = -\rho, \quad \frac{r}{R} \frac{d}{dr} \left( r \frac{dR}{dr} \right) + (-\rho + r^2\lambda) R = 0.
\]

where \( \lambda \) and \( \rho \) are unknown constants. Solving these ordinary differential equations yields the general solution,

\[
u(t, r, \theta) = \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\rho \ e^{-\lambda t} \left[ B(\lambda, \rho) \sin(\sqrt{\rho}\theta) + C(\lambda, \rho) \cos(\sqrt{\rho}\theta) \right] \]

\[
\times \left[ D(\lambda, \rho) J_{\sqrt{\rho}}(\sqrt{\lambda} r) + E(\lambda, \rho) Y_{\sqrt{\rho}}(\sqrt{\lambda} r) \right].
\]

Boundary conditions are required to determine \( \{B, C, D, E\} \).

- A necessary and sufficient condition for a system with Hamiltonian \( H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y) \), to be separable in elliptic, polar, parabolic, or Cartesian coordinates is that the expression,

\[
(V_{yy} - V_{xx})(-2axy - b'y - bx + d) + 2V_{xy} (ay^2 - ax^2 + by - bx' + c - c')
\]

\[
+ V_x (6ay + 3b) + V_y (-6ax - 3b')
\]

vanishes for some constants \( (a, b, b', c, c', d) \neq (0, 0, 0, c, c, 0) \).

- Consider the orthogonal coordinate system \( \{u^1, u^2, u^3\} \), with the metric \( \{g_{ij}\} \), and \( g = g_{11}g_{22}g_{33} \). The Stäckel matrix is defined as
\[
S = \begin{bmatrix}
\Phi_{11}(u^1) & \Phi_{12}(u^1) & \Phi_{13}(u^1) \\
\Phi_{21}(u^2) & \Phi_{22}(u^2) & \Phi_{23}(u^2) \\
\Phi_{31}(u^3) & \Phi_{32}(u^3) & \Phi_{33}(u^3)
\end{bmatrix},
\]

where the \{\Phi_{ij}\} are specified. The determinant of \(S\) can be written as
\[
s = \frac{1}{g_{ii}} \frac{1}{g_{jj}} (M_{11} + \Phi_{12}M_{21} + \Phi_{13}M_{31}),
\]
where
\[
M_{11} = \begin{bmatrix}
\Phi_{22} & \Phi_{23} \\
\Phi_{32} & \Phi_{33}
\end{bmatrix}, \quad M_{21} = \begin{bmatrix}
\Phi_{12} & \Phi_{13} \\
\Phi_{32} & \Phi_{33}
\end{bmatrix}, \quad M_{31} = \begin{bmatrix}
\Phi_{12} & \Phi_{13} \\
\Phi_{22} & \Phi_{23}
\end{bmatrix}.
\]

If \(g_{ii} = s/M_{11}\) and \(\sqrt{g}/s = f_1(u^1)f_2(u^2)f_3(u^3)\) then the Helmholtz equation \(\nabla^2 W + \lambda^2 W = 0\) separates with \(W = X_1(u^1)X_2(u^2)X_3(u^3)\). Here the \{\Phi_i\} are defined by
\[
\frac{1}{f_i} \frac{d}{du^i} \left( f_i \frac{dX_i}{du^i} \right) + X_i \sum_{j=1}^3 \alpha_j \Phi_{ij} = 0,
\]
with \(\alpha_1 = \lambda^2\), and \(\alpha_2\) and \(\alpha_3\) arbitrary. For example, in parabolic coordinates \(\{\mu, \nu, \psi\}\) the metric coefficients are \(g_{11} = g_{22} = \mu^2 + \nu^2\) and \(g_{33} = \mu^2\nu^2\). Hence, \(\sqrt{g} = \mu\nu(\mu^2 + \nu^2)\). For the Stäckel matrix
\[
S = \begin{bmatrix}
\mu^2 & -1 & 1/\mu^2 \\
\nu^2 & 1 & 1/\nu^2 \\
0 & 0 & 1
\end{bmatrix}
\]
(for which \(s = \mu^2 + \nu^2\), \(M_{11} = M_{21} = 1\), and \(M_{21} = \mu^{-2} + \nu^{-2}\)), the separation condition holds with \(f_1 = \mu\), \(f_2 = f_3 = 1\). Hence, the Helmholtz equation separates in parabolic coordinates. The separated equations are
\[
\begin{align*}
\frac{1}{\mu} \frac{d}{d\mu} \left( \mu \frac{dX_1}{d\mu} \right) + X_1 \left( \alpha_1 \mu^2 - \alpha_2 + \frac{\alpha_1}{\mu^2} \right) &= 0, \\
\frac{1}{\nu} \frac{d}{d\nu} \left( \nu \frac{dX_2}{d\nu} \right) + X_2 \left( \alpha_1 \nu^2 + \alpha_2 + \frac{\alpha_1}{\nu^2} \right) &= 0, \\
\frac{d^2X_3}{d\psi^2} + \alpha_3 X_3 &= 0,
\end{align*}
\]
where \(W = X_1(\mu)X_2(\nu)X_3(\psi)\).

- **Necessary and sufficient conditions for separation of the Laplace equation** \((\nabla^2 W = 0)\) are
\[
\frac{g_{ii}}{g_{jj}} = \frac{M_{11}}{M_{11}} \quad \text{and} \quad \frac{\sqrt{g}}{g_{ii}} = f_1(u^1)f_2(u^2)f_3(u^3)M_{11}.
\]

**Particular solutions**

In these tables, we assume that \(P(x)\) is a polynomial of degree \(n\).
If \( R(x) \) is a particular solution to \( z_x + m z_y = R(x, y) \) is

\[
1. e^{ax+by} \quad e^{ax+by}/(a + mb).
2. f(ax + by) \quad \int f(y) \, du/(a + mb), \quad u = ax + by.
3. f(y - mx) \quad xf(y - mx).
4. \phi(x, y) f(y - mx) \quad f(y - mx) \int \phi(x, a + mx) \, dx; \text{ then substitute } a = y - mx.
\]

If \( R(x) \) is a particular solution to \( z_x + m z_y - kz = R(x, y) \) is

\[
5. e^{ax+by} \quad e^{ax+by}/(a + mb - k).
6. \sin(ax + by) \quad -(a + bm) \cos(ax + by) + k \sin(ax + by)/(a + bm)^2 + k^2.
7. e^{ax+by} \sin(ax + by) \quad \text{Replace } k \text{ by } k - \alpha - m\beta \text{ in formula 6 and multiply by } e^{ax+by}.
8. e^{kx} f(ax + by) \quad e^{kx} \int f(y) \, du/(a + mb), \quad u = ax + by.
9. f(y - mx) \quad -f(y - mx)/k.
10. P(x) f(y - mx) \quad -1/k f(y - mx) \left[ P(x) + \frac{P'(x)}{k} + \frac{P''(x)}{2k^2} + \cdots + \frac{P^n(x)}{k^n} \right].
11. e^{kx} f(y - mx) \quad xe^{kx} f(y - mx).
\]

### 5.7.9 TRANSFORMING PARTIAL DIFFERENTIAL EQUATIONS

To transform a partial differential equation, construct a new function which depends upon new variables, and then differentiate with respect to the old variables to see how the derivatives transform. Consider transforming

\[ f_{xx} + f_{yy} + xf_x = 0, \quad (5.7.7) \]

from the \( \{x, y\} \) variables to the \( \{u, v\} \) variables, where \( \{u = x, v = x/y\} \). Note that the inverse transformation is given by \( \{x = u, y = u/v\} \).

First, define \( g(u, v) \) as the function \( f(x, y) \) when written in the new variables, that is

\[ f(x, y) = g(u, v) = g \left( x, \frac{x}{y} \right). \quad (5.7.8) \]

Now create the needed derivative terms, carefully applying the chain rule. For example, differentiating Equation (5.7.8) with respect to \( x \) results in

\[
f_x(x, y) = g_u \frac{\partial}{\partial x} (u) + g_v \frac{\partial}{\partial x} (v) = g_1 \frac{\partial}{\partial x} (x) + g_2 \frac{\partial}{\partial x} \left( \frac{x}{y} \right) \\
= g_1 + g_2 \frac{1}{y} = g_1 + \frac{v}{u} g_2.
\]

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where a subscript of “1” (“2”) indicates a derivative with respect to the first (second) argument of the function $g(u, v)$, that is, $g_1(u, v) = g_u(u, v)$. Use of this “slot notation” tends to minimize errors. In like manner

$$f_y(x, y) = g_x \frac{\partial}{\partial y} (u) + g_v \frac{\partial}{\partial y} (v) = g_1 \frac{\partial}{\partial y} (x) + g_2 \frac{\partial}{\partial y} \left( \frac{x}{y} \right)$$

$$= -\frac{x}{y^2} g_2 = -u \frac{v^2}{u} g_2.$$

The second order derivatives can be calculated similarly:

$$f_{xx}(x, y) = \frac{\partial}{\partial x} \left( f_x(x, y) \right) = \frac{\partial}{\partial x} \left( g_1 + \frac{1}{y} g_2 \right),$$

$$= g_{11} + \frac{2v}{u} g_{12} + \frac{v^2}{u^2} g_{22},$$

$$f_{xy}(x, y) = \frac{\partial}{\partial x} \left( -\frac{x}{y^2} g_2 \right) = -\frac{u^2}{v^2} g_2 - \frac{u^3}{v^3} g_{12} - \frac{u^2}{v^2} g_{22}, \quad \text{and}$$

$$f_{yy}(x, y) = \frac{\partial}{\partial y} \left( -\frac{x}{y^2} g_2 \right) = \frac{2v^3}{u^2} g_2 + \frac{v^4}{u^2} g_{22}.$$

Finally, Equation (5.7.7) in the new variables has the form,

$$0 = f_{xx} + f_{yy} + xf_y,$$

$$= \left( g_{11} + \frac{2v}{u} g_{12} + \frac{v^2}{u^2} g_{22} \right) + \left( \frac{2v^3}{u^2} g_2 + \frac{v^4}{u^2} g_{22} \right) + \left( \frac{u^2}{v} g_2 \right),$$

$$= \frac{v^2 (2v - u^2)}{u^2} g_v + \frac{2v}{u} g_{uv} + \frac{2v}{u} g_{uv} + \frac{v^2 (1 + v^2)}{u^2} g_{vv}.$$

### 5.8 EIGENVALUES

The eigenvalues $\{\lambda_i\}$ of the differential operator $L$ are the solutions of $L[u_i] = \lambda_i u_i$. Given a geometric shape, the eigenvalues of the Dirichlet problem are called eigenfrequencies (that is, $\nabla^2 u = \lambda u$ with $u = 0$ on the boundary).

- You cannot hear the shape of a drum. The following figures have the same eigenfrequencies.
You cannot hear the shape of a two-piece band. The following pairs of figures have the same eigenfrequencies.

\[ \begin{align*}
1 & \quad \sqrt{8} \\
1 & \quad 2
\end{align*} \quad \begin{align*}
2 & \quad \sqrt{2} \\
2 & \quad 2
\end{align*} \]

5.9 INTEGRAL EQUATIONS

5.9.1 DEFINITIONS

\[ h(x)u(x) = f(x) + \lambda \int_{a}^{b(x)} k(x, t)G[u(t); t] \, dt. \]  (5.9.1)

- $k(x, t)$ kernel
- $u(x)$ function to be determined
- $h(x)$, $f(x)$ given functions
- $\lambda$ eigenvalue

Classification of integral equations

- **Linear** \( G[u(x); x] = u(x) \).
- **Volterra** \( b(x) = x \).
- **Fredholm** \( b(x) = b \).
- **First kind** \( h(x) = 0 \).
- **Second kind** \( h(x) = 1 \).
- **Third kind** \( h(x) \neq 0, 1 \).
- **Homogeneous** \( f(x) = 0 \).
- **Singular** \( a = -\infty, b = \infty \).
Classification of kernels

Symmetric \[ k(x, t) = k(t, x). \]
Hermitian \[ k(x, t) = k(t, x). \]
Separable/degenerate \[ k(x, t) = \sum_{i=1}^{n} a_i(x) b_i(t), \text{ n} < \infty. \]
Difference \[ k(x, t) = k(x - t). \]
Cauchy \[ k(x, t) = \frac{1}{x - t}. \]
Singular \[ k(x, t) \to \infty \text{ as } t \to x. \]
Hilbert–Schmidt \[ \int_{a}^{b} \int_{a}^{b} |k(x, t)|^2 \, dx \, dt < \infty. \]

5.9.2 CONNECTION TO DIFFERENTIAL EQUATIONS

The initial value problem
\[ u''(x) + A(x)u'(x) + B(x)u(x) = g(x), \quad x > a, \]
\[ u(a) = c_1, \quad u'(a) = c_2, \quad (5.9.2) \]
is equivalent to the Volterra integral equation,
\[ u(x) = f(x) + \int_{a}^{x} k(x, t)u(t) \, dt, \quad x \geq a, \]
\[ f(x) = \int_{a}^{x} (x - t)g(t) \, dt + (x - a)[A(a)c_1 + c_2] + c_1, \]
\[ k(x, t) = (t - x)[B(t) - A'(t)] - A(t). \quad (5.9.3) \]

The boundary value problem
\[ u''(x) + A(x)u'(x) + B(x)u(x) = g(x), \quad a < x < b, \]
\[ u(a) = c_1, \quad u(b) = c_2, \quad (5.9.4) \]
is equivalent to the Fredholm integral equation
\[ u(x) = f(x) + \int_{a}^{b} k(x, t)u(t) \, dt, \quad a \leq x \leq b, \]
\[ f(x) = c_1 + \int_{a}^{x} (x - t)g(t) \, dt + \frac{x - a}{b - a} \left[ c_2 - c_1 - \int_{a}^{b} (b - t)g(t) \, dt \right], \]
\[ k(x, t) = \begin{cases} \frac{x - t}{b - a} [A(t) - (a - t)[A'(t) - B(t)]], & x > t, \\ \frac{x - t}{b - a} [A(t) - (b - t)[A'(t) - B(t)]], & x < t. \end{cases} \quad (5.9.5) \]

5.9.3 FREDHOLM ALTERNATIVE

For \[ u(x) = f(x) + \lambda \int_{a}^{b} k(x, t)u(t) \, dt \text{ with } \lambda \neq 0, \text{ consider the solutions to } u_H(x) = \lambda \int_{a}^{b} k(x, t)u_H(t) \, dt. \]

- If the only solution is \[ u_H(x) = 0, \text{ then there is a unique solution } u(x). \]
• If \( u_H(x) \neq 0 \), then there is no solution unless \( \int_a^b u_H(t) f(t) \, dt = 0 \) for all \( u_H(x) \), so that \( u_H^*(x) = \lambda \int_a^b k(t, x) u_H^*(t) \, dt \). In this case, there are infinitely many solutions.

### 5.9.4 SPECIAL EQUATIONS WITH SOLUTIONS

1. **Generalized Abel equation:**

   \[
   \int_0^x \frac{u(t) \, dt}{[h(x) - h(t)]^a} = f(x), \\
   u(x) = \frac{\sin(\alpha \pi)}{\pi} \frac{d}{dx} \int_0^x \frac{h'(t) f(t) \, dt}{[h(x) - h(t)]^{1-a}}
   \]

   where \( 0 \leq x \leq 1, 0 \leq a < 1, 0 \leq h(x) \leq 1, h'(x) > 0, \) and \( h'(x) \) is continuous.

2. **Cauchy equation:**

   \[
   \mu u(x) = f(x) + \int_0^1 \frac{u(t) \, dt}{t - x}, \\
   u(x) = \begin{cases} 
   \frac{x^\gamma \sin(\pi \gamma)}{\pi^2} \int_1^x \frac{ds}{(x-s)^\gamma} \int_0^1 \frac{f(t) \, dt}{(t-s)^\gamma} & \mu < 0, \\
   \frac{(1-x)^\gamma \sin(\pi \gamma)}{\pi^2} \int_0^x \frac{ds}{(x-s)^\gamma} \int_0^x \frac{f(t) \, dt}{(t-s)^\gamma} & \mu > 0,
   \end{cases}
   \]

   where \( 0 < x < 1, \mu \) is real, \( \mu \neq 0, |\mu| = \pi \cot(\pi \gamma), 0 < \gamma < 1/2, \) and the integral is a Cauchy principal value integral.

3. **Volterra equation with difference kernel:**

   \[
   u(x) = f(x) + \lambda \int_0^x k(x-t) u(t) \, dt, \quad x \geq 0 \\
   u(x) = L^{-1} \left[ \frac{F(s)}{1 - \lambda K(s)} \right],
   \]

   where \( L[f(x)] = F(s) \) and \( L[k(x)] = K(s) \) (see Section 6.26).
4. Fredholm equation with difference kernel:

\[ u(x) = f(x) + \lambda \int_{-\infty}^{\infty} k(x-t)u(t) \, dt, \]

\[ u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\alpha x} \frac{\mathcal{F}(\alpha)}{1 - \lambda K(\alpha)} \, d\alpha \]  

(5.9.9)

where \(-\infty < x < \infty, \mathcal{F} < [f(x)]= \mathcal{F}(\alpha)\) and \(\mathcal{F}[k(x)]= K(\alpha)\) (see Sections 5.9.3 and 6.22).

5. Fredholm equation with separable kernel:

\[ u(x) = f(x) + \lambda \sum_{k=1}^{n} a_k(x)b_k(t)u(t) \, dt, \]

\[ u(x) = f(x) + \lambda \sum_{k=1}^{n} c_k a_k(x), \text{ with} \]

\[ c_m = \int_{a}^{b} b_m(t) f(t) dt + \lambda \sum_{k=1}^{n} c_k \int_{a}^{b} b_m(t)a_k(t) \, dt \]  

(5.9.10)

where \(a \leq x \leq b, n < \infty, \) and \(m = 1, 2, \ldots n \) (see Section 5.9.3).

6. Fredholm equation with symmetric kernel:

\[ u(x) = f(x) + \lambda \int_{a}^{b} k(x,t)u(t) \, dt. \]

Solve \(u_n(x) = \lambda_n \int_{a}^{b} k(x,t)u_n(t) \, dt\) for \(\{u_n, \lambda_n\}.\)

For \(\lambda \neq \lambda_n, \)

\[ u(x) = f(x) + \lambda \sum_{n=1}^{\infty} \frac{u_n(x) \int_{a}^{b} f(t)u_n(t) \, dt}{(\lambda_n - \lambda) \int_{a}^{b} u_n^2(t) \, dt}. \]

(5.9.11)

For \(\lambda = \lambda_n\) and \(\int_{a}^{b} f(t)u_m(t) \, dt = 0,\)

\[ u(x) = f(x) + cu_m(x) + \lambda_m \sum_{n \neq m}^{\infty} \frac{u_n(x) \int_{a}^{b} f(t)u_n(t) \, dt}{(\lambda_n - \lambda_m) \int_{a}^{b} u_n^2(t) \, dt}, \]

where \(a \leq x \leq b,\) and \(k(x,t) = k(t,x)\) (see Section 5.9.3).
7. Volterra equation of second kind:

\[ u(x) = f(x) + \int_a^x k(x, t)u(t) dt, \]

\[ u(x) = f(x) + \lambda \int_a^x \sum_{n=0}^\infty \lambda^n k_{n+1}(x, t) f(t) dt, \]

\[ k_1(x, t) = k(x, t), \]

and \( k_{n+1}(x, t) = \int_t^x k(x, s)k_n(s, t) ds \)  \hspace{1cm} (5.9.12)

where \( k(x, t) \) and \( f(x) \) are continuous, \( \lambda \neq 0 \), and \( x \geq a \).

8. Fredholm equation of second kind: resolvent kernel:

\[ u(x) = f(x) + \lambda \int_a^b k(x, t)u(t) dt, \]

\[ u(x) = f(x) + \lambda \int_a^b \frac{D(x, t; \lambda)}{D(\lambda)} f(t) dt, \]

\[ D(\lambda) = \sum_{n=0}^\infty \frac{(-\lambda)^n c_n}{n!}, \]

\[ c_0 = 1, \quad c_n = \int_a^b A_{n-1}(t, t) dt, \quad n = 1, 2, \ldots, \]  \hspace{1cm} (5.9.13)

\[ D(x, t; \lambda) = k(x, t) + \sum_{n=1}^\infty \frac{(-\lambda)^n}{n!} A_n(x, t), \]

\[ A_0(x, t) = k(x, t), \quad \text{and} \quad A_n(x, t) = c_n k(x, t) - n \int_a^b k(x, s)A_{n-1}(s, t) ds \]

where \( k(x, t) \) and \( f(x) \) are continuous, \( \lambda \neq 0 \), \( a \leq x \leq b \), and \( D(\lambda) \neq 0 \) (see Section 5.9.3).

9. Fredholm equation of second kind (Neumann series):

\[ u(x) = f(x) + \lambda \int_a^b k(x, t)u(t) dt, \]

\[ u(x) = f(x) + \sum_{n=1}^\infty \lambda^n \phi_n(x), \]

\[ \phi_n(x) = \int_a^b k_n(x, s)f(s) ds, \]  \hspace{1cm} (5.9.14)

\[ k_1(x, s) = k(x, s), \]

and \( k_n(x, s) = \int_a^b k(x, t)k_{n-1}(t, s) dt, \quad n = 2, 3, \ldots, \)
where \( |\lambda| < \left( \int_a^b \int_a^b k^2(x, t) \, dx \, dt \right)^{-1/2} \), \( \lambda \neq 0 \), and \( a \leq x \leq b \) (see Section 5.9.3).

## 5.10 TENSOR ANALYSIS

### 5.10.1 DEFINITIONS

1. An \( n \)-dimensional coordinate manifold of class \( C^k \), \( k \geq 1 \), is a point set \( M \) together with the totality of allowable coordinate systems on \( M \). An allowable coordinate system \( (\phi, U) \) on \( M \) is a one-to-one mapping \( \phi : U \to M \), where \( U \) is an open subset of \( \mathbb{R}^n \). The \( n \)-tuple \( (x^1, \ldots, x^n) \in U \) give the coordinates of the corresponding point \( \phi(x^1, \ldots, x^n) \in M \). If \( (\tilde{\phi}, \tilde{U}) \) is a second coordinate system on \( M \), then the one-to-one correspondence \( \tilde{\phi}^{-1} \circ \phi : U \to \tilde{U} \), called a coordinate transformation on \( M \), is assumed to be of class \( C^k \). It may be written as
   \[
   \tilde{x}^i = \tilde{f}^i(x^1, \ldots, x^n), \quad i = 1, \ldots, n, \tag{5.10.1}
   \]
   where the \( \tilde{f} \) are defined by \( (\tilde{\phi}^{-1} \circ \phi)(x^1, \ldots, x^n) = (\tilde{f}^1(x^1, \ldots, x^n), \ldots, \tilde{f}^n(x^1, \ldots, x^n)) \). The coordinate transformation \( \tilde{\phi}^{-1} \circ \phi \) has inverse \( \phi^{-1} \circ \tilde{\phi} \), expressible in terms of the coordinates as
   \[
   x^i = f^i(\tilde{x}^1, \ldots, \tilde{x}^n), \quad i = 1, \ldots, n. \tag{5.10.2}
   \]

2. The Jacobian matrix \( \frac{\partial \tilde{x}^i}{\partial x^j} \) of the transformation satisfies \( \frac{\partial \tilde{x}^i}{\partial x^k} \frac{\partial x^k}{\partial x^j} = \delta^i_j \) and \( \frac{\partial \tilde{x}^i}{\partial x^k} \frac{\partial \tilde{x}^k}{\partial x^j} = \delta^i_j \), where a repeated upper and lower index signifies summation over the range \( k = 1, \ldots, n \) (the Einstein summation convention) and \( \delta^i_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \) denotes the Kronecker delta. Note also that \( \det \left( \frac{\partial \tilde{x}^i}{\partial x^j} \right) \neq 0 \).

3. A function \( F : M \to \mathbb{R} \) is called a scalar invariant on \( M \). The coordinate representation of \( F \) in any coordinate system \( (\phi, U) \) is defined by \( f := F \circ \phi \). The coordinate representations \( \tilde{f} \) of \( F \) with respect to a second coordinate system \( (\tilde{\phi}, \tilde{U}) \) is related to \( f \) by \( \tilde{f}(\tilde{x}^1, \ldots, \tilde{x}^n) = f(f^1(\tilde{x}^1, \ldots, \tilde{x}^n), \ldots, f^n(\tilde{x}^1, \ldots, \tilde{x}^n)) \).

4. A parameterized curve on \( M \) is a mapping \( \gamma : I \to M \), where \( I \subset \mathbb{R} \) is some interval. The coordinate representation of \( \gamma \) in any coordinate system \( (\phi, U) \) is defined by \( \gamma = \phi^{-1} \circ \gamma \). The mapping \( \gamma \) defines a parameterized curve in \( \mathbb{R}^n \). The component functions of \( \gamma \) denoted by \( \gamma^i \) (for \( i = 1, \ldots, n \)) are defined by \( \gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t)) \). The curve \( \gamma \) is \( C^k \) if, and only if, the functions \( \gamma^i \) are \( C^k \) for every coordinate system on \( M \).
5.10.2 ALGEBRAIC TENSOR OPERATIONS

1. Permutation of indices: The components of the sum of the tensors $T_1$ and $T_2$ of type $(r, s)$ are given by

$$T^i_{j_1 \cdots j_r} = T^i_{h_1 \cdots h_r} + T^i_{h_1 \cdots h_r},$$

(5.10.3)

2. Multiplication: The components of the tensor or outer product of a tensor $T_1$ of type $(r, s)$ and a tensor $T_2$ of type $(t, u)$ are given by

$$T^i_{j_1 \cdots j_r, k_1 \cdots k_t} = T^r_{i_1 \cdots i_r} T^s_{j_1 \cdots j_t} T^{h_1 \cdots h_s},$$

(5.10.4)

3. Contraction: The components of the contraction of the $t$th contravariant index with the $r$th covariant index of a tensor $T$ of type $(r, s)$, with $rs \geq 1$, is given by

$$T^i_{j_1 \cdots j_r, k_1 \cdots k_r} = T^i_{j_1 \cdots j_r} T^h_{k_1 \cdots k_r}.$$

4. Permutation of indices: Let $T$ be any tensor of type $(0, r)$ and $S_r$ the group of permutations of the set $\{1, \cdots, r\}$. The components of the tensor, obtained by permuting the indices of $T$ with any $\sigma \in S_r$, are given by $(\sigma T)_{i_1 \cdots i_r} = T_{(\sigma)_{i_1 \cdots i_r}}$. The symmetric part of $T$, denoted by $S(T)$, is the tensor whose components are given by

$$S(T)_{i_1 \cdots i_r} = T_{(i_1 \cdots i_r)} = \frac{1}{r!} \sum_{\sigma \in S_r} T_{(\sigma)_{i_1 \cdots i_r}}.$$  

(5.10.5)

The tensor $T$ is said to be symmetric if, and only if, $T_{i_1 \cdots i_r} = T_{(i_1 \cdots i_r)}$. The skew symmetric part of $T$, denoted by $A(T)$, is the tensor whose components are given by

$$A(T)_{i_1 \cdots i_r} = T_{(i_1 \cdots i_r)} = \frac{1}{r!} \sum_{\sigma \in S_r} \text{sgn}(\sigma) T_{(\sigma)_{i_1 \cdots i_r}}.$$  

(5.10.6)
where \( \text{sgn}(\sigma) = \pm 1 \) according to whether \( \sigma \) is an even or odd permutation. The tensor \( T \) is said to be skew symmetric if, and only if, \( T_{\alpha \beta} = -T_{\beta \alpha} \). If \( r = 2 \), \( S(T)_{\alpha \beta} = \frac{1}{2}(T_{\alpha \beta} + T_{\beta \alpha}) \) and \( A(T)_{\alpha \beta} = \frac{1}{2}(T_{\alpha \beta} - T_{\beta \alpha}) \).

### 5.10.3 DIFFERENTIATION OF TENSORS

1. A linear connection \( \nabla \) at \( p \in M \) is an object which, with respect to each coordinate system on \( M \), is represented by \( n^3 \) real numbers \( \Gamma^{i}_{\ell k} \), called the connection coefficients, whose values in any two coordinate systems \( \phi \) and \( \tilde{\phi} \) are related by

\[
\tilde{\Gamma}^{i}_{\ell k} = \Gamma^{i}_{\ell m} \frac{\partial \tilde{x}^{m}}{\partial x^{\ell}} \frac{\partial x^{\ell}}{\partial \tilde{x}^{k}} + \frac{\partial^{2} x^{\ell}}{\partial \tilde{x}^{i} \partial \tilde{x}^{k}} \frac{\partial \tilde{x}^{i}}{\partial x^{\ell}}. \tag{5.10.7}
\]

The quantities \( \Gamma^{i}_{\ell k} \) are not the components of a tensor of type \((1, 2)\). A linear connection \( \nabla \) on \( M \) is an assignment of a linear connection to each point of \( M \). A connection \( \nabla \) is \( C^{k} \) if its connection coefficients \( \Gamma^{i}_{\ell k} \) are \( C^{k} \) in every coordinate system on \( M \).

2. The components of the covariant derivative of a tensor field \( T \) of type \((r, s)\), with respect to a connection \( \nabla \), are given by

\[
\nabla_{k} T^{i_{1} \cdots i_{r}}_{j_{1} \cdots j_{s}} = \partial_{k} T^{i_{1} \cdots i_{r}}_{j_{1} \cdots j_{s}} + \Gamma_{\ell k}^{i} T^{i_{1} \cdots i_{r}}_{\ell j_{1} \cdots j_{s}} + \cdots
\]

\[
\cdots + \Gamma_{\ell k}^{i} T^{i_{1} \cdots i_{r} \ell}_{\ell j_{1} \cdots j_{s}} - \Gamma_{\ell k}^{j} T^{i_{1} \cdots i_{r} j}_{\ell j_{1} \cdots j_{s}} \cdots - \Gamma_{\ell k}^{j} T^{i_{1} \cdots i_{r} \ell}_{j_{1} \cdots j_{s}}. \tag{5.10.8}
\]

where

\[
\partial_{k} T^{i_{1} \cdots i_{r}}_{j_{1} \cdots j_{s}} = T^{i_{1} \cdots i_{r}}_{j_{1} \cdots j_{s}, k} = \frac{\partial T^{i_{1} \cdots i_{r}}_{j_{1} \cdots j_{s}}}{\partial x^{k}}. \tag{5.10.9}
\]

This formula has this structure:

- Apart from the partial derivative term, there is a negative affine term for each covariant index and a positive affine term for each contravariant index.

- The second subscript in the \( \Gamma \)-symbols is always the differentiated index \( (k \text{ in this case}) \).

3. In tensor analysis, a comma is used to denote partial differentiation.

4. The quantity \( \delta A_{i} = dA_{i} - \Gamma_{ij}^{k} A_{k} dx^{j} \) is called the covariant differential of \( A_{i} \).

If \( A_{i} \) is displaced in such a way that \( \delta A_{i} = 0 \), the displacement is said to be parallel with respect to the condition \( \nabla \).

5. A vector field \( Y^{i}(t) \) is said to be parallel along a parameterized curve \( \gamma \) if the component functions satisfy the differential equation \( \frac{dY^{i}}{dt} + Y^{j} \frac{dx^{j}}{dt} \Gamma^{i}_{jk} = 0 \), where \( x^{i} \) denotes the component functions of \( \gamma \) in the coordinate system \( \phi \).
6. A parameterized curve \( \gamma \) in \( M \) is said to be an *affinely parameterized geodesic* if the component functions of \( \gamma \) satisfy the differential equation 
\[
\frac{dx^i}{dt} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0,
\]
which is equivalent to the statement that the tangent vector \( \frac{d\gamma}{dt} \) to \( \gamma \) is parallel along \( \gamma \).

7. The components of the *torsion tensor* \( S \) of \( \nabla \) on \( M \) are defined by
\[
S^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}.
\] (5.10.10)

8. The components of the *curvature tensor* \( R \) of \( \nabla \) on \( M \) are defined by
\[
R^i_{jk\ell} = \partial_k \Gamma^i_{j\ell} - \partial_\ell \Gamma^i_{jk} + \Gamma^m_{jk} \Gamma^i_{m\ell} - \Gamma^m_{j\ell} \Gamma^i_{mk}.
\] (5.10.11)

In some references \( R \) is defined with the opposite sign.

9. The *Ricci tensor* of \( \nabla \) is defined by
\[
R_{jk} = R^\ell_{jk\ell}.
\]

5.10.4 METRIC TENSOR

1. A *covariant metric tensor field* on \( M \) is a tensor field \( g_{ij} \) which satisfies \( g_{ij} = g_{ji} \) and \( g = |g_{ij}| \neq 0 \) on \( M \). The *contravariant metric* \( g^{ij} \) satisfies \( g^{ik} g_{kj} = \delta_i^j \).

The *line element* is expressible in terms of the metric tensor as
\[
ds^2 = g_{ij} dx^i dx^j.
\]

2. *Signature of the metric:* For each \( p \in M \), a coordinate system exists such that \( g_{ij}(p) = \text{diag}(1, \cdots, 1, -1, \cdots, -1) \). The *signature* of \( g_{ij} \) is defined by
\[
s = 2r - n.
\]
It is independent of the coordinate system in which \( g_{ij}(p) \) has the above diagonal form and is the same at every \( p \in M \). A metric is said to be *positive definite* if \( s = n \). A manifold, admitting a positive definite metric, is called a *Riemannian manifold*. A metric is said to be *indefinite* if \( s \neq n \) and \( s \neq -n \). A manifold, admitting an indefinite metric, is called a *pseudo-Riemannian manifold*. If \( s = 2 - n \) or \( n - 2 \), the metric is said to Lorentzian and the corresponding manifold is called a *Lorentzian manifold*.

3. The *inner product* of a pair of vectors \( X^i \) and \( Y^j \) is given by \( g_{ij} X^i Y^j \). If \( X^i = Y^i \), then \( g_{ij} X^i X^j \) defines the “square” of the length of \( X^i \). If \( g_{ij} \) is positive definite, then \( g_{ij} X^i X^j \geq 0 \) for all \( X^i \), and \( g_{ij} X^i X^j = 0 \) if, and only if, \( X^i = 0 \). In the positive definite case, the *angle* \( \theta \) between two tangent vectors \( X^i \) and \( Y^j \) is defined by
\[
\cos \theta = g_{ij} X^i Y^j / (g_{k\ell} X^k X^\ell \ g_{mn} Y^m Y^n)^{1/2}.
\]
If \( g \) is indefinite, \( g_{ij} X^i X^j \) may have a positive, negative, or zero value. A nonzero vector \( X^i \), satisfying \( g_{ij} X^i X^j = 0 \), is called a *null vector*. If \( g_{ij} \) is indefinite, it is not possible in general to define the angle between two tangent vectors.

4. *Operation of lowering indices:* The components of the tensor resulting from *lowering the* \( t \)th *contravariant index* of a tensor \( T \) of type \((r,s)\), with \( r \geq 1 \), are given by
\[
T^a_{\cdot a \cdots a} b_{b \cdots b} = g_{ik} T^i_{j_1 \cdots j_r} g_{k}^{a_1 \cdots a_r-ir}.
\] (5.10.12)
5. **Operation of raising indices**: The components of the tensor from raising the \( t^{th} \) covariant index of a tensor \( T \) of type \((r,s)\), with \( s \geq 1 \), is given by

\[
T^{{i_1 \cdots i_r j_1 \cdots j_r}}_{{j_1 \cdots j_{r-1} k_{r+1} \cdots k_s}} = g^{i_k j_k} T_{{i_1 \cdots i_r j_1 \cdots j_r \cdots k_{r+1} \cdots k_s}}. \tag{5.10.13}
\]

6. The arc length of a parameterized curve \( \gamma : I \to M \), where \( I = [a, b] \), and \( \phi \) is any coordinate system, is defined by

\[
L = \int_a^b \sqrt{\epsilon g_{ij}(x^1(t), \ldots, x^n(t)) \dot{x}^i \dot{x}^j} \, dt, \tag{5.10.14}
\]

where \( \epsilon = \text{sgn}(g_{ij} \dot{x}^i \dot{x}^j) = \pm 1 \) and \( \dot{x}^i = \frac{dx^i}{dt} \).

### 5.10.5 RESULTS

The following results hold on any manifold \( M \) admitting any connection \( \nabla \):

1. The covariant derivative operator \( \nabla_k \) is linear with respect to tensor addition, satisfies the product rule with respect to tensor multiplication, and commutes with contractions.

2. If \( T \) is any tensor of type \((0, r)\), then

\[
\nabla_k T_{{i_1 \cdots i_r}} = T_{{i_1 \cdots i_r, k}} - \frac{1}{2} \left( S_{{[i_k] j_1 \cdots j_r}^i} + \cdots + S_{{[i_r] j_1 \cdots j_r}^i} \right),
\]

where \( || \) indicates that the enclosed indices are excluded from the symmetrization. Thus \( T_{{i_1 \cdots i_r, k}} \) defines a tensor of type \((0, r+1)\), and in the torsion free case, \( \nabla_k \nabla_j f = \nabla_j \nabla_k f \).

3. If \( X^i \) is any vector field on \( M \), then the identity \( 2 \nabla_1 \nabla_2 X^j + \nabla_2 X^j S_{jk} \) reduces to the Ricci identity, which may be extended to tensor fields of type \((r,s)\). For the tensor field \( T^i_{jk} \), it has the form

\[
2 \nabla_1 T^k_{\ell m} + \nabla_2 T^k_{\ell m} S_{ij} = T^k_{\ell m} R^i_{nmij} - T^k_{\ell n} R^i_{mij} - T^k_{\ell m} R^i_{nij}.
\]

4. The torsion tensor \( S \) and curvature tensor \( R \) satisfy the following identities:

\[
S_{(jk)}^i = 0, \quad 0 = R^j_{{[i; k, m]}} + R^j_{{m[k} l_{i]}} S_{\ell m}^n, \quad (5.10.17)
\]

\[
R^i_{{j(k\ell)}} = 0, \quad R^i_{{[j\ell \cdot k]}} = S^i_{{[j\ell \cdot k]}}, \quad (5.10.18)
\]

In the torsion free case, these identities reduce to the cyclical identity \( R^i_{{[j\ell \cdot k]}} = 0 \) and Bianchi’s identity \( R^i_{{j[k\ell \cdot m]}} = 0 \).
The following results hold for any pseudo-Riemannian manifold $M$ with metric tensor field $g_{ij}$:

1. A unique connection $\nabla$ called the Levi–Civita or pseudo-Riemannian connection with vanishing torsion ($S^i_{jk} = 0$) exists that satisfies $\nabla_i g_{jk} = 0$. It follows that $\nabla_i g^{jk} = 0$. The connection coefficients of $\nabla$, called the Christoffel symbols of the second kind, are given by $\Gamma^i_{jk} = \frac{1}{2}(g^{i\ell}_{jk} + g^{jk}_{i\ell} - g_{jk}^{\ell})$ are the Christoffel symbols of the first kind.

2. The operations of raising and lowering indices commute with the covariant derivative. For example if $X^i = g_{ij}X^j$, then $\nabla_k X^i = g_{ij}\nabla_k X^j$.

3. The divergence of a vector $X^i$ is given by $\nabla_i X^i = |g|^{-\frac{1}{2}}\partial_i (|g|^\frac{1}{2} X^i)$. The Laplacian of a scalar invariant $f$ is given by $\Gamma f = g^{ij}\nabla_i \nabla_j f = \nabla_i (g^{ij} \nabla_j f) = |g|^{-\frac{1}{2}}\partial_i (|g|^\frac{1}{2} g^{ij}\partial_j f)$.

4. The equations of an affinely parameterized geodesic may be written as $\frac{d}{dt}(g^{ij}_{\dot{x}j}) - \frac{1}{2}g^{jk}_{\dot{x}i}\dot{x}^j\dot{x}^k = 0$.

5. Let $X^i$ and $Y^i$ be the components of any vector fields which are propagated in parallel along any parameterized curve $\gamma$. Then $\frac{d}{dt}(g_{ij} X^i Y^j) = 0$, which implies that the inner product $g_{ij} X^i Y^j$ is constant along $\gamma$. In particular, if $\dot{x}^i$ are the components of the tangent vector to $\gamma$, then $g_{ij} \dot{x}^i \dot{x}^j$ is constant along $\gamma$.

6. The Riemann tensor, defined by $R^m_{ijk} = g^{mn} R^n_{ijk}$, is given by

$$R^m_{ijk} = g^{mn} ([i\ell, m][jk, n] - [ik, m][j\ell, n]).$$

(5.10.19)

It has the following symmetries:

$$R^m_{ij(k\ell)} = R^m_{ij(k\ell)} = 0, \quad R_{ij(k\ell)} = R_{k\ell ij}, \quad \text{and} \quad R^m_{ij(k\ell)} = 0.$$  

(5.10.20)

Consequently it has a maximum of $n(n^2 - 1)/12$ independent components.

7. The equations $R_{ij(k\ell)} = 0$ are necessary and sufficient conditions for $M$ to be a flat pseudo-Riemannian manifold, that is, a manifold for which a coordinate system exists so that the components $g_{ij}$ are constant on $M$.

8. The Ricci tensor is given by

$$R_{ij} = \partial_j \Gamma^k_{ik} - \partial_k \Gamma^k_{ij} + \Gamma^k_{ij} \Gamma^\ell_{k\ell} - \Gamma^k_{ij} \Gamma^\ell_{k\ell}$$

(5.10.21)

$$= \frac{1}{2} \partial_i \partial_j (\log |g|) - \frac{1}{2} \Gamma^k_{ij} \partial_k (\log |g|) - \partial_k \Gamma^k_{ij} + \Gamma^k_{im} \Gamma^m_{kj}.$$

It possesses the symmetry $R_{ij} = R_{ji}$, and thus has a maximum of $n(n + 1)/2$ independent components.

9. The scalar curvature or curvature invariant is defined by $R = g^{ij} R_{ij}$. 

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10. The Einstein tensor is defined by \( G_{ij} = R_{ij} - \frac{1}{2} R g_{ij} \). In view of the Bianchi identity, it satisfies: 
\[ g^{ij} \nabla_j G_{ki} = 0. \]

11. A normal coordinate system with origin \( x_0 \in M \) is defined by 
\[ g^{ij}(x) = g^{ij}(x_0), \]
where a “0” affixed over a quantity indicates that the quantity is evaluated at \( x_0 \).

The connection coefficients satisfy \( \Gamma_{ij}^{(k)}(x_j, x_j, \ldots, x_j) = 0 \) (for \( r = 2, 3, 4, \ldots \)) in any normal coordinate system. The equations of the geodesics through \( x_0 \) are given by 
\[ x_i = s k_i, \] where \( s \) is an affine parameter and \( k_i \) is any constant vector.

5.10.6 EXAMPLES

1. The components of the gradient of a scalar invariant \( \frac{\partial f}{\partial x^i} \) define a tensor of type \( (0,1) \), since they transform as 
\[ \frac{\partial \tilde{f}}{\partial \tilde{x}^i} = \frac{\partial f}{\partial x^j} \frac{\partial x^j}{\partial \tilde{x}^i}. \]

2. The components of the tangent vector to a parameterized curve \( \frac{dx^i}{dt} \) define a tensor of type \( (1,0) \), because they transform as 
\[ \frac{d\tilde{x}^i}{dt} = \frac{dx^j}{dt} \frac{\partial \tilde{x}^i}{\partial x^j}. \]

3. The determinant of the metric tensor \( g \) defines a relative scalar invariant of weight \( w = 2 \), because it transforms as 
\[ \tilde{g} = \left| \frac{\partial x^i}{\partial \tilde{x}^j} \right|^2 g. \]

4. The Kronecker deltas \( \delta^i_j \) are the components of a constant absolute tensor of type \( (1,1) \), because 
\[ \delta^i_j = \delta^k_l \frac{\partial \tilde{x}^i}{\partial x^k} \frac{\partial \tilde{x}^j}{\partial x^l}. \]

5. The permutation symbol defined by 
\[ e_{i_1 \cdots i_n} = \begin{cases} 1, & \text{if } i_1 \cdots i_n \text{ is an even permutation of } 1 \cdots n, \\ -1, & \text{if } i_1 \cdots i_n \text{ is an odd permutation of } 1 \cdots n, \\ 0, & \text{otherwise,} \end{cases} \] (5.10.22)
satisfies 
\[ \left| \frac{\partial x^i}{\partial \tilde{x}^j} \right| e_{j_1 \cdots j_n} = e_{i_1 \cdots i_n} \frac{\partial x^{j_1}}{\partial \tilde{x}^i} \cdots \frac{\partial x^{j_n}}{\partial \tilde{x}^i}. \] Hence it defines a tensor of type \( (0,n,-1) \), that is, it is a relative tensor of weight \( w = -1 \). The contravariant permutation symbol \( e^{i_1 \cdots i_n} \), defined in a similar way, is a relative tensor of weight \( w = 1 \).

6. The Levi–Civita symbol, \( e_{i_1 \cdots i_n} = |g|^{\frac{1}{2}} e_{i_1 \cdots i_n} \), defines a covariant absolute tensor of valence \( n \). The contravariant Levi–Civita tensor satisfies 
\[ e^{i_1 \cdots i_n} = g^{i_1 j_1} \cdots g^{i_n j_n} e_{j_1 \cdots j_n} = (-1)^{\frac{n(n-1)}{2}} |g|^{-\frac{1}{2}} e^{i_1 \cdots i_n}. \] (5.10.23)

Using this symbol, the dual of a covariant skew-symmetric tensor of valence \( r \) is defined by 
\[ *T_{i_1 \cdots i_r} = \frac{1}{r! e_{i_1 \cdots i_r}} T_{j_1 \cdots j_r}. \]

7. Cartesian tensors: Let \( M = E^3 \) (i.e., Euclidean 3 space) with metric tensor \( g_{ij} = \delta_{ij} \) with respect to Cartesian coordinates. The components of a Cartesian tensor of valence \( r \) transform as 
\[ T_{i_1 \cdots i_r} = T_{j_1 \cdots j_r} O_{i_1 j_1} \cdots O_{i_r j_r}, \] (5.10.24)
where $O_{ij}$ are the components of a constant orthogonal matrix which satisfies $(O^{-1})_{ij} = (O')_{ij} = O_{ji}$. For Cartesian tensors, all indices are written as covariant, because no distinction is required between covariant and contravariant indices.

An oriented Cartesian tensor is a Cartesian tensor where the orthogonal matrix in the transformation law is restricted by $\det(O_{ij}) = 1$. The Levi–Civita symbol $\epsilon_{ijk}$ is an example of an oriented Cartesian tensor as is the cross product, $(X \times Y)_i = \epsilon_{ijk}X_jY_k$, of two vectors. The connection coefficients satisfy $\Gamma'_{ijk} = 0$ in every Cartesian coordinate system on $E^3$. Thus the partial derivatives of Cartesian tensors are themselves Cartesian tensors, that is, if $T_{i\cdots i}$ is a Cartesian tensor, then so is $\partial_k T_{i\cdots i}$. A particular example is the curl of a vector field $X_i$ given by $(\text{curl } X)_i = \epsilon_{ijk}\partial_j X_k$ which defines an oriented Cartesian tensor.

8. Note the useful relations: $\epsilon_{ijk} \epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$, and $\epsilon_{ijk} \epsilon_{lmn} = \delta_{im}\delta_{jn}\delta_{kl} - \delta_{in}\delta_{jm}\delta_{kl}$ for any $i, j, k, l, m, n$, and $\epsilon_{ijkl} = \epsilon_{klmj} = 2\delta_{im}\delta_{jn}$, and $\epsilon_{ijkl} = \epsilon_{jikl} = \epsilon_{ijlk}$ are examples of Cartesian tensors.

9. The stress tensor $E_{ij}$ and the strain tensor $e_{ij}$ are examples of Cartesian tensors.

10. Orthogonal curvilinear coordinates: Let $M$ be a 3-dimensional Riemannian manifold admitting a coordinate system $[x^1, x^2, x^3]$ such that the metric tensor has the form $g_{ii} = h_i^2(x^1, x^2, x^3)$ for $i = 1, \ldots, 3$ with $g_{ij} = g^{ij} = 0$ for $i \neq j$. The metric tensor on $E^3$ has this form with respect to orthogonal curvilinear coordinates. The non-zero components of various corresponding quantities corresponding to this metric are as follows:

- **Covariant metric tensor,**
  $$g_{11} = h_1^2, \quad g_{22} = h_2^2, \quad g_{33} = h_3^2.$$

- **Contravariant metric tensor,**
  $$g^{11} = h_1^{-2}, \quad g^{22} = h_2^{-2}, \quad g^{33} = h_3^{-2}.$$

- **Christoffel symbols of the first kind** (note that $[ij, k] = 0$ if $i, j, k$ are all different),
  $$\begin{align*}
  [11, 1] &= h_1 h_{1,1}, & [11, 2] &= -h_1 h_{1,2}, & [11, 3] &= -h_1 h_{1,3} \\
  [12, 1] &= h_1 h_{1,2}, & [12, 2] &= h_2 h_{2,1}, & [13, 1] &= h_1 h_{3,1} \\
  [13, 2] &= h_2 h_{3,1}, & [22, 1] &= -h_2 h_{2,1}, & [22, 2] &= h_2 h_{2,2} \\
  [23, 1] &= -h_3 h_{3,1}, & [23, 2] &= h_2 h_{3,2}, & [23, 3] &= h_3 h_{3,2} \\
  [33, 1] &= -h_3 h_{3,1}, & [33, 2] &= -h_3 h_{3,2}, & [33, 3] &= h_3 h_{3,3}.
  \end{align*}$$

- **Christoffel symbols of the second kind** (note that $\Gamma^k_{ij} = 0$ if $i, j, k$ are all different),
  $$\begin{align*}
  \Gamma^1_{11} &= h_1^{-1} h_{1,1}, & \Gamma^1_{12} &= h_1^{-1} h_{1,2}, & \Gamma^1_{13} &= h_1^{-1} h_{1,3} \\
  \Gamma^1_{22} &= -h_1^{-2} h_{2,1}, & \Gamma^1_{23} &= -h_1^{-2} h_{3,1}, & \Gamma^1_{11} &= h_1^{-1} h_{2,1} \\
  \Gamma^2_{12} &= h_2^{-1} h_{3,1}, & \Gamma^2_{22} &= h_2^{-1} h_{2,2}, & \Gamma^2_{23} &= h_2^{-1} h_{2,3} \\
  \Gamma^2_{33} &= -h_2^{-2} h_{3,1}, & \Gamma^2_{11} &= -h_2^{-2} h_{1,1}, & \Gamma^2_{13} &= h_2^{-1} h_{1,3} \\
  \Gamma^3_{22} &= -h_3^{-2} h_{2,1}, & \Gamma^3_{23} &= -h_3^{-2} h_{3,2}, & \Gamma^3_{33} &= h_3^{-1} h_{3,3}.
  \end{align*}$$

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Vanishing Riemann tensor conditions (Lamé equations),
\[ h_{1,2,3} - h_2^{-1}h_{1,2}h_{2,3} - h_3^{-1}h_{1,3}h_{3,2} = 0, \]
\[ h_{2,1,3} - h_1^{-1}h_{1,3}h_{2,1} - h_3^{-1}h_{1,1}h_{2,3} = 0, \]
\[ h_{3,1,2} - h_1^{-1}h_{1,2}h_{3,1} - h_2^{-1}h_{2,1}h_{3,2} = 0, \]
\[ h_2h_{2,3,3} + h_3h_{3,2,2} + h_1^{-2}h_3h_{2,1}h_{3,1} - h_2^{-1}h_2h_{2,2}h_{3,2} \]
\[ - h_2h_3^{-1}h_{2,3}h_{3,3} = 0, \]
\[ h_1h_{1,3,3} + h_3h_{3,1,1} + h_1^{-2}h_3h_{1,2}h_{3,2} - h_1^{-1}h_1h_{1,1}h_{3,1} \]
\[ - h_1h_3^{-1}h_{1,3}h_{3,3} = 0, \]
\[ h_1h_{1,2,2} + h_2h_{2,1,1} + h_1h_3^{-2}h_{1,3}h_{2,3} - h_1^{-1}h_2h_{1,1}h_{2,1} \]
\[ - h_1h_2^{-1}h_{1,2}h_{2,2} = 0. \]

11. The 2-sphere: A coordinate system \([\theta, \phi]\) for the 2-sphere \(x^2 + y^2 + z^2 + w^2 = r^2\) is given by \(x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta,\) where \([\theta, \phi] \in U = (0, \pi) \times (0, 2\pi)\). This is a non-Euclidean space. The nonzero independent components of various quantities defined on the sphere are given below:

- Covariant metric tensor components are \(g_{11} = r^2, \quad g_{22} = r^2 \sin^2 \theta.\)
- Contravariant metric tensor components are \(g^{11} = r^{-2}, \quad g^{22} = r^{-2} \csc^2 \theta.\)
- Christoffel symbols of the first kind are \([12, 2] = r^2 \sin \theta \cos \phi, \quad [22, 1] = -r^2 \sin \theta \cos \phi.\)
- Christoffel symbols of the second kind are \(\Gamma^1_{12} = -\sin \theta \cos \phi, \quad \Gamma^2_{12} = -\cos \theta \csc \phi.\)
- Covariant Riemann tensor components are \(R_{1212} = r^2 \sin^2 \theta.\)
- Covariant Ricci tensor components are \(R_{11} = -1, \quad R_{22} = -\sin^2 \theta.\)
- Ricci scalar is \(R = -2r^2.\)

12. The 3-sphere: A coordinate system \([\psi, \theta, \phi]\) for the 3-sphere \(x^2 + y^2 + z^2 + w^2 = r^2\) is given by \(x = r \sin \psi \sin \theta \cos \phi, y = r \sin \psi \sin \theta \sin \phi, z = r \sin \psi \cos \theta,\) and \(w = r \cos \psi,\) where \([\psi, \theta, \phi] \in U = (0, \pi) \times (0, \pi) \times (0, 2\pi)\). The nonzero components of various quantities defined on the sphere are given below:

- Covariant metric tensor components,
\[ g_{11} = r^2, \quad g_{22} = r^2 \sin^2 \psi, \quad g_{33} = r^2 \sin^2 \psi \sin^2 \theta. \]
- Contravariant metric tensor components,
\[ g^{11} = r^{-2}, \quad g^{22} = r^{-2} \csc^2 \psi, \quad g^{33} = r^{-2} \csc^2 \psi \csc^2 \theta. \]
- Christoffel symbols of the first kind,
\[ [22, 1] = -r^2 \sin \psi \cos \psi \quad [33, 1] = -r^2 \sin \psi \cos \psi \sin^2 \theta \]
\[ [12, 2] = r^2 \sin \psi \cos \psi \quad [33, 2] = -r^2 \sin^2 \psi \sin \theta \cos \theta \]
\[ [13, 3] = r^2 \sin \psi \cos \psi \sin^2 \theta \quad [23, 3] = r^2 \sin^2 \psi \sin \theta \cos \theta. \]
• Christoffel symbols of the second kind,

\[
\begin{align*}
\Gamma^1_{22} &= -\sin \psi \cos \psi & \Gamma^1_{33} &= -\sin \psi \cos \psi \sin^2 \theta \\
\Gamma^2_{12} &= \cot \psi & \Gamma^2_{33} &= -\sin \theta \cos \theta \\
\Gamma^3_{33} &= -\cot \psi.
\end{align*}
\]

• Covariant Riemann tensor components,

\[
\begin{align*}
R_{1212} &= r^2 \sin^2 \psi, & R_{1313} &= r^2 \sin^2 \psi \sin^2 \theta, & R_{2323} &= r^2 \sin^2 \psi \sin^2 \theta.
\end{align*}
\]

• Covariant Ricci tensor components,

\[
\begin{align*}
R_{11} &= -2, & R_{22} &= -2 \sin^2 \psi, & R_{33} &= -2 \sin^2 \psi \sin^2 \theta.
\end{align*}
\]

The Ricci scalar is \( R = -6r^{-2}. \)

• Covariant Einstein tensor components,

\[
\begin{align*}
G_{11} &= 1, & G_{22} &= \sin^2 \psi, & G_{33} &= \sin^2 \psi \sin^2 \theta.
\end{align*}
\]

13. Polar coordinates: The line element is given by \( ds^2 = dr^2 + r^2 d\theta^2. \) Thus the metric tensor is \( g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \) and the nonzero Christoffel symbols are \([21, 2] = [12, 2] = -[22, 1] = r.\)

## 5.11 ORTHOGONAL COORDINATE SYSTEMS

In an orthogonal coordinate system, let \( \{a_i\} \) denote the unit vectors in each of the three coordinate directions, and let \( \{u_i\} \) denote distance along each of these axes. The coordinate system may be designated by the metric coefficients \( \{g_{ii}, g_{22}, g_{33}\}, \) defined by

\[
g_{ii} = \left( \frac{\partial x_1}{\partial u_i} \right)^2 + \left( \frac{\partial x_2}{\partial u_i} \right)^2 + \left( \frac{\partial x_3}{\partial u_i} \right)^2, \tag{5.11.1}
\]

where \( \{x_1, x_2, x_3\} \) represent rectangular coordinates. With these, we define \( g = g_{11}g_{22}g_{33}. \)

Operations for orthogonal coordinate systems are sometimes written in terms of \( \{h_i\} \) functions, instead of the \( \{g_{ii}\} \) terms. Here, \( h_i = \sqrt{g_{ii}}, \) so that \( \sqrt{g} = h_1h_2h_3. \) For example, in cylindrical polar coordinates, \( \{x_1 = r \cos \theta, x_2 = r \sin \theta, x_3 = z\}, \) so that \( \{h_1 = 1, h_2 = r, h_3 = 1\}. \)

In the following, \( \phi \) represents a scalar, \( \mathbf{E} = E_1a_1 + E_2a_2 + E_3a_3, \) and \( \mathbf{F} = F_1a_1 + F_2a_2 + F_3a_3 \) represent vectors.
\[
\text{grad } \phi = \nabla \phi = \frac{a_1}{\sqrt{g_{11}}} \frac{\partial \phi}{\partial u_1} + \frac{a_2}{\sqrt{g_{22}}} \frac{\partial \phi}{\partial u_2} + \frac{a_3}{\sqrt{g_{33}}} \frac{\partial \phi}{\partial u_3}, \quad (5.11.2)
\]

\[
\text{div } \mathbf{E} = \nabla \cdot \mathbf{E} = \frac{1}{\sqrt{g}} \left[ \frac{\partial}{\partial u_1} \left( g E_1 \frac{\partial \phi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left( g E_2 \frac{\partial \phi}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left( g E_3 \frac{\partial \phi}{\partial u_3} \right) \right], \quad (5.11.3)
\]

\[
\text{curl } \mathbf{E} = \nabla \times \mathbf{E} = a_1 \frac{\Gamma_1}{\sqrt{g_{11}}} + a_2 \frac{\Gamma_2}{\sqrt{g_{22}}} + a_3 \frac{\Gamma_3}{\sqrt{g_{33}}}, \quad (5.11.4)
\]

\[
\left[ (\mathbf{F} \cdot \nabla) \mathbf{E} \right]_j = \frac{1}{h_i} \sum_{i=1}^3 \left[ \frac{F_i}{h_i} \frac{\partial E_j}{\partial u_i} + \frac{E_i}{h_i h_j} \left( F_j \frac{\partial h_j}{\partial u_i} - F_i \frac{\partial h_i}{\partial u_j} \right) \right], \quad (5.11.5)
\]

\[
\nabla^2 \phi = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1} \left[ \frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial u_1} \right] + \frac{\partial}{\partial u_2} \left[ \frac{h_3 h_1}{h_2} \frac{\partial \phi}{\partial u_2} \right] + \frac{\partial}{\partial u_3} \left[ \frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial u_3} \right] \right\}, \quad (5.11.7)
\]

\[
\text{grad } \text{div } \mathbf{E} = \nabla (\nabla \cdot \mathbf{E}) = \frac{a_1}{\sqrt{g_{11}}} \frac{\partial \Upsilon}{\partial x_1} + \frac{a_2}{\sqrt{g_{22}}} \frac{\partial \Upsilon}{\partial x_2} + \frac{a_3}{\sqrt{g_{33}}} \frac{\partial \Upsilon}{\partial x_3}, \quad (5.11.8)
\]

\[
\text{curl } \text{curl } \mathbf{E} = \nabla \times (\nabla \times \mathbf{E}) = a_1 \sqrt{\frac{g_{11}}{g}} \left[ \frac{\partial \Gamma_3}{\partial x_2} - \frac{\partial \Gamma_2}{\partial x_3} \right] + a_2 \sqrt{\frac{g_{22}}{g}} \left[ \frac{\partial \Gamma_1}{\partial x_3} - \frac{\partial \Gamma_3}{\partial x_1} \right] + a_3 \sqrt{\frac{g_{33}}{g}} \left[ \frac{\partial \Gamma_2}{\partial x_1} - \frac{\partial \Gamma_1}{\partial x_2} \right], \quad (5.11.9)
\]

\[
\Psi = \text{grad } \text{div } \mathbf{E} - \text{curl } \text{curl } \mathbf{E} = \nabla (\nabla \cdot \mathbf{E}) - \nabla \times (\nabla \times \mathbf{E}) = \frac{a_1}{\sqrt{g_{11}}} \frac{1}{\sqrt{g}} \left[ \frac{\partial \Upsilon}{\partial x_1} + \frac{g_{11}}{\sqrt{g}} \left[ \frac{\partial \Gamma_2}{\partial x_2} - \frac{\partial \Gamma_3}{\partial x_3} \right] \right] + a_2 \frac{1}{\sqrt{g_{22}}} \frac{g_{22}}{\sqrt{g}} \left[ \frac{\partial \Gamma_1}{\partial x_3} - \frac{\partial \Gamma_3}{\partial x_1} \right] + a_3 \frac{1}{\sqrt{g_{33}}} \frac{g_{33}}{\sqrt{g}} \left[ \frac{\partial \Gamma_2}{\partial x_1} - \frac{\partial \Gamma_1}{\partial x_2} \right], \quad (5.11.10)
\]
where $\Upsilon$ and $\mathbf{\Gamma} = (\Gamma_1, \Gamma_2, \Gamma_3)$ are defined by

\[
\Upsilon = \frac{1}{\sqrt{g}} \left\{ \frac{\partial}{\partial x_1} \left[ E_1 \sqrt{g_{11}} \right] + \frac{\partial}{\partial x_2} \left[ E_2 \sqrt{g_{22}} \right] + \frac{\partial}{\partial x_3} \left[ E_3 \sqrt{g_{33}} \right] \right\},
\]
\[
\Gamma_1 = \frac{g_{11}}{\sqrt{g}} \left\{ \frac{\partial}{\partial x_2} \left( \sqrt{g_{33}} E_3 \right) - \frac{\partial}{\partial x_3} \left( \sqrt{g_{22}} E_2 \right) \right\},
\]
\[
\Gamma_2 = \frac{g_{22}}{\sqrt{g}} \left\{ \frac{\partial}{\partial x_3} \left( \sqrt{g_{11}} E_1 \right) - \frac{\partial}{\partial x_1} \left( \sqrt{g_{33}} E_3 \right) \right\},
\]
\[
\Gamma_3 = \frac{g_{22}}{\sqrt{g}} \left\{ \frac{\partial}{\partial x_1} \left( \sqrt{g_{22}} E_2 \right) - \frac{\partial}{\partial x_2} \left( \sqrt{g_{11}} E_1 \right) \right\}. 
\]

5.11.1 LIST OF ORTHOGONAL COORDINATE SYSTEMS

1. Rectangular coordinates $\{x, y, z\}$
   Ranges: $-\infty < x < \infty, -\infty < y < \infty, -\infty < z < \infty$.
   \[
g_{11} = g_{22} = g_{33} = \sqrt{g} = 1,
\]
   $h_1 = h_2 = h_3 = 1$.

\[
\text{grad } f = a_x \frac{\partial f}{\partial x} + a_y \frac{\partial f}{\partial y} + a_z \frac{\partial f}{\partial z},
\]
\[
\text{div } \mathbf{E} = \frac{\partial}{\partial x} (E_x) + \frac{\partial}{\partial y} (E_y) + \frac{\partial}{\partial z} (E_z),
\]
\[
\text{curl } \mathbf{E} = \left( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) a_x + \left( \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) a_y + \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) a_z,
\]
\[
\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}, \text{ and }
\]
\[
[(\mathbf{F} \cdot \nabla) \mathbf{E}]_x = F_x \frac{\partial E_x}{\partial x} + F_y \frac{\partial E_x}{\partial y} + F_z \frac{\partial E_x}{\partial z}.
\]

In this coordinate system the following notation is sometimes used: $\mathbf{i} = a_x, \mathbf{j} = a_y, \mathbf{k} = a_z$. 

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2. Circular cylinder coordinates \( \{r, \theta, z\} \)

Relations: \( x = r \cos \theta, \ y = r \sin \theta, \ z = z. \)

Ranges: \( 0 < r < \infty, \ 0 < \theta < 2\pi, \ -\infty < z < \infty. \)
\( g_{11} = g_{33} = 1, \ g_{22} = r^2, \sqrt{g} = r, \)
\( h_1 = r, \ h_2 = h_3 = 1. \)

\[
\text{grad } f = \mathbf{a}_r \frac{\partial f}{\partial r} + \mathbf{a}_\theta \frac{\partial f}{\partial \theta} + \mathbf{a}_z \frac{\partial f}{\partial z}, \quad (5.11.17)
\]
\[
\text{div } \mathbf{E} = \frac{1}{r} \frac{\partial}{\partial r} (r E_r) + \frac{1}{r} \frac{\partial E_\theta}{\partial \theta} + \frac{\partial E_z}{\partial z}, \quad (5.11.18)
\]
\[
(\text{curl } \mathbf{E})_r = \frac{1}{r} \frac{\partial E_z}{\partial \theta} - \frac{\partial E_\theta}{\partial z}, \quad (5.11.19)
\]
\[
(\text{curl } \mathbf{E})_\theta = \frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}, \quad (5.11.20)
\]
\[
(\text{curl } \mathbf{E})_z = \frac{1}{r} \frac{\partial (r E_\theta)}{\partial r} - \frac{1}{r} \frac{\partial E_r}{\partial \theta}, \quad \text{and} \quad (5.11.21)
\]
\[
\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}. \quad (5.11.22)
\]

3. Elliptic cylinder coordinates \( \{\eta, \psi, z\} \)

Relations: \( x = a \cosh \eta \cos \psi, \ y = a \sinh \eta \sin \psi, \ z = z. \)

Ranges: \( 0 < \eta < \infty, \ 0 < \psi < 2\pi, \ -\infty < z < \infty. \)
\( g_{11} = g_{22} = a^2 (\cosh^2 \eta - \cos^2 \psi), \ g_{33} = 1, \sqrt{g} = a^2 (\cosh^2 \eta - \cos^2 \psi), \)
\( h_1 = h_2 = h_3 = 1. \)

4. Parabolic cylinder coordinates \( \{\mu, v, z\} \)

Relations: \( x = \frac{1}{2} (\mu^2 - v^2), \ y = \mu v, \ z = z. \)

Ranges: \( 0 \leq \mu < \infty, \ -\infty < v < 2\pi, \ -\infty < z < \infty. \)
\( g_{11} = g_{22} = \mu^2 + v^2, \ g_{33} = 1, \sqrt{g} = \mu^2 + v^2, \)
\( h_1 = h_2 = h_3 = 1. \)

5. Spherical coordinates \( \{r, \theta, \psi\} \)

Relations: \( x = r \sin \theta \cos \phi, \ y = r \sin \theta \sin \phi, \ z = r \cos \theta. \)

Ranges: \( 0 \leq r < \infty, \ 0 \leq \theta \leq \pi, \ 0 \leq \psi < 2\pi. \)
\( g_{11} = 1, \ g_{22} = r^2, \ g_{33} = r^2 \sin^2 \theta, \sqrt{g} = r^2 \sin \theta, \)
\( h_1 = r^2, \ h_2 = \sin \theta, \ h_3 = 1. \)
9. Conical coordinates \( \{r, \theta, \lambda\} \)

Relations: \( x = r \sin \theta \cos \lambda, y = r \sin \theta \sin \psi, z = r \cos \theta \).

Ranges: \( 0 \leq r < \infty, 0 \leq \theta < \pi, 0 \leq \psi < 2\pi. \)

\[
g_{11} = g_{22} = a^2(\sin^2 \theta + \sin^2 \psi) \quad g_{33} = a^2 \sin^2 \theta, \quad \sqrt{g} = a^2(\sin^2 \theta + \sin^2 \psi) \sin \theta \sin \lambda. \]

\( h_1 = \sin \theta, h_2 = \sin \psi, h_3 = a. \)

8. Parabolic coordinates \( \{\mu, v, \psi\} \)

Relations: \( x = \mu \cos \psi, y = \mu \sin \psi, z = \frac{1}{2}(\mu^2 - v^2). \)

Ranges: \( 0 \leq \mu < \infty, 0 \leq v \leq \infty, 0 \leq \psi < 2\pi. \)

\[
g_{11} = g_{22} = \mu^2 + v^2 \quad g_{33} = \mu^2 v^2, \quad \sqrt{g} = \mu v (\mu^2 + v^2). \]

\( h_1 = \mu, h_2 = v, h_3 = 1. \)

7. Oblate spheroidal coordinates \( \{\eta, \theta, \psi\} \)

Relations: \( x = a \cosh \eta \sin \theta \cos \psi, y = a \cosh \eta \sin \theta \sin \psi, z = a \sinh \eta \sin \theta. \)

Ranges: \( 0 \leq \eta < \infty, 0 \leq \theta \leq \pi, 0 \leq \psi < 2\pi. \)

\[
g_{11} = g_{22} = a^2(\sinh^2 \eta - \sin^2 \theta) \quad g_{33} = a^2 \sinh^2 \eta \sin^2 \theta, \quad \sqrt{g} = a^2(\sinh^2 \eta - \sin^2 \theta) \sinh \eta \sin \theta. \]

\( h_1 = \sinh \eta, h_2 = \sin \theta, h_3 = a. \)

6. Prolate spheroidal coordinates \( \{\eta, \theta, \psi\} \)

Relations: \( x = a \sinh \eta \sin \theta \cos \psi, y = a \sinh \eta \sin \theta \sin \psi, z = a \cosh \eta \cos \theta. \)

Ranges: \( 0 \leq \eta < \infty, 0 \leq \theta \leq \pi, 0 \leq \psi < 2\pi. \)

\[
g_{11} = g_{22} = a^2(\sinh^2 \eta + \sin^2 \theta) \quad g_{33} = a^2 \sinh^2 \eta \sin^2 \theta, \quad \sqrt{g} = a^2(\sinh^2 \eta + \sin^2 \theta) \sinh \eta \sin \theta. \]

\( h_1 = \sinh \eta, h_2 = \sin \theta, h_3 = a. \)

\[
\begin{align*}
\text{grad } f & = e_r \frac{\partial f}{\partial r} + e_\theta \frac{\partial f}{\partial \theta} + e_\phi \frac{\partial f}{\partial \phi}, \\
\text{div } E & = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 E_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (E_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} E_\phi, \\
(\text{curl } E)_r & = \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (E_\phi \sin \theta) - \frac{\partial}{\partial \phi} A_\theta \right], \\
(\text{curl } E)_\theta & = \frac{1}{r} \frac{\partial}{\partial r} (r E_\theta) - \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} E_\phi, \\
(\text{curl } E)_\phi & = \frac{1}{r} \frac{\partial}{\partial r} (r E_\phi) - \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} E_\theta, \\
\nabla^2 f & = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}.
\end{align*}
\]
\[ h_1 = r^2, \quad h_2 = \sqrt{(\theta^2 - b^2)(c^2 - \theta^2)}, \quad h_3 = \sqrt{(b^2 - \lambda^2)(c^2 - \lambda^2)}. \]

10. Ellipsoidal coordinates \( \{n, \theta, \lambda\} \)

- Relations: \( x^2 = (n \theta \lambda / b c)^2 \), \( y^2 = (n^2 - b^2)(\theta^2 - b^2)(b^2 - \lambda^2) / [b^2(c^2 - b^2)] \), \( z^2 = (n^2 - c^2)(c^2 - \theta^2)(c^2 - \lambda^2) / [c^2(c^2 - b^2)] \).
- Ranges: \( c^2 \leq n^2 < \infty, \ b^2 < \theta^2 < c^2, \ 0 < \lambda^2 < b^2 \).
- Relations: \( g_{11} = (n^2 - \theta^2)(n^2 - \lambda^2) / ((n^2 - b^2)(n^2 - c^2)) \), \( g_{22} = (\theta^2 - \lambda^2)(\theta^2 - \lambda^2) / ((\theta^2 - b^2)(\theta^2 - c^2)) \), \( g_{33} = (n^2 - \lambda^2)(\theta^2 - \lambda^2) / ((b^2 - \lambda^2)(c^2 - \lambda^2)) \).
- \( \sqrt{\mathcal{E}} = \frac{1}{\sqrt{(n^2 - b^2)(n^2 - c^2)(\theta^2 - b^2)(\theta^2 - c^2)}}. \)
- \( h_1 = \sqrt{(n^2 - b^2)(n^2 - c^2)}, \quad h_2 = \sqrt{(\theta^2 - b^2)(\theta^2 - c^2)}, \quad h_3 = \sqrt{(b^2 - \lambda^2)(c^2 - \lambda^2)}. \)

11. Paraboloidal coordinates \( \{\mu, \nu, \lambda\} \)

- Relations: \( x^2 = 4(\mu - b)(\mu - c)/b - c \), \( y^2 = 4(\mu - c)(\nu - \lambda)/b - c \), \( z^2 = \mu + \nu + \lambda - b - c \).
- Ranges: \( b < \mu < \infty, \ 0 < \nu < c, \ c < \lambda < b \).
- Relations: \( g_{11} = (\mu - \nu)(\mu - \lambda)/((\mu - b)(\mu - c)) \), \( g_{22} = (\mu - \nu)(\lambda - \nu)/((\nu - c)(\nu - c)) \), \( g_{33} = (\lambda - \nu)(\mu - \lambda)/((b - \lambda)(c - \lambda)) \).
- \( h_1 = \sqrt{(\mu - b)/\mu - c)}, \quad h_2 = \sqrt{(\nu - c)/\nu - c)}, \quad h_3 = \sqrt{(\lambda - \nu)/\lambda - \nu}). \)

### 5.12 CONTROL THEORY

Let \( x \) be a state vector, let \( y \) be an observable vector, and let \( u \) be the control. Each of \( x, y, \) and \( u \) has \( n \) components. If a system evolves as

\[ \dot{x} = A x + B u, \]

and

\[ y = C x + D u, \]

then, taking Laplace transforms, \( \tilde{y} = G(s)\tilde{u} \) where \( G(s) \) is the transfer function given by \( G(s) = C(s I - A)^{-1} B + D \).

A system is said to be controllable if, and only if, for any times \( t_0, t_1 \) and any outputs \( \{y_0, y_1\} \), a control \( u(t) \) exists so that \( y(t_0) = y_0 \) and \( y(t_1) = y_1 \). The system is not controllable if \( \text{rank}[B \ A B \ A^2 B \ \ldots \ A^{n-1} B] < n \).

If, given \( u(t) \) and \( y(t) \) on some interval \( t_0 < t < t_1 \), the value of \( x(t) \) can be deduced on that interval, then the system is said to be observable. Observability requires \( \text{rank}[C^T \ A^T C^T \ \ldots \ (A^{n-1})^T C^T] = n \).

If the control is bounded (say \( u^- < u_i < u^+ \)), then a “bang–bang” control is one for which \( u_i = u^- \) or \( u_i = u^+ \). A “bang–off–bang” control is one for which \( u_i = 0 \), \( u_i = u^- \) or \( u_i = u^+ \).

A second frequently studied control problem is \( \dot{x} = f(x, u, t) \), where \( x(t_0) \) and \( x(t_f) \) are specified, and there is a cost function, \( J = \int_{t_0}^{t_f} \phi(x, u, t) \). The goal is to
minimize the cost function. Defining the Hamiltonian \( H(x, u, t) = \phi + z \cdot f \), the optimal control satisfies

\[
\dot{x} = \frac{\partial H}{\partial z}, \quad \dot{z} = -\frac{\partial H}{\partial x}, \quad 0 = \frac{\partial H}{\partial u}.
\]

Example: In the one-dimensional case, with \( \dot{x} = -ax + u, x(0) = x_0, x(\infty) = 0 \), and \( J = \int_0^\infty (x^2 + u^2) \, dt \); the optimal control is given by \( u = (a - \sqrt{1 + a^2})x^*(t) \) where \( x^*(t) = x_0 e^{-\sqrt{1 + a^2}t} \).

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Chapter 6

Special Functions

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6.1.1 DEFINITION OF ANGLES

If two lines intersect and one line is rotated about the point of intersection, the angle of rotation is designated positive if the angle of rotation is counterclockwise. Angles are commonly measured in units of radians or degrees. Degrees are a historical unit related to the calendar defined by a complete revolution equalling 360 degrees (the approximate number of days in a year). Radians are the angular unit usually used for mathematics and science. Radians are specified by the arc length traced by the tip of a rotating line divided by the length of that line. Thus a complete rotation of a line about the origin corresponds to $2\pi$ radians of rotation. It is a convenient convention that a full rotation of $2\pi$ radians is divided into four angular segments of $\pi/2$ each and that these are referred to as the four quadrants using Roman numerals I, II, III, and IV to designate them (see Figure 6.1.1).

6.1.2 CHARACTERIZATION OF ANGLES

A “right” angle is the angle between two perpendicular lines. It is equal to $\pi/2$ radians or 90 degrees. An acute angle is an angle less than $\pi/2$ radians. An obtuse angle is one between $\pi/2$ and $\pi$ radians.

6.1.3 RELATION BETWEEN RADIANS AND DEGREES

The angle $\pi$ radians corresponds to 180 degrees. Therefore,
\[
\text{one radian} = \frac{180}{\pi} = 57.30 \text{ degrees},
\]
\[
\text{one degree} = \frac{\pi}{180} = 0.01745 \text{ radians}.
\]

6.1.4 CIRCULAR FUNCTIONS

Consider the rectangular coordinate system shown in Figure 6.1.1. The coordinate $x$ is positive to the right of the origin and the coordinate $y$ is positive above the origin. The radius vector $r$ shown terminating on the point $P(x, y)$ is shown rotated by the angle $\alpha$ up from the $x$ axis. The radius vector $r$ has component vectors $x$ and $y$.

The trigonometric or circular functions of the angle $\alpha$ are defined in terms of the signed coordinates $x$ and $y$ and the length $r$, always positive. Note that the coordinate $x$ is negative in quadrants II and III and the coordinate $y$ is negative in quadrants III and IV. The definitions of the trigonometric functions in terms of the Cartesian coordinates $x$ and $y$ of the point $P(x, y)$ are shown below. The angle $\alpha$ can be specified in radians, degrees, or any other unit.
FIGURE 6.1.1
The four quadrants (left) and notation for trigonometric functions (right).

<table>
<thead>
<tr>
<th>Quadrants</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>II</td>
<td>1</td>
</tr>
<tr>
<td>III</td>
<td>IV</td>
</tr>
</tbody>
</table>

There are also the following seldom used functions:

versed sine of $A = \text{vers} A = 1 - \cos A$,

covered sine of $A = \text{vers} \cos A = \text{covers} A = 1 - \sin A$,

exsecant of $A = \text{exsec} A = \sec A - 1$,

haversine of $A = \frac{1}{2} \text{vers} A$.

6.1.5 PERIODICITY RELATIONSHIPS

When $n$ is any integer,

\[
sin (\alpha + n2\pi) = \sin \alpha,
\cos (\alpha + n2\pi) = \cos \alpha, \tag{6.1.2}
\tan (\alpha + n\pi) = \tan \alpha.
\]

6.1.6 SYMMETRY RELATIONSHIPS

\[
sin (-\alpha) = -\sin \alpha, \quad \cos (-\alpha) = +\cos \alpha, \quad \tan (-\alpha) = -\tan \alpha.
\]
6.1.7 SIGNS IN THE FOUR QUADRANTS

<table>
<thead>
<tr>
<th>Quadrant</th>
<th>sin</th>
<th>cos</th>
<th>tan</th>
<th>csc</th>
<th>sec</th>
<th>cot</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>II</td>
<td>+</td>
<td>−</td>
<td>−</td>
<td>+</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td>III</td>
<td>−</td>
<td>−</td>
<td>+</td>
<td>−</td>
<td>−</td>
<td>+</td>
</tr>
<tr>
<td>IV</td>
<td>−</td>
<td>+</td>
<td>−</td>
<td>−</td>
<td>+</td>
<td>−</td>
</tr>
</tbody>
</table>

6.1.8 FUNCTIONS IN TERMS OF ANGLES IN THE FIRST QUADRANT

For $0 \leq \alpha \leq \pi/2$, with $n$ any integer

<table>
<thead>
<tr>
<th>$-\alpha$</th>
<th>$\pi/2 \pm \alpha$</th>
<th>$\pi \pm \alpha$</th>
<th>$3\pi/2 \pm \alpha$</th>
<th>$2n\pi \pm \alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>sin</td>
<td>$-\sin \alpha$</td>
<td>$\cos \alpha$</td>
<td>$\mp \sin \alpha$</td>
<td>$-\cos \alpha$</td>
</tr>
<tr>
<td>cos</td>
<td>$\cos \alpha$</td>
<td>$\mp \sin \alpha$</td>
<td>$-\cos \alpha$</td>
<td>$\pm \sin \alpha$</td>
</tr>
<tr>
<td>tan</td>
<td>$-\tan \alpha$</td>
<td>$\mp \cot \alpha$</td>
<td>$\pm \tan \alpha$</td>
<td>$\mp \cot \alpha$</td>
</tr>
<tr>
<td>csc</td>
<td>$-\csc \alpha$</td>
<td>$\mp \sec \alpha$</td>
<td>$\pm \csc \alpha$</td>
<td>$\pm \sec \alpha$</td>
</tr>
<tr>
<td>sec</td>
<td>$\sec \alpha$</td>
<td>$\mp \csc \alpha$</td>
<td>$-\sec \alpha$</td>
<td>$\pm \csc \alpha$</td>
</tr>
<tr>
<td>cot</td>
<td>$-\cot \alpha$</td>
<td>$\mp \tan \alpha$</td>
<td>$\pm \cot \alpha$</td>
<td>$\mp \tan \alpha$</td>
</tr>
</tbody>
</table>
6.1.9 CIRCULAR FUNCTIONS OF SOME SPECIAL ANGLES

<table>
<thead>
<tr>
<th>Angle</th>
<th>0 = 0°</th>
<th>π/12 = 15°</th>
<th>π/6 = 30°</th>
<th>π/4 = 45°</th>
<th>π/3 = 60°</th>
</tr>
</thead>
<tbody>
<tr>
<td>sin</td>
<td>0</td>
<td>2√2/4(√3 - 1)</td>
<td>1/2</td>
<td>√2/2</td>
<td>√3/2</td>
</tr>
<tr>
<td>cos</td>
<td>1</td>
<td>2√2/4(√3 + 1)</td>
<td>√3/2</td>
<td>√2/2</td>
<td>1/2</td>
</tr>
<tr>
<td>tan</td>
<td>0</td>
<td>2 - √3</td>
<td>√3/3</td>
<td>1</td>
<td>√3</td>
</tr>
<tr>
<td>csc</td>
<td>∞</td>
<td>√2(√3 + 1)</td>
<td>2</td>
<td>√2/2</td>
<td>2√3/3</td>
</tr>
<tr>
<td>sec</td>
<td>1</td>
<td>√2(√3 - 1)</td>
<td>2√3/3</td>
<td>√2</td>
<td>2</td>
</tr>
<tr>
<td>cot</td>
<td>∞</td>
<td>2 + √3</td>
<td>√3</td>
<td>1</td>
<td>√3/3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Angle</th>
<th>5π/12 = 75°</th>
<th>π/2 = 90°</th>
<th>7π/12 = 105°</th>
<th>2π/3 = 120°</th>
</tr>
</thead>
<tbody>
<tr>
<td>sin</td>
<td>2√2/4(√3 + 1)</td>
<td>1</td>
<td>2√2/4(√3 + 1)</td>
<td>1/2</td>
</tr>
<tr>
<td>cos</td>
<td>2√2/4(√3 - 1)</td>
<td>0</td>
<td>-2√2/4(√3 - 1)</td>
<td>-1/2</td>
</tr>
<tr>
<td>tan</td>
<td>2 + √3</td>
<td>∞</td>
<td>-2 + √3</td>
<td>-√3</td>
</tr>
<tr>
<td>csc</td>
<td>√2(√3 - 1)</td>
<td>1</td>
<td>√2(√3 - 1)</td>
<td>2√3/3</td>
</tr>
<tr>
<td>sec</td>
<td>√2(√3 + 1)</td>
<td>∞</td>
<td>-2(√3 + 1)</td>
<td>-2</td>
</tr>
<tr>
<td>cot</td>
<td>2 - √3</td>
<td>0</td>
<td>-(2 - √3)</td>
<td>-√3/3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Angle</th>
<th>3π/4 = 135°</th>
<th>5π/6 = 150°</th>
<th>11π/12 = 165°</th>
<th>π = 180°</th>
</tr>
</thead>
<tbody>
<tr>
<td>sin</td>
<td>√2/2</td>
<td>√3/2</td>
<td>√2(√3 - 1)</td>
<td>0</td>
</tr>
<tr>
<td>cos</td>
<td>-√2/2</td>
<td>-√3/2</td>
<td>-√2(√3 + 1)</td>
<td>-1</td>
</tr>
<tr>
<td>tan</td>
<td>-1</td>
<td>-√3/3</td>
<td>-2(√3 - 1)</td>
<td>0</td>
</tr>
<tr>
<td>csc</td>
<td>√2</td>
<td>2</td>
<td>√2(√3 + 1)</td>
<td>∞</td>
</tr>
<tr>
<td>sec</td>
<td>-√2</td>
<td>-2√3/3</td>
<td>-2√3(√3 - 1)</td>
<td>-1</td>
</tr>
<tr>
<td>cot</td>
<td>-1</td>
<td>-√3</td>
<td>-(2 + √3)</td>
<td>∞</td>
</tr>
</tbody>
</table>

6.1.10 ONE CIRCULAR FUNCTION IN TERMS OF ANOTHER

For 0 ≤ x ≤ π/2,
6.1.11 DEFINITIONS IN TERMS OF EXPONENTIALS

\[
\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad e^{iz} = \cos z + i \sin z.
\]
\[
\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad e^{-iz} = \cos z - i \sin z.
\]
\[
\tan z = \frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}.
\]
6.1.12 FUNDAMENTAL IDENTITIES

Reciprocal relations

\[
\begin{align*}
\sin \alpha &= \frac{1}{\csc \alpha}, & \cos \alpha &= \frac{1}{\sec \alpha}, & \tan \alpha &= \frac{\sin \alpha}{\cos \alpha} = \frac{1}{\cot \alpha}, \\
\csc \alpha &= \frac{1}{\sin \alpha}, & \sec \alpha &= \frac{1}{\cos \alpha}, & \cot \alpha &= \frac{\cos \alpha}{\sin \alpha} = \frac{1}{\tan \alpha}.
\end{align*}
\]

Pythagorean theorem

\[
\begin{align*}
\sin^2 z + \cos^2 z &= 1. \\
\sec^2 z - \tan^2 z &= 1. \\
\csc^2 z - \cot^2 z &= 1.
\end{align*}
\]

Product relations

\[
\begin{align*}
\sin \alpha &= \tan \alpha \cos \alpha, & \cos \alpha &= \cot \alpha \sin \alpha, \\
\tan \alpha &= \sin \alpha \sec \alpha, & \cot \alpha &= \cos \alpha \csc \alpha, \\
\sec \alpha &= \csc \alpha \tan \alpha, & \csc \alpha &= \sec \alpha \cot \alpha.
\end{align*}
\]

Quotient relations

\[
\begin{align*}
\sin \alpha &= \frac{\tan \alpha}{\sec \alpha}, & \cos \alpha &= \cot \alpha \sec \alpha, & \tan \alpha &= \frac{\sin \alpha}{\cos \alpha}, \\
\csc \alpha &= \frac{\sec \alpha}{\tan \alpha}, & \sec \alpha &= \frac{\csc \alpha}{\cot \alpha}, & \cot \alpha &= \frac{\cos \alpha}{\sin \alpha}.
\end{align*}
\]

6.1.13 ANGLE SUM AND DIFFERENCE RELATIONSHIPS

\[
\begin{align*}
\sin (\alpha \pm \beta) &= \sin \alpha \cos \beta \pm \cos \alpha \sin \beta, \\
\cos (\alpha \pm \beta) &= \cos \alpha \cos \beta \mp \sin \alpha \sin \beta, \\
\tan (\alpha \pm \beta) &= \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}, \\
\cot (\alpha \pm \beta) &= \frac{\cot \alpha \cot \beta \mp 1}{\cot \beta \pm \cot \alpha}.
\end{align*}
\]
6.1.14 **DOUBLE ANGLE FORMULAE**

\[
\sin 2\alpha = 2 \sin \alpha \cos \alpha = \frac{2 \tan \alpha}{1 + 2 \tan^2 \alpha}.
\]

\[
\cos 2\alpha = 2 \cos^2 \alpha - 1 = 1 - 2 \sin^2 \alpha = \cos^2 \alpha - \sin^2 \alpha = \frac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha}.
\]

\[
\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}.
\]

\[
\cot 2\alpha = \frac{\cot^2 \alpha - 1}{2 \cot \alpha}.
\]

6.1.15 **MULTIPLE ANGLE FORMULAE**

\[
\sin 3\alpha = -4 \sin^3 \alpha + 3 \sin \alpha.
\]

\[
\sin 4\alpha = -8 \sin^3 \alpha \cos \alpha + 4 \sin \alpha \cos \alpha.
\]

\[
\sin 5\alpha = 16 \sin^5 \alpha - 20 \sin^3 \alpha + 5 \sin \alpha.
\]

\[
\sin 6\alpha = 32 \sin^5 \alpha - 32 \sin \alpha \cos^4 \alpha + 6 \sin \alpha \cos \alpha.
\]

\[
\sin n\alpha = 2 \sin (n - 1)\alpha \cos \alpha - \sin (n - 2)\alpha.
\]

\[
\cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha.
\]

\[
\cos 4\alpha = 8 \cos^4 \alpha - 8 \cos^2 \alpha + 1.
\]

\[
\cos 5\alpha = 16 \cos^5 \alpha - 20 \cos^3 \alpha + 5 \cos \alpha.
\]

\[
\cos 6\alpha = 32 \cos^5 \alpha - 48 \cos^4 \alpha + 18 \cos^2 \alpha - 1.
\]

\[
\cos n\alpha = 2 \cos (n - 1)\alpha \cos \alpha - \cos (n - 2)\alpha.
\]

\[
\tan 3\alpha = \frac{- \tan^3 \alpha + 3 \tan \alpha}{-3 \tan^3 \alpha + 1}.
\]

\[
\tan 4\alpha = \frac{-4 \tan^3 \alpha + 4 \tan \alpha}{\tan^3 \alpha - 6 \tan^2 \alpha + 1}.
\]

\[
\tan n\alpha = \frac{\tan (n - 1)\alpha + \tan \alpha}{- \tan (n - 1)\alpha \tan \alpha + 1}.
\]
6.1.16 HALF ANGLE FORMULAE

\[
\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}
\]
(positive if \(\alpha/2\) is in quadrant I or IV, negative if in II or III).

\[
\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}
\]
(positive if \(\alpha/2\) is in quadrant I or II, negative if in III or IV).

\[
\tan \frac{\alpha}{2} = \frac{1 - \cos \alpha}{\sin \alpha} = \frac{\sin \alpha}{1 + \cos \alpha} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}
\]
(positive if \(\alpha/2\) is in quadrant I or III, negative if in II or IV).

\[
\cot \frac{\alpha}{2} = \frac{1 + \cos \alpha}{\sin \alpha} = \frac{\sin \alpha}{1 - \cos \alpha} = \pm \sqrt{\frac{1 + \cos \alpha}{1 - \cos \alpha}}
\]
(positive if \(\alpha/2\) is in quadrant I or III, negative if in II or IV).

6.1.17 POWERS OF CIRCULAR FUNCTIONS

\[
\sin^2 \alpha = \frac{1}{2} (1 - \cos 2\alpha).
\]
\[
\cos^2 \alpha = \frac{1}{2} (1 + \cos 2\alpha).
\]

\[
\sin^3 \alpha = \frac{1}{4} (-\sin 3\alpha + 3 \sin \alpha).
\]
\[
\cos^3 \alpha = \frac{1}{4} (\cos 3\alpha + 3 \cos \alpha).
\]

\[
\sin^4 \alpha = \frac{1}{8} (3 - 4 \cos 2\alpha + \cos 4\alpha).
\]
\[
\cos^4 \alpha = \frac{1}{8} (3 + 4 \cos 2\alpha + \cos 4\alpha).
\]

\[
\tan^2 \alpha = \frac{1 - \cos 2\alpha}{1 + \cos 2\alpha}.
\]
\[
\cot^2 \alpha = \frac{1 + \cos 2\alpha}{1 - \cos 2\alpha}.
\]

6.1.18 PRODUCTS OF SINE AND COSINE

\[
\cos \alpha \cos \beta = \frac{1}{2} \cos (\alpha - \beta) + \frac{1}{2} \cos (\alpha + \beta).
\]
\[
\sin \alpha \sin \beta = \frac{1}{2} \cos (\alpha - \beta) - \frac{1}{2} \cos (\alpha + \beta).
\]
\[
\sin \alpha \cos \beta = \frac{1}{2} \sin (\alpha - \beta) + \frac{1}{2} \sin (\alpha + \beta).
\]
### 6.1.19 SUMS OF CIRCULAR FUNCTIONS

\[
\begin{align*}
sin \alpha \pm sin \beta &= 2 sin \frac{\alpha \pm \beta}{2} \cos \frac{\alpha \mp \beta}{2}.
&\cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}.
&\cos \alpha - \cos \beta = -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}.
&\tan \alpha \pm \tan \beta = \frac{\sin \alpha \pm \beta}{\cos \alpha \cos \beta}.
&\cot \alpha \pm \cot \beta = \frac{\sin \alpha \pm \alpha}{\sin \alpha \sin \beta}.
&\sin \alpha + \sin \beta = \tan \frac{\alpha + \beta}{2}.
&\sin \alpha - \sin \beta = \tan \frac{\alpha - \beta}{2}.
&\sin \alpha + \sin \beta = \cot \frac{-\alpha + \beta}{2}.
&\cos \alpha - \cos \beta = \tan \frac{\alpha + \beta}{2}.
&\cos \alpha + \cos \beta = \tan \frac{\alpha - \beta}{2}.
\end{align*}
\]

(6.1.4)

### 6.1.20 EVALUATING SINES AND COSINES

The following table is useful for evaluating sines and cosines in multiples of \(\pi\):

<table>
<thead>
<tr>
<th>(n) an integer</th>
<th>(n) even</th>
<th>(n) odd</th>
<th>(\frac{n}{2}) odd</th>
<th>(\frac{n}{2}) even</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sin n\pi)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\cos n\pi)</td>
<td>((-1)^n)</td>
<td>+1</td>
<td>-1</td>
<td>+1</td>
</tr>
<tr>
<td>(\sin \frac{n\pi}{2})</td>
<td>0</td>
<td>((-1)^{(n-1)/2})</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\cos \frac{n\pi}{2})</td>
<td>((-1)^{n/2})</td>
<td>0</td>
<td>-1</td>
<td>+1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(n) odd</th>
<th>(n/2) odd</th>
<th>(n/2) even</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sin n\pi/4)</td>
<td>((-1)^{(n^2+4n+11)/8}/\sqrt{2})</td>
<td>((-1)^{(n-2)/4})</td>
</tr>
</tbody>
</table>

Note the useful formulae (where \(i^2 = -1\))

\[
\begin{align*}
\sin \frac{n\pi}{2} &= \frac{i^{n+1}}{2} \left[ (-1)^n - 1 \right], \text{ and } \\
\cos \frac{n\pi}{2} &= \frac{i^n}{2} \left[ (-1)^n + 1 \right].
\end{align*}
\]

(6.1.5)
6.2 CIRCULAR FUNCTIONS AND PLANAR TRIANGLES

6.2.1 RIGHT TRIANGLES

Let $A$, $B$, and $C$ designate the vertices of a right triangle with $C$ the right angle and $a$, $b$, and $c$ the lengths of the sides opposite the corresponding vertices:

\[
\sin A = \frac{a}{c} = \frac{1}{\csc A}, \\
\cos A = \frac{b}{c} = \frac{1}{\sec A}, \\
\tan A = \frac{a}{b} = \frac{1}{\cot A}.
\]

The Pythagorean theorem states that $a^2 + b^2 = c^2$.

The sum of the interior angles equals $\pi$, i.e., $A + B + C = \pi$.

6.2.2 GENERAL PLANE TRIANGLES

Let $A$, $B$, and $C$ designate the interior angles of a general triangle and let $a$, $b$, and $c$ be the length of the sides opposite those angles.

Radius of the inscribed circle:

\[
r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},
\]

where $s = \frac{1}{2} (a + b + c)$, the semiperimeter.

Radius of the circumscribed circle:

\[
R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C} = \frac{abc}{4 \text{(Area)}}. \tag{6.2.1}
\]

©1996 CRC Press LLC
Law of sines:

\[
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.
\]

Law of cosines:

\[
a^2 = c^2 + b^2 - 2bc \cos A, \quad \cos A = \frac{c^2 + b^2 - a^2}{2bc},
\]
\[
b^2 = a^2 + c^2 - 2ca \cos B, \quad \cos B = \frac{a^2 + c^2 - b^2}{2ca},
\]
\[
c^2 = b^2 + a^2 - 2ab \cos C, \quad \cos C = \frac{b^2 + a^2 - c^2}{2ab}.
\]

Triangle sides in terms of other components:

\[
a = b \cos C + c \cos B, \\
c = a \cos B + b \cos C, \\
b = c \cos B + a \cos A.
\]

Law of tangents:

\[
\frac{a + b}{a - b} = \tan \frac{A + B}{2}, \quad \frac{a + b}{a - b} = \tan \frac{A - B}{2}, \\
\frac{b + c}{b - c} = \tan \frac{B + C}{2}, \quad \frac{b + c}{b - c} = \tan \frac{B - C}{2}, \\
\frac{a + c}{a - c} = \tan \frac{A + C}{2}, \quad \frac{a + c}{a - c} = \tan \frac{A - C}{2}, \quad \text{(6.2.2)}
\]

Area of general triangle:

\[
\text{Area} = \frac{bc \sin A}{2} = \frac{ac \sin B}{2} = \frac{ab \sin C}{2},
\]
\[
= \frac{c^2 \sin A \sin B}{2 \sin C} = \frac{b^2 \sin A \sin C}{2 \sin B} = \frac{a^2 \sin B \sin C}{2 \sin A},
\]
\[
= \sqrt{s(s - a)(s - b)(s - c)} = rs = \frac{abc}{4R} \quad \text{(Heron’s formula)}.
\]
Mollweide’s formulae:

\[
\frac{b - c}{a} = \frac{\sin \frac{1}{2}(B - C)}{\cos \frac{1}{2}A},
\]

\[
\frac{c - a}{b} = \frac{\sin \frac{1}{2}(C - A)}{\cos \frac{1}{2}B},
\]

\[
\frac{a - b}{c} = \frac{\sin \frac{1}{2}(A - B)}{\cos \frac{1}{2}C}.
\]

Newton’s formulae:

\[
\frac{b + c}{a} = \frac{\cos \frac{1}{2}(B - C)}{\sin \frac{1}{2}A},
\]

\[
\frac{c + a}{b} = \frac{\cos \frac{1}{2}(C - A)}{\sin \frac{1}{2}B},
\]

\[
\frac{a + b}{c} = \frac{\cos \frac{1}{2}(A - B)}{\sin \frac{1}{2}C}.
\]

### 6.2.3 HALF ANGLE FORMULAE

\[
\tan \frac{A}{2} = \frac{r}{s - a} \quad \tan \frac{B}{2} = \frac{r}{s - b} \quad \tan \frac{C}{2} = \frac{r}{s - c}
\]

\[
\sin \frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{bc}} \quad \cos \frac{A}{2} = \sqrt{\frac{s(s - a)}{bc}}
\]

\[
\sin \frac{B}{2} = \sqrt{\frac{(s - c)(s - a)}{ca}} \quad \cos \frac{B}{2} = \sqrt{\frac{s(s - b)}{ca}}
\]

\[
\sin \frac{C}{2} = \sqrt{\frac{(s - a)(s - b)}{ab}} \quad \cos \frac{C}{2} = \sqrt{\frac{s(s - c)}{ab}}
\]

### 6.2.4 SOLUTION OF TRIANGLES

A triangle is totally described by specifying any side and two additional parameters: either the remaining two sides, another side and the included angle, or two specified angles. Two angles alone specify the shape of a triangle, but not its size, which requires specification of a side.
Three sides given

Formulae for any one of the angles:

\[
\begin{align*}
\cos A &= \frac{c^2 + b^2 - a^2}{2bc}, & \sin A &= \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}, \\
\sin \frac{A}{2} &= \sqrt{\frac{(s-b)(s-c)}{bc}}, & \cos \frac{A}{2} &= \sqrt{\frac{s(s-a)}{bc}}, \\
\tan \frac{A}{2} &= \frac{\sqrt{(s-b)(s-c)}}{s(s-a)} = \frac{r}{s-a}.
\end{align*}
\]

Given two sides \((b, c)\) and the included angle \((A)\)

See Figure 6.2.2, left. The remaining side and angles can be determined by repeated use of the law of cosines. For example,

Nonlogarithmic solution; perform these steps sequentially:

\[
\begin{align*}
a^2 &= b^2 + c^2 - 2bc \cos A \\
\cos B &= (a^2 + c^2 - b^2)/2ca \\
\cos C &= (a^2 + b^2 - c^2)/2ba
\end{align*}
\]

Logarithmic solution; perform these steps sequentially:

\[
\begin{align*}
B + C &= \pi - A \\
\tan \frac{(B - C)}{2} &= \frac{b - c}{b + c} \tan \frac{(B + C)}{2} \\
B &= \frac{B + C}{2} + \frac{B - C}{2} \\
C &= \frac{B + C}{2} - \frac{B - C}{2} \\
a &= \frac{b \sin A}{\sin B}
\end{align*}
\]
Given two sides \((b, c)\) and an angle \((C)\), not the included angle
See Figure 6.2.2, middle. The remaining angles and side are determined by use of the law of sines, and the fact that the sum of the angles is \(\pi\), \(A + B + C = \pi\).

\[
\sin B = \frac{b \sin C}{c}, \quad A = \pi - B - C, \quad a = \frac{b \sin A}{\sin B}.
\]

Given one side \((b)\) and two angles \((B, C)\)
See Figure 6.2.2, right. The third angle is specified by \(A = \pi - B - C\). The remaining sides are found by

\[
a = \frac{b \sin A}{\sin B}, \quad c = \frac{b \sin C}{\sin B}.
\]

### 6.3 INVERSE CIRCULAR FUNCTIONS

#### 6.3.1 DEFINITION IN TERMS OF AN INTEGRAL

\[
\sin^{-1} z = \int_0^z \frac{dt}{\sqrt{1 - t^2}},
\]

\[
\cos^{-1} z = \int_z^1 \frac{dt}{\sqrt{1 - t^2}} = \frac{\pi}{2} - \sin^{-1} z,
\]

\[
\tan^{-1} z = \int_0^z \frac{dt}{1 + t^2} = \frac{\pi}{2} - \cot^{-1} z,
\]

where \(z\) can be complex. The path of integration must not cross the real axis in the first two cases. In the third case, it must not cross the imaginary axis except possibly inside the unit circle. If \(-1 \leq x \leq 1\), then \(\sin^{-1} x\) and \(\cos^{-1} x\) are real, \(-\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}\), and \(0 \leq \cos^{-1} x \leq \pi\).

\[
csc^{-1} z = \sin^{-1}(1/z),
\]

\[
sec^{-1} z = \cos^{-1}(1/z),
\]

\[
\cot^{-1} z = \tan^{-1}(1/z),
\]

\[
sec^{-1} z + \csc^{-1} z = \pi/2.
\]
6.3.2 FUNDAMENTAL PROPERTIES

The general solutions of the equations \{\sin t = z, \cos t = z, \tan t = z\} are, respectively, (where \(k\) is an arbitrary integer):

\[
\begin{align*}
  t &= \sin^{-1} z = (-1)^k \sin^{-1} z + k\pi, \\
  t &= \cos^{-1} z = \pm \cos^{-1} z + 2k\pi, \\
  t &= \tan^{-1} z = \tan^{-1} z + k\pi, (z^2 \neq -1).
\end{align*}
\]

6.3.3 PRINCIPAL VALUES OF THE INVERSE CIRCULAR FUNCTIONS

The notation \(\sin^{-1} x\) is used to denote any angle whose \(\sin\) is \(x\). The function \(\sin^{-1} x\) is usually used to denote the principal value. Similar notation is used for the other inverse trigonometric functions. The principal values of the inverse trigonometric functions are defined as follows:

1. When \(-1 \leq x \leq 1\), then \(-\pi/2 \leq \sin^{-1} x \leq \pi/2\).
2. When \(-1 \leq x \leq 1\), then \(0 \leq \cos^{-1} x \leq \pi\).
3. When \(-\infty \leq x \leq \infty\), then \(-\pi/2 \leq \tan^{-1} x \leq \pi/2\).
4. When \(1 \leq x\), then \(0 \leq \csc^{-1} x \leq \pi/2\).
   \hspace{1cm} When \(x \leq -1\), then \(-\pi/2 \leq \csc^{-1} x \leq 0\).
5. When \(1 \leq x\), then \(0 \leq \sec^{-1} x \leq \pi/2\).
   \hspace{1cm} When \(x \leq -1\), then \(\pi/2 \leq \sec^{-1} x \leq \pi\).
6. When \(-\infty \leq x \leq \infty\), then \(0 \leq \cot^{-1} x \leq \pi\).

6.3.4 FUNDAMENTAL IDENTITIES

\[
\begin{align*}
\sin^{-1} x + \cos^{-1} x &= \pi/2. \\
\tan^{-1} x + \cot^{-1} x &= \pi/2.
\end{align*}
\]

(6.3.1)

If \(\alpha = \sin^{-1} x\), then

\[
\begin{align*}
  \sin \alpha &= x, & \cos \alpha &= \sqrt{1 - x^2}, & \tan \alpha &= \frac{x}{\sqrt{1 - x^2}}, \\
  \csc \alpha &= \frac{1}{x}, & \sec \alpha &= \frac{1}{\sqrt{1 - x^2}}, & \cot \alpha &= \frac{\sqrt{1 - x^2}}{x}.
\end{align*}
\]

If \(\alpha = \cos^{-1} x\), then

\[
\begin{align*}
  \sin \alpha &= \sqrt{1 - x^2}, & \cos \alpha &= x, & \tan \alpha &= \frac{\sqrt{1 - x^2}}{x}, \\
  \csc \alpha &= \frac{1}{\sqrt{1 - x^2}}, & \sec \alpha &= \frac{1}{x}, & \cot \alpha &= \frac{x}{\sqrt{1 - x^2}}.
\end{align*}
\]
If \( \alpha = \tan^{-1} x \), then

\[
\begin{align*}
\sin \alpha &= \frac{x}{\sqrt{1 + x^2}}, \\
\cos \alpha &= \frac{1}{\sqrt{1 + x^2}}, \\
\tan \alpha &= x, \\
\csc \alpha &= \frac{1}{x}, \\
\sec \alpha &= \sqrt{1 + x^2}, \\
\cot \alpha &= \frac{1}{x}.
\end{align*}
\]

6.3.5 FUNCTIONS OF NEGATIVE ARGUMENTS

\[
\begin{align*}
\sin^{-1} (-z) &= -\sin^{-1} z, \\
\sec^{-1} (-z) &= \pi - \sec^{-1} z, \\
\cos^{-1} (-z) &= \pi - \cos^{-1} z, \\
\csc^{-1} (-z) &= -\csc^{-1} z, \\
\tan^{-1} (-z) &= -\tan^{-1} z, \\
\cot^{-1} (-z) &= \pi - \cot^{-1} z.
\end{align*}
\]

6.3.6 RELATIONSHIP TO INVERSE HYPERBOLIC FUNCTIONS

\[
\begin{align*}
\sin^{-1} z &= -i \sinh^{-1} (iz), \\
\sec^{-1} z &= \pm i \sech^{-1} (iz), \\
\cos^{-1} z &= \pm i \cosh^{-1} (iz), \\
\csc^{-1} z &= i \csch^{-1} (iz), \\
\tan^{-1} z &= -i \tanh^{-1} (iz), \\
\cot^{-1} z &= i \coth^{-1} (iz).
\end{align*}
\]

6.3.7 SUM AND DIFFERENCE OF TWO INVERSE TRIGONOMETRIC FUNCTIONS

\[
\begin{align*}
\sin^{-1} z_1 \pm \sin^{-1} z_2 &= \sin^{-1} \left( z_1 \sqrt{1 - z_2^2} \pm z_2 \sqrt{1 - z_1^2} \right), \\
\cos^{-1} z_1 \pm \cos^{-1} z_2 &= \cos^{-1} \left( z_1 z_2 \mp \sqrt{(1 - z_2^2)(1 - z_1^2)} \right), \\
\tan^{-1} z_1 \pm \tan^{-1} z_2 &= \tan^{-1} \left( \frac{z_1 \pm z_2}{1 \mp z_1 z_2} \right), \\
\sin^{-1} z_1 \pm \cos^{-1} z_2 &= \sin^{-1} \left( z_1 z_2 \pm \sqrt{(1 - z_1^2)(1 - z_2^2)} \right), \\
&= \cos^{-1} \left( z_2 \sqrt{1 - z_1^2} \mp z_1 \sqrt{1 - z_2^2} \right), \\
\tan^{-1} z_1 \pm \cot^{-1} z_2 &= \tan^{-1} \left( \frac{z_1 z_2 \pm 1}{z_2 \mp z_1} \right) = \cot^{-1} \left( \frac{z_2 \mp z_1}{z_1 z_2 \pm 1} \right).
\end{align*}
\]
6.4 SPHERICAL GEOMETRY AND TRIGONOMETRY

6.4.1 RIGHT SPHERICAL TRIANGLES

Let $a$, $b$, and $c$ be the sides of a right spherical triangle with opposite angles $A$, $B$, and $C$, respectively, where each side is measured by the angle subtended at the center of the sphere. Assume that $C = \pi/2 = 90^\circ$ (see Figure 6.4.3). Then,

$$\sin a = \tan b \cot B,$$
$$\sin b = \tan a \cot A,$$
$$\cos A = \tan b \cot c,$$
$$\cos B = \tan a \cot c,$$
$$\cos c = \cos A \cot B,$$
$$\sin a = \sin A \sin c,$$
$$\sin b = \sin B \sin c,$$
$$\cos A = \cos a \sin B,$$
$$\cos B = \cos b \sin A,$$
$$\cos c = \cos a \cos b.$$

**FIGURE 6.4.3**
Right spherical triangle (left) and diagram for Napier’s rule (right).

**Napier’s rules of circular parts**

Arrange the five quantities $a$, $b$, $\text{co-}A$ (complement of $A$), $\text{co-}C$, $\text{co-}B$ of a right spherical triangle with right angle at $C$, in cyclic order as pictured in Figure 6.4.3. If any one of these quantities is designated a middle part, then two of the other parts are adjacent to it, and the remaining two parts are opposite to it. The formulæ above for a right spherical triangle may be recalled by the following two rules:

1. The sine of any middle part is equal to the product of the tangents of the two adjacent parts.
2. The sine of any middle part is equal to the product of the cosines of the two opposite parts.
Rules for determining quadrant
1. A leg and the angle opposite to it are always of the same quadrant.
2. If the hypotenuse is less than 90°, the legs are of the same quadrant.
3. If the hypotenuse is greater than 90°, the legs are of unlike quadrants.

6.4.2 OBLIQUE SPHERICAL TRIANGLES
In the following:
• $a$, $b$, $c$ represent the sides of any spherical triangle.
• $A$, $B$, $C$ represent the corresponding opposite angles.
• $a'$, $b'$, $c'$, $A'$, $B'$, $C'$ are the corresponding parts of the polar triangle.
• $s = (a + b + c)/2$.
• $S = (A + B + C)/2$.
• $\Delta$ is the area of spherical triangle.
• $E$ is the spherical excess of the triangle.
• $R$ is the radius of the sphere upon which the triangle lies.

\[
0^\circ < a + b + c < 360^\circ, \quad 180^\circ < A + B + C < 540^\circ,
\]
\[
E = A + B + C - 180^\circ, \quad \Delta = \pi R^2 E / 180.
\]
\[
\tan \frac{1}{4} E = \sqrt{\tan \frac{s}{2} \tan \frac{1}{2} (s - a) \tan \frac{1}{2} (s - b) \tan \frac{1}{2} (s - c)}.
\]
\[
A = 180^\circ - a', \quad B = 180^\circ - b', \quad C = 180^\circ - c',
\]
\[
a = 180^\circ - A', \quad b = 180^\circ - B', \quad c = 180^\circ - C'.
\]

Spherical law of sines
\[
\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.
\]

Spherical law of cosines for sides
\[
\cos a = \cos b \cos c + \sin b \sin c \cos A,
\]
\[
\cos b = \cos c \cos a + \sin c \sin a \cos B, \quad (6.4.1)
\]
\[
\cos c = \cos a \cos b + \sin a \sin b \cos C.
\]
Spherical law of cosines for angles

\[
\cos A = -\cos B \cos C + \sin B \sin C \cos a,
\]
\[
\cos B = -\cos C \cos A + \sin C \sin A \cos b,
\]
\[
\cos C = -\cos A \cos B + \sin A \sin B \cos c.
\]

Spherical law of tangents

\[
\tan \frac{1}{2}(B - C) = \tan \frac{1}{2}(b - c)
\]
\[
\tan \frac{1}{2}(B + C) = \tan \frac{1}{2}(b + c)
\]
\[
\tan \frac{1}{2}(C - A) = \tan \frac{1}{2}(c - a)
\]
\[
\tan \frac{1}{2}(C + A) = \tan \frac{1}{2}(c + a)
\]
\[
\tan \frac{1}{2}(A - B) = \tan \frac{1}{2}(a - b)
\]
\[
\tan \frac{1}{2}(A + B) = \tan \frac{1}{2}(a + b).
\] (6.4.2)

Spherical half angle formulae

Define \( k^2 = (\tan r)^2 = \frac{\sin(s - a) \sin(s - b) \sin(s - c)}{\sin s} \). Then
\[
\tan \left( \frac{A}{2} \right) = \frac{k}{\sin(s - a)},
\]
\[
\tan \left( \frac{B}{2} \right) = \frac{k}{\sin(s - b)},
\]
\[
\tan \left( \frac{C}{2} \right) = \frac{k}{\sin(s - c)}.
\] (6.4.3)

Spherical half side formulae

Define \( K^2 = (\tan R)^2 = \frac{-\cos S}{\cos(S - A) \cos(S - B) \cos(S - C)} \). Then
\[
\tan(a/2) = K \cos(S - A),
\]
\[
\tan(b/2) = K \cos(S - B),
\]
\[
\tan(c/2) = K \cos(S - C).
\] (6.4.4)

Gauss’s formulae

\[
\frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}c} = \frac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2}C}, \quad \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}c} = \frac{\sin \frac{1}{2}(A+B)}{\cos \frac{1}{2}C},
\]
\[
\frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}c} = \frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2}C}, \quad \frac{\cos \frac{1}{2}(a+b)}{\sin \frac{1}{2}c} = \frac{\cos \frac{1}{2}(A+B)}{\sin \frac{1}{2}C}.
\]
Napier’s analogs

\[
\frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} = \frac{\tan \frac{1}{2}(a - b)}{\tan \frac{1}{2}c}, \quad \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} = \frac{\tan \frac{1}{2}(A - B)}{\cot \frac{1}{2}C}, \quad \frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} = \frac{\tan \frac{1}{2}(a + b)}{\tan \frac{1}{2}c}.
\]

Haversine formulae

\[
hav a = hav(b - c) + \sin b \sin c \hav A.
\]

\[
hav A = \frac{\sin(s - b) \sin(s - c)}{\sin b \sin c}, = \frac{\hav a - hav(b - c)}{\sin b \sin c}, = \hav[180^\circ - (B + C)] + \sin B \sin C \hav a.
\]

Rules for determining quadrant

1. If \( A > B > C \), then \( a > b > c \).
2. A side (angle) which differs by more than 90° from another side (angle) is in the same quadrant as its opposite angle (side).
3. Half the sum of any two sides and half the sum of the opposite angles are in the same quadrant.

Summary of solution of oblique spherical triangles

<table>
<thead>
<tr>
<th>Given</th>
<th>Solution</th>
<th>Check</th>
</tr>
</thead>
<tbody>
<tr>
<td>Three sides</td>
<td>Half-angle formulae</td>
<td>Law of sines</td>
</tr>
<tr>
<td>Three angles</td>
<td>Half-side formulae</td>
<td>Law of sines</td>
</tr>
<tr>
<td>Two sides and included angle</td>
<td>Napier’s analogies (to find sum and difference of unknown angles); then law of sines (to find remaining side).</td>
<td>Gauss’s formulae</td>
</tr>
<tr>
<td>Two angles and included side</td>
<td>Napier’s analogies (to find sum and difference of unknown sides); then law of sines (to find remaining angle).</td>
<td>Gauss’s formulae</td>
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<tr>
<td>Two sides and an opposite angle</td>
<td>Law of sines (to find an angle); then Napier’s analogies (to find remaining angle and side). Note number of solutions.</td>
<td>Gauss’s formulae</td>
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<tr>
<td>Two angles and an opposite side</td>
<td>Law of sines (to find a side); then Napier’s analogies (to find remaining side and angle). Note number of solutions.</td>
<td>Gauss’s formulae</td>
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</table>
### 6.4.3 TABLE OF TRIGONOMETRIC FUNCTIONS

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<tr>
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<th>$\cos x$</th>
<th>$\tan x$</th>
<th>$\cot x$</th>
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</table>
6.5 EXPONENTIAL FUNCTION

6.5.1 EXPONENTIATION

For any real number and $m$ a positive integer, the exponential $a^m$ is defined as

$$a^m = a \cdot a \cdot a \ldots a$$

(6.5.1)

The following three laws of exponents follow:

1. $a^n \cdot a^m = a^{n+m}$.
2. \( \frac{a^m}{a^n} = \begin{cases} \frac{a^{m-n}}{a^n}, & \text{if } m > n, \\ 1, & \text{if } m = n, \\ a^{m-n}, & \text{if } m < n. \end{cases} \)
3. $(a^m)^n = a^{mn}$.

The $n$th root function is defined as the inverse of the $n$th power function:

If $b^n = a$, then $b = \sqrt[n]{a} = a^{(1/n)}$.

If $n$ is odd, there will be a unique real number satisfying the above definition of $\sqrt[n]{a}$, for any real value of $a$. If $n$ is even, for positive values of $a$ there will be two real values for $\sqrt[n]{a}$, one positive and one negative. By convention, the symbol $\sqrt[n]{a}$ means the positive value. If $n$ is even and $a$ is negative, then there are no real values for $\sqrt[n]{a}$.

To extend the definition to include $a^t$ (for $t$ not necessarily an integer), so as to maintain the laws of exponents, the following definitions are required (where we now restrict $a$ to be positive, $p$ to be an odd number, and $q$ to be an even number):

$$a^0 = 1 \quad a^{p/q} = \sqrt[q]{a^p} \quad a^{-t} = \frac{1}{a^t}$$

With these restrictions, the second law of exponents can be written as $\frac{a^m}{a^n} = a^{m-n}$.

If $a > 1$, then the function $a^x$ is monotone increasing while, if $0 < a < 1$ then the function $a^x$ is monotone decreasing.

6.5.2 DEFINITION OF $e^z$

$$\exp(z) = e^z = \lim_{m \to \infty} \left(1 + \frac{z}{m}\right)^m$$

$$= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \ldots$$

If $z = x + iy$, then $e^z = e^x e^{iy} = e^x (\cos y + i \sin y)$. 

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6.5.3 DERIVATIVE AND INTEGRAL OF $e^x$

The derivative of $e^x$ is $e^x$. The integral of $e^x$ is $e^x$.

6.5.4 CIRCULAR FUNCTIONS IN TERMS OF EXPONENTIALS

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad e^{iz} = \cos z + i \sin z.$$  
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad e^{-iz} = \cos z - i \sin z.$$ 

6.6 LOGARITHMIC FUNCTIONS

6.6.1 DEFINITION OF THE NATURAL LOG

The natural logarithm (also known as the Napierian logarithm) of $z$ is written as $\ln z$ or as $\log_e z$. It is sometimes written $\log z$ (this is also used to represent a “generic” logarithm, a logarithm to any base). One definition is

$$\ln z = \int_1^z \frac{dt}{t},$$

where the integration path from 1 to $z$ does not cross the origin or the negative real axis.
For complex values of $z$ the natural logarithm, as defined above, can be represented in terms of its magnitude and phase. If $z = x + iy = re^{i\theta}$, then $\ln z = \ln r + i\theta$, where $r = \sqrt{x^2 + y^2}$, $x = r \cos \theta$, and $y = r \sin \theta$.

### 6.6.2 SPECIAL VALUES

\[
\begin{align*}
\ln 0 &= -\infty, \\
\ln 1 &= 0, \\
\ln e &= 1, \\
\ln (-1) &= i\pi + 2\pi ik, \\
\ln (\pm i) &= \pm \frac{i\pi}{2} + 2\pi ik,
\end{align*}
\]

($e$ is given numerically on page 14).

### 6.6.3 LOGARITHMS TO A BASE OTHER THAN $e$

The logarithmic function to the base $a$, written $\log_a$, is defined as

\[
\log_a z = \frac{\log_b z}{\log_b a} = \frac{\ln z}{\ln a}
\]

Note the properties:

- $\log_a a^p = p$.
- $\log_a b = \frac{1}{\log_b a}$.
- $\log_{10} z = \frac{\ln z}{\ln 10} = (\log_{10} e) \ln z \approx (0.4342944819 \ldots) \ln z$.
- $\ln z = (\ln 10) \log_{10} z \approx (2.3025850929 \ldots) \log_{10} z$.

### 6.6.4 RELATIONSHIP OF THE LOGARITHM TO THE EXPONENTIAL

For real values of $z$ the logarithm is a monotonic function, as is the exponential. Any monotonic function has a single-valued inverse function; the natural logarithm is the inverse of the exponential. If $x = e^y$, then $y = \ln x$, and $x = e^{\ln x}$. The same inverse relations exist for bases other than $e$. For example, if $u = a^w$, then $w = \log_a u$, and $u = a^{\log_a u}$.
6.6.5 IDENTITIES

\[
\begin{align*}
\log_a z_1 z_2 &= \log_a z_1 + \log_a z_2, & \text{for } (-\pi < \arg z_1 + \arg z_2 < \pi). \\
\log_a \frac{z_1}{z_2} &= \log_a z_1 - \log_a z_2, & \text{for } (-\pi < \arg z_1 - \arg z_2 < \pi). \\
\log_a z^n &= n \log_a z, & \text{for } (-\pi < n \arg z < \pi), \text{ when } n \text{ is an integer.}
\end{align*}
\]

6.6.6 SERIES EXPANSIONS FOR THE NATURAL LOGARITHM

\[
\begin{align*}
\ln(1 + z) &= z - \frac{1}{2} z^2 + \frac{1}{3} z^3 - \ldots, & \text{for } |z| < 1. \\
\ln z &= \left(\frac{z - 1}{z}\right) + \frac{1}{2} \left(\frac{z - 1}{z}\right)^2 + \frac{1}{3} \left(\frac{z - 1}{z}\right)^3 + \ldots, & \text{for } \Re z \geq \frac{1}{2}.
\end{align*}
\]

6.6.7 DERIVATIVE AND INTEGRATION FORMULAE

\[
\frac{d\ln z}{dz} = \frac{1}{z}, \quad \int \frac{dz}{z} = \ln z, \quad \int \ln z \, dz = z \ln z - z.
\]

6.7 HYPERBOLIC FUNCTIONS

6.7.1 DEFINITIONS OF THE HYPERBOLIC FUNCTIONS

\[
\begin{align*}
\sinh z &= \frac{e^z - e^{-z}}{2}, \\
\cosh z &= \frac{e^z + e^{-z}}{2}, \\
\tanh z &= \frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{\sinh z}{\cosh z}, \\
\coth z &= \frac{1}{\tanh z}.
\end{align*}
\]

The curve \( y = \cosh x \) is called a catenary.
When $z = x + iy$,

\[
\begin{align*}
\sinh z &= \sinh x \cos y + i \cosh x \sin y, \\
\cosh z &= \cosh x \cos y + i \sinh x \sin y, \\
\tanh z &= \frac{\sinh 2x + i \sin 2y}{\cosh 2x + \cos 2y}, \\
\coth z &= \frac{\sinh 2x - i \sin 2y}{\cosh 2x - \cos 2y}.
\end{align*}
\]
6.7.2 RANGE OF VALUES

<table>
<thead>
<tr>
<th>Function</th>
<th>Domain (interval of ( u ))</th>
<th>Range (interval of function)</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
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<td>( (-\infty, +\infty) )</td>
<td></td>
</tr>
<tr>
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<td>( [1, +\infty) )</td>
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<td>( (-\infty, 0) )</td>
<td>( (0, -\infty) )</td>
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<td>( (+\infty, 1) )</td>
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6.7.3 SERIES EXPANSIONS

\[
\begin{align*}
cosh z &= 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \ldots, \quad |z| < \infty. \\
\sinh z &= z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \ldots, \quad |z| < \infty. \\
tanh z &= z - \frac{z^3}{3} + \frac{2z^5}{15} + \ldots, \quad |z| < \frac{\pi}{2}.
\end{align*}
\]

6.7.4 SYMMETRY RELATIONSHIPS

\[\cosh (-z) = + \cosh z, \quad \sinh (-z) = - \sinh z, \quad \tanh (-z) = - \tanh z.\]

6.7.5 INTERRELATIONSHIPS AMONG THE HYPERBOLIC FUNCTIONS

\[e^z = \cosh z + \sinh z, \quad e^{-z} = \cosh z - \sinh z,\]

\[(\cosh z)^2 - (\sinh z)^2 = (\tanh z)^2 + (\sech z)^2 = (\coth z)^2 - (\csch z)^2 = 1.\]

6.7.6 RELATIONSHIP TO CIRCULAR FUNCTIONS

\[\cosh z = \cos iz, \quad \sinh z = -i \sin iz, \quad \tanh z = -i \tan iz.\]

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### 6.7.7 Hyperbolic Functions in Terms of One Another

<table>
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<tr>
<th>Function</th>
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<th>cosh ( x )</th>
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<tr>
<td>cosh ( x ) = ( \sqrt{1 + (\sinh x)^2} )</td>
<td>cosh ( x )</td>
<td>tanh ( x )</td>
<td>±( \frac{1}{\sqrt{1 - (\tanh x)^2}} )</td>
</tr>
<tr>
<td>tanh ( x ) = ( \frac{\sinh x}{\sqrt{1 + (\sinh x)^2}} )</td>
<td>±( \frac{\sqrt{(\cosh x)^2 - 1}}{\cosh x} )</td>
<td>tanh ( x )</td>
<td>tanh ( x )</td>
</tr>
<tr>
<td>csch ( x ) = ( \frac{1}{\sinh x} )</td>
<td>±( \frac{1}{\sqrt{(\cosh x)^2 - 1}} )</td>
<td>tanh ( x )</td>
<td>tanh ( x )</td>
</tr>
<tr>
<td>sech ( x ) = ( \frac{1}{\sqrt{1 + (\sinh x)^2}} )</td>
<td>tanh ( x )</td>
<td>tanh ( x )</td>
<td>±( \sqrt{1 - (\tanh x)^2} )</td>
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<tr>
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</tr>
</tbody>
</table>

<table>
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<tr>
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<th>sech ( x )</th>
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</tr>
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<tr>
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<td>( \frac{1}{\coth x} )</td>
</tr>
<tr>
<td>cosh ( x ) = ( \frac{\sqrt{(\cosh x)^2 + 1}}{\cosh x} )</td>
<td>( \frac{1}{\tanh x} )</td>
<td>( \frac{1}{\cosh x} )</td>
<td>( \frac{1}{\cosh x} )</td>
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<tr>
<td>tanh ( x ) = ( \frac{1}{\sqrt{(\cosh x)^2 + 1}} )</td>
<td>±( \frac{\sqrt{(\tanh x)^2 - 1}}{\tanh x} )</td>
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<td>( \frac{1}{\coth x} )</td>
</tr>
<tr>
<td>csch ( x ) = ( \frac{\cosh x}{\sqrt{(\cosh x)^2 + 1}} )</td>
<td>( \frac{1}{\tanh x} )</td>
<td>( \frac{1}{\cosh x} )</td>
<td>( \frac{1}{\cosh x} )</td>
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<tr>
<td>sech ( x ) = ( \frac{\sqrt{(\cosh x)^2 + 1}}{(\cosh x)^2 + 1} )</td>
<td>±( \frac{\sqrt{(\tanh x)^2 - 1}}{\tanh x} )</td>
<td>( \frac{1}{\coth x} )</td>
<td>( \frac{1}{\coth x} )</td>
</tr>
<tr>
<td>coth ( x ) = ( \frac{\sqrt{(\cosh x)^2 + 1}}{(\cosh x)^2 + 1} )</td>
<td>±( \frac{\sqrt{(\tanh x)^2 - 1}}{\tanh x} )</td>
<td>( \frac{1}{\coth x} )</td>
<td>( \frac{1}{\coth x} )</td>
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</table>

### 6.7.8 Sum and Difference Formulae

\[
\begin{align*}
\cosh (z_1 \pm z_2) &= \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2, \\
\sinh (z_1 \pm z_2) &= \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2, \\
\tanh (z_1 \pm z_2) &= \frac{\tanh z_1 \pm \tanh z_2}{1 \pm \tanh z_1 \tanh z_2} = \frac{\sinh 2z_1 \pm \sinh 2z_2}{\cosh 2z_1 \pm \cosh 2z_2}, \\
\coth (z_1 \pm z_2) &= \frac{1 \pm \coth z_1 \coth z_2}{\coth z_1 \pm \coth z_2} = \frac{\sinh 2z_1 \mp \sinh 2z_2}{\cosh 2z_1 - \cosh 2z_2}.
\end{align*}
\]
6.7.9 MULTIPLE ARGUMENT RELATIONS

\[ \sinh 2\alpha = 2 \sinh \alpha \cosh \alpha = \frac{2 \tanh \alpha}{1 - \tanh^2 \alpha}. \]
\[ \sinh 3\alpha = +3 \sinh \alpha + 4 \sinh^3 \alpha = \sinh \alpha (4 \cosh^2 \alpha - 1). \]
\[ \sinh 4\alpha = 4 \sinh^3 \alpha \cosh \alpha + 4 \cosh \alpha \sinh \alpha. \]
\[ \cosh 2\alpha = \cosh^2 \alpha + \sinh^2 \alpha = 2 \cosh^2 \alpha - 1, \]
\[ = 1 + 2 \sinh^2 \alpha = \frac{1 + \tanh^2 \alpha}{1 - \tanh^2 \alpha}. \]
\[ \cosh 3\alpha = -3 \cosh \alpha + 4 \cosh^3 \alpha = \cosh \alpha (4 \sinh^2 \alpha + 1). \]
\[ \cosh 4\alpha = \cosh^4 \alpha + 6 \sinh^2 \alpha \cosh^2 \alpha + 6 \sinh^4 \alpha. \]
\[ \tanh 2\alpha = \frac{2 \tanh \alpha}{1 + \tanh^2 \alpha}. \]
\[ \tanh 3\alpha = \frac{3 \tanh \alpha + \tanh^3 \alpha}{1 + 3 \tanh^2 \alpha}. \]
\[ \coth 2\alpha = \frac{1 + \coth^2 \alpha}{2 \coth \alpha}. \]
\[ \coth 3\alpha = \frac{3 \coth \alpha + \coth^3 \alpha}{1 + 3 \coth^2 \alpha}. \]

6.7.10 SUMS OF FUNCTIONS

\[ \sinh u \pm \sinh w = 2 \sinh \frac{u \pm w}{2} \cosh \frac{u \mp w}{2}, \]
\[ \cosh u + \cosh w = 2 \cosh \frac{u + w}{2} \cosh \frac{u - w}{2}, \]
\[ \cosh u - \cosh w = 2 \sinh \frac{u + w}{2} \sinh \frac{u - w}{2}, \]
\[ \tanh u \pm \tanh w = \frac{\sinh u \pm w}{\cosh u \cosh w}, \]
\[ \coth u \pm \coth w = \frac{\sinh u \pm w}{\sinh u \sinh w}. \]

6.7.11 PRODUCTS OF FUNCTIONS

\[ \sinh u \sinh w = \frac{1}{2} (\cosh(u + w) - \cosh(u - w)), \]
\[ \sinh u \cosh w = \frac{1}{2} (\sinh(u + w) + \sinh(u - w)), \]
\[ \cosh u \cosh w = \frac{1}{2} (\cosh(u + w) + \cosh(u - w)). \]
6.7.12 HALF–ARGUMENT FORMULAE

\[
\sinh \frac{z}{2} = \pm \sqrt{\frac{\cosh z - 1}{2}}, \\
\cosh \frac{z}{2} = \pm \sqrt{\frac{\cosh z + 1}{2}}, \\
\tanh \frac{z}{2} = \pm \sqrt{\frac{\cosh z - 1}{\cosh z + 1}} = \frac{\sinh z}{\cosh z + 1}, \\
\coth \frac{z}{2} = \pm \sqrt{\frac{\cosh z + 1}{\cosh z - 1}} = \frac{\sinh z}{\cosh z - 1}.
\]

6.7.13 DIFFERENTIATION FORMULAE

\[
\frac{d\sinh z}{dz} = \cosh z, \\
\frac{d\cosh z}{dz} = \sinh z, \\
\frac{dtanh z}{dz} = (\sech z)^2, \\
\frac{d\sech z}{dz} = -\sech z \tanh z, \\
\frac{dcsech z}{dz} = -\csch z \coth z, \\
\frac{d\coth z}{dz} = -(\csch z)^2.
\]

6.8 INVERSE HYPERBOLIC FUNCTIONS

When \(z = x + iy\),

\[
\cosh^{-1} z = \int_{0}^{z} \frac{dt}{\sqrt{t^2 - 1}}, \\
\sinh^{-1} z = \int_{0}^{z} \frac{dt}{\sqrt{1 + t^2}}, \\
\tanh^{-1} z = \int_{0}^{z} \frac{dt}{1 - t^2}.
\]

6.8.1 RANGE OF VALUES

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<tr>
<th>Function</th>
<th>Domain</th>
<th>Range</th>
<th>Remarks</th>
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<td>((-\infty, +\infty))</td>
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<td>(\cosh^{-1} u)</td>
<td>([1, +\infty))</td>
<td>((-\infty, +\infty))</td>
<td>Even function, double valued</td>
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<td>((-\infty, +\infty))</td>
<td>Odd function</td>
</tr>
<tr>
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<td>((0, -\infty), (\infty, 0))</td>
<td>Odd function, two branches, Pole at (u = 0)</td>
</tr>
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<td>((-\infty, +\infty))</td>
<td>Double valued</td>
</tr>
<tr>
<td>(\coth^{-1} u)</td>
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<td>Odd function, two branches</td>
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</table>
### 6.8.2 RELATIONSHIPS AMONG INVERSE HYPERBOLIC FUNCTIONS

<table>
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<td>(\cosh^{-1} x)</td>
<td>(\pm \sinh^{-1} \frac{\sqrt{x^2 - 1}}{x})</td>
<td>(\cosh^{-1} x)</td>
<td>(\pm \tanh^{-1} \frac{\sqrt{x^2 - 1}}{x})</td>
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<tr>
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<td>(\pm \cosh^{-1} \frac{1}{\sqrt{1 - x^2}})</td>
<td>(\tanh^{-1} \frac{1}{\sqrt{1 + x^2}})</td>
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<td>(\pm \cosh^{-1} \frac{1}{\sqrt{x^2 - 1}})</td>
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<tr>
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<tr>
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<td>(\pm \coth^{-1} \frac{x}{\sqrt{x^2 - 1}})</td>
</tr>
<tr>
<td>(\tanh^{-1} x)</td>
<td>(\csch^{-1} \frac{1}{\sqrt{1 - x^2}})</td>
<td>(\pm \sech^{-1} \frac{1}{\sqrt{1 - x^2}})</td>
<td>(\coth^{-1} \frac{1}{x})</td>
</tr>
<tr>
<td>(\csch^{-1} x)</td>
<td>(\csch^{-1} x)</td>
<td>(\pm \sech^{-1} \frac{x}{\sqrt{1 + x^2}})</td>
<td>(\coth^{-1} \frac{\sqrt{x^2 + 1}}{x})</td>
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<tr>
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<td>(\pm \csch^{-1} \frac{x}{\sqrt{1 - x^2}})</td>
<td>(\sech^{-1} x)</td>
<td>(\pm \coth^{-1} \frac{1}{\sqrt{1 - x^2}})</td>
</tr>
<tr>
<td>(\coth^{-1} x)</td>
<td>(\csch^{-1} \frac{1}{\sqrt{x^2 - 1}})</td>
<td>(\sech^{-1} \frac{\sqrt{x^2 - 1}}{x})</td>
<td>(\coth^{-1} x)</td>
</tr>
</tbody>
</table>
6.8.3 RELATIONSHIPS WITH LOGARITHMIC FUNCTIONS

\[
\begin{align*}
\sinh^{-1} x &= \log \left( x + \sqrt{x^2 + 1} \right), & \cosh^{-1} x &= \log \left( \frac{1 + \sqrt{1 + x^2}}{x} \right), \\
\cosh^{-1} x &= \log \left( x \pm \sqrt{x^2 - 1} \right), & \text{sech}^{-1} x &= \log \left( \frac{1 \pm \sqrt{1 - x^2}}{x} \right), \\
\tanh^{-1} x &= \frac{1}{2} \log \left( \frac{1 + x}{1 - x} \right), & \coth^{-1} x &= \frac{1}{2} \log \left( \frac{x + 1}{x - 1} \right).
\end{align*}
\]

6.8.4 RELATIONSHIPS WITH CIRCULAR FUNCTIONS

\[
\begin{align*}
\sinh^{-1} x &= -i \sin^{-1} ix, & \sinh^{-1} ix &= +i \sin^{-1} x, \\
\cosh^{-1} x &= \pm i \cos^{-1} ix, & \cosh^{-1} ix &= \pm i \cos^{-1} x, \\
\tanh^{-1} x &= -i \tan^{-1} ix, & \tanh^{-1} ix &= +i \tan^{-1} x, \\
\text{csch}^{-1} x &= +i \csc^{-1} ix, & \text{csch}^{-1} ix &= -i \csc^{-1} x, \\
\text{sech}^{-1} x &= \pm i \sec^{-1} ix, & \text{sech}^{-1} ix &= \pm i \sec^{-1} x, \\
\cot^{-1} x &= +i \cot^{-1} ix, & \cot^{-1} ix &= -i \cot^{-1} x.
\end{align*}
\]

6.8.5 SUM AND DIFFERENCE OF FUNCTIONS

\[
\begin{align*}
\sinh^{-1} x \pm \sinh^{-1} y &= \sinh^{-1} x \sqrt{1 + y^2} \pm y \sqrt{1 + x^2}, \\
\cosh^{-1} x \pm \cosh^{-1} y &= \cosh^{-1} xy \pm \sqrt{(y^2 - 1)(x^2 - 1)}, \\
\tanh^{-1} x \pm \tanh^{-1} y &= \tanh^{-1} \frac{x \pm y}{xy \pm 1}, \\
\sinh^{-1} x \pm \cosh^{-1} y &= \sinh^{-1} y \sqrt{1 + x^2} \pm x \sqrt{y^2 - 1}, \\
\cosh^{-1} x \pm \cosh^{-1} y &= \cosh^{-1} \left( \frac{xy + 1}{y \pm x} \right), \\
\tanh^{-1} x \pm \coth^{-1} y &= \tanh^{-1} \left( \frac{y \pm x}{xy \pm 1} \right).
\end{align*}
\]
6.9 GUDERMANNIAN FUNCTION

This function relates circular and hyperbolic functions without the use of functions of imaginary argument. The Gudermannian is a monotonic odd function which is asymptotic to $\pm \frac{\pi}{2}$ at $x = \pm \infty$. It is zero at the origin.

\[ \text{gd} x = \text{the Gudermannian of } x \]
\[ = \int_0^x \frac{dt}{\cosh t} = 2 \tan^{-1} \left( \tanh \frac{x}{2} \right) = 2 \tan^{-1} e^x - \frac{\pi}{2}. \]

\[ \text{gd}^{-1} x = \text{the inverse Gudermannian of } x \]
\[ = \int_0^x \frac{dt}{\cos t} = \log \left[ \tan \left( \frac{\pi}{4} + \frac{x}{2} \right) \right] = \log (\sec x + \tan x). \]

If \( \text{gd}(x + iy) = \alpha + i\beta \), then

\[ \tan \alpha = \frac{\sinh x}{\cos y}, \quad \tanh x = \frac{\sin \alpha}{\cosh \beta}, \]
\[ \tan \beta = \frac{\sin y}{\cosh x}, \quad \tan y = \frac{\sin \beta}{\cosh \alpha}. \]
6.9.1 FUNDAMENTAL IDENTITIES

\[
\tanh \left( \frac{x}{2} \right) = \tan \left( \frac{\text{gd} \, x}{2} \right),
\]
\[
e^x = \cosh x + \sinh x = \sec \, \text{gd} \, x + \tan \, \text{gd} \, x,
\]
\[
= \tan \left( \frac{\pi}{4} + \frac{\text{gd} \, x}{2} \right) = \frac{1 + \sin \, (\text{gd} \, x)}{\cos(\text{gd} \, x)},
\]
\[
i \, \text{gd}^{-1} \, x = \text{gd}^{-1}(ix), \quad \text{where} \quad i = \sqrt{-1}.
\]

6.9.2 DERIVATIVES OF GUDERMANNIAN

\[
\frac{d(\text{gd} \, x)}{dx} = \text{sech} \, x \quad \frac{d(\text{gd}^{-1} \, x)}{dx} = \sec \, x.
\]

6.9.3 RELATIONSHIP TO HYPERBOLIC AND CIRCULAR FUNCTIONS

\[
\sinh x = \tan \, (\text{gd} \, x), \quad \csc h \, x = \cot \, (\text{gd} \, x),
\]
\[
\cosh x = \sec \, (\text{gd} \, x), \quad \sec h \, x = \cos \, (\text{gd} \, x),
\]
\[
\tanh x = \sin \, (\text{gd} \, x), \quad \coth \, x = \cosec \, (\text{gd} \, x).
\]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( e^x )</th>
<th>( \log x )</th>
<th>( \text{gd} , x )</th>
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6.10 ORTHOGONAL POLYNOMIALS

6.10.1 HERMITE POLYNOMIALS
Symbol: $H_n(x)$.
Interval: $[-\infty, \infty]$.
Differential Equation: $y'' - 2xy' + 2ny = 0$.
Explicit Expression: $H_n(x) = \sum_{m=0}^{[n/2]} (-1)^m n! (2x)^{n-2m} / m!(n-2m)!$.
Recurrence Relation: $H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x)$.
Weight: $e^{-x^2}$.
Standardization: $H_n(x) = 2^n x^n + \ldots$.
Norm: $\int_{-\infty}^{\infty} e^{-x^2} [H_n(x)]^2 \, dx = 2^n n! \sqrt{\pi}$.
Rodrigues' Formula: $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2})$.
Generating Function: $\sum_{n=0}^{\infty} H_n(x) \frac{z^n}{n!} = e^{xz + 2xz}$.
Inequality: $|H_n(x)| < 2^n e^{x^2} n!$.

6.10.2 JACOBI POLYNOMIALS
Symbol: $P_n^{(\alpha, \beta)}(x)$.
Interval: $[-1, 1]$.
Differential Equation: $(1 - x^2)y'' + [\beta - \alpha - (\alpha + \beta + 2)x]y' + n(n + \alpha + \beta + 1)y = 0$.
Explicit Expression:
$$P_n^{(\alpha, \beta)}(x) = \frac{1}{2^n} \sum_{m=0}^{n} \binom{n + \alpha}{m} \binom{n + \beta}{n - m} (x - 1)^{n-m} (x + 1)^m.$$ Recurrence Relation: $2(n+1)(n + \alpha + \beta + 1)(2n + \alpha + \beta) P_{n+1}^{(\alpha, \beta)}(x) = (2n + \alpha + \beta + 1)(\alpha^2 - \beta^2) + (2n + \alpha + \beta + 2)(2n + \alpha + \beta + 1)x P_{n}^{(\alpha, \beta)}(x) - 2(n + \alpha)(n + \beta)(2n + \alpha + \beta + 2) P_{n-1}^{(\alpha, \beta)}(x)$.
Weight: $(1 - x)^{\alpha}(1 + x)^{\beta}$.
Standardization: $P_n^{(\alpha, \beta)}(1) = \binom{n + \alpha}{n}$.
Norm:
$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} [P_n^{(\alpha, \beta)}(x)]^2 \, dx = \frac{2^{n+\beta+1} \Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)}{(2n + \alpha + \beta + 1)n! \Gamma(n + \alpha + \beta + 1)}.$$
Rodrigues’ Formula:
\[ p_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n! (1 - x)^{\alpha} (1 + x)^{\beta}} \frac{d^n}{dx^n} \left[ (1 - x)^{n+\alpha} (1 + x)^{n+\beta} \right]. \]

Generating Function:
\[ \sum_{n=0}^{\infty} p_n^{(\alpha, \beta)}(x) z^n = 2^{\alpha+\beta} R^{-1} (1 - z + R)^{-\alpha} (1 + z + R)^{-\beta}, \]
where \( R = \sqrt{1 - 2xz + z^2} \) and \(|z| < 1\).

Inequality:
\[ \max_{-1 \leq x \leq 1} |p_n^{(\alpha, \beta)}(x)| = \begin{cases} \binom{n+q}{n} \sim n^q, & \text{if } q = \max(\alpha, \beta) \geq -\frac{1}{2}, \\ \binom{p_n^{(\alpha, \beta)}(x')}{n} \sim n^{-1/2}, & \text{if } q = \max(\alpha, \beta) < -\frac{1}{2}, \end{cases} \]
where \( \alpha, \beta > 1 \) and \( x' \) (in the second result) is one of the two maximum points nearest \((\beta - \alpha)/(\alpha + \beta + 1)\).

### 6.10.3 LAGUERRE POLYNOMIALS

Symbol: \( L_n(x) \).
Interval: \([0, \infty]\).

\( L_n(x) \) is the same as \( L_n^{(0)}(x) \) (see the generalized Laguerre polynomials).

### 6.10.4 GENERALIZED LAGUERRE POLYNOMIALS

Symbol: \( L_n^{(\alpha)}(x) \).
Interval: \([0, \infty]\).

Differential Equation:
\[ xy'' + (\alpha + 1 - x)y' + ny = 0. \]

Explicit Expression:
\[ L_n^{(\alpha)}(x) = \sum_{m=0}^{n} \frac{(-1)^m}{m!} \binom{n+\alpha}{n-m} x^m. \]

Recurrence Relation:
\[ (n + 1) L_{n+1}^{(\alpha)}(x) = [(2n + \alpha + 1) - x] L_n^{(\alpha)}(x) - (n + \alpha) L_{n-1}^{(\alpha)}(x). \]

Weight:
\[ x^{\alpha} e^{-x}. \]

Standardization:
\[ L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!} x^n + \ldots. \]

Norm:
\[ \int_0^{\infty} x^{\alpha} e^{-x} \left[ L_n^{(\alpha)}(x) \right]^2 dx = \frac{\Gamma(n + \alpha + 1)}{n!}. \]

Rodrigues’ Formula:
\[ L_n^{(\alpha)}(x) = \frac{1}{n! x^{\alpha} e^{-x}} \frac{d^n}{dx^n} [x^{n+\alpha} e^{-x}]. \]

Generating Function:
\[ \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) z^n = (1 - z)^{-\alpha-1} \exp \left( \frac{xz}{z - 1} \right). \]

Inequality:
\[ |L_n^{(\alpha)}(x)| \leq \begin{cases} \frac{\Gamma(n+\alpha+1)}{n! \Gamma(\alpha+1)} e^{x/2}, & \text{if } x \geq 0 \text{ and } \alpha > 0, \\ \frac{2 - \frac{\Gamma(n+\alpha+1)}{n! \Gamma(\alpha+1)}}{e^{x/2}}, & \text{if } x \geq 0 \text{ and } -1 < \alpha < 0, \end{cases} \]

Note that \( L_n^{(m)}(x) = (-1)^m \frac{d^m}{dx^m} \left[ L_n^{(m)}(x) \right]. \)

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6.10.5 LEGENDRE POLYNOMIALS

Symbol: \( P_n(x) \).
Interval: \([-1, 1]\).
Differential Equation: \((1 - x^2)y'' - 2xy' + n(n+1)y = 0\).

Explicit Expression: \( P_n(x) = \frac{1}{2^n} \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^m \binom{n}{m} \binom{2n-2m}{n} x^{n-2m} \).

Recurrence Relation: \((n+1)P_{n+1}(x) = (2n+1)x P_n(x) - n P_{n-1}(x)\).

Weight: 1.

Standardization: \( P_n(1) = 1 \).

Norm: \( \int_{-1}^{1} [P_n(x)]^2 \, dx = \frac{2}{2n+1} \).

Rodrigues’ Formula: \( P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} [(1-x^2)^n] \).

Generating Function: \( \sum_{n=0}^{\infty} P_n(x) z^n = (1 - 2xz + z^2)^{-1/2}, \) and \(-1 < x < 1, |z| < 1\).

Inequality: \(|P_n(x)| \leq 1\) for \(-1 \leq x \leq 1\).

The Legendre polynomials satisfy \( \int_{-1}^{1} P_{n}(x) P_{m}(x) \, dx = \frac{2}{2m+1} \delta_{nm} \).

The Legendre series representation is
\[
f(x) = \sum_{n=0}^{\infty} A_n P_n(x), \quad A_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) \, dx. \tag{6.10.1}
\]

The associated Legendre functions \( P_\ell^m(x) \) are
\[
P_\ell^m(x) = (-1)^m (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x), \quad \ell \geq m.
\]

6.10.6 TSCHEBYSHEFF POLYNOMIALS, FIRST KIND

Symbol: \( T_n(x) \).
Interval: \([-1, 1]\).
Differential Equation: \((1 - x^2)y'' - xy' + n^2 y = 0\).

Explicit Expression: \( T_n(x) = \cos (n \cos^{-1} x) \),
\[
= \frac{n}{2} \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^m \frac{(n-m-1)!}{m!(n-2m)!} (2x)^{n-2m}.
\]

Recurrence Relation: \( T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x)\).

Weight: \( (1 - x^2)^{-1/2} \).

Standardization: \( T_n(1) = 1 \).
Norm: \[ \int_{-1}^{1} (1-x^2)^{-1/2} [T_n(x)]^2 \, dx = \begin{cases} \pi, & n = 0, \\ \pi/2, & n \neq 0. \end{cases} \]

Rodrigues' Formula: \[ T_n(x) = \frac{\sqrt{\pi} (1-x^2)^{n-1/2}}{(-2)^n n! (n+\frac{1}{2})} \frac{d^n}{dx^n} [(1-x^2)^n]. \]

Generating Function: \[ \sum_{n=0}^{\infty} T_n(x) z^n = \frac{1}{1-2xz+z^2}, \text{ for } -1 < x < 1 \text{ and } |z| < 1. \]

Inequality: \[ |T_n(x)| \leq n+1, \text{ for } -1 \leq x \leq 1. \]

Note that \[ T_n(x) = \frac{n!}{\Gamma(n+\frac{1}{2})} P_n^{(1/2,-1/2)}(x). \]

### 6.10.7 TCHEBYSHEFF POLYNOMIALS, SECOND KIND

Symbol: \( U_n(x) \).

Interval: \([-1, 1]\).

Differential Equation: \((1-x^2) y'' - 3xy' + n(n+2)y = 0.\)

Explicit Expression: \( U_n(x) = \sum_{m=0}^{[n/2]} (-1)^m (m-n)! (2x)^{n-2m} \frac{m!}{m!(n-2m)!} \)

\[ U_n(\cos \theta) = \frac{\sin((n+1) \theta)}{\sin \theta}. \]

Recurrence Relation: \( U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x). \)

Weight: \((1-x^2)^{1/2}.\)

Standardization: \( U_n(1) = n+1. \)

Norm: \[ \int_{-1}^{1} (1-x^2)^{1/2} [U_n(x)]^2 \, dx = \pi^2. \]

Rodrigues’ Formula: \[ U_n(x) = \frac{(-1)^n (n+1) \sqrt{\pi}}{(1-x^2)^{1/2} 2^{n+1} \Gamma(n+\frac{3}{2})} \frac{d^n}{dx^n} [(1-x^2)^n]. \]

Generating Function: \[ \sum_{n=0}^{\infty} U_n(x) z^n = \frac{1}{1-2xz+z^2}, \text{ for } -1 < x < 1 \text{ and } |z| < 1. \]

Inequality: \[ |U_n(x)| \leq n+1, \text{ for } -1 \leq x \leq 1. \]

Note that \( U_n(x) = \frac{(n+1)! \sqrt{\pi}}{2^{n} \Gamma(n+\frac{3}{2})} P_n^{(1,2,1/2)}(x). \)
6.10.8 TABLES OF ORTHOGONAL POLYNOMIALS

$$\begin{align*}
H_0 &= 1 \\
H_1 &= 2x \\
H_2 &= 4x^2 - 2 \\
H_3 &= 8x^3 - 12x \\
H_4 &= 16x^4 - 48x^2 + 12 \\
H_5 &= 32x^5 - 160x^3 + 120x \\
H_6 &= 64x^6 - 480x^4 + 720x^2 - 120 \\
H_7 &= 128x^7 - 1344x^5 + 3360x^3 - 1680x \\
H_8 &= 256x^8 - 3584x^6 + 13440x^4 - 13440x^2 + 1680 \\
H_9 &= 512x^9 - 9216x^7 + 68640x^5 - 80640x^3 + 30240x \\
H_{10} &= 1024x^{10} - 23040x^8 + 161280x^6 - 403200x^4 + 302400x^2 - 30240 \\

L_0 &= 1 \\
L_1 &= x + 1 \\
L_2 &= (x^2 - 4x + 2)/2 \\
L_3 &= (x^3 - 9x^2 + 18x + 6)/6 \\
L_4 &= (x^4 - 16x^3 + 72x^2 - 96x + 24)/24 \\
L_5 &= (x^5 - 25x^4 + 200x^3 - 600x^2 + 600x + 120)/120 \\
L_6 &= (x^6 - 36x^5 + 450x^4 - 2400x^3 + 5400x^2 - 4320x + 720)/720 \\

P_0 &= 1 \\
P_1 &= (4199P_0 + 16150P_1 + 15504P_2 + 7904P_3 + 2176P_4 + 256P_5)/46189 \\
P_2 &= x \\
P_3 &= (3315P_1 + 4760P_2 + 2992P_3 + 960P_4 + 128P_5)/12155 \\
P_4 &= (3x^2 - 1)/2 \\
P_5 &= (5x^3 - 3x)/2 \\
P_6 &= (35x^4 - 30x^3 + 3x)/8 \\
P_7 &= (63x^5 - 70x^4 + 15x^3)/8 \\
P_8 &= (231x^6 - 315x^5 + 105x^4 - 5x)/16 \\
P_9 &= (429x^7 - 693x^6 + 315x^5 - 35x)/16 \\
P_{10} &= (6435x^8 - 12012x^7 + 6930x^6 - 1260x^2 + 35)/128 \\
P_{11} &= (121553x^9 - 25740x^8 + 18010x^7 - 4620x^6 + 315x)/128 \\
P_{12} &= (46189x^{10} - 109395x^9 + 90090x^8 - 30030x^7 + 3465x^6 - 63)/256 \\

T_0 &= 1 \\
T_1 &= x \\
T_2 &= (126T_0 + 210T_1 + 120T_2 + 45T_3 + 10T_4 + T_5)/512 \\
T_3 &= x^3 = (126T_0 + 84T_1 + 36T_2 + 9T_3 + T_4)/256 \\
T_4 &= x^4 - 1 \\
T_5 &= x^5 = (35T_0 + 21T_1 + T_2 + T_3)/64 \\
T_6 &= x^6 - 2x^5 + 5x^4 - 1 \\
T_7 &= x^7 = (107T_0 + 157T_1 + 67T_2 + 16T_3)/32 \\
T_8 &= x^8 - 3x^7 + 6x^6 - 7x^5 + 5x^4 - 1 \\
T_9 &= x^9 = (31T_0 + 41T_1 + 16T_2 + 6T_3)/16 \\
T_{10} &= x^{10} - 4x^9 + 10x^8 - 12x^7 + 9x^6 - 5x^5 + 2x^4 - x^3 + 1 \\
T_{11} &= x^{11} = (14T_0 + 14T_1 + 6T_2 + U_3)/128 \\
T_{12} &= x^{12} = (5U_0 + 9U_1 + 35U_2 + 90U_3 + 120U_4 + U_5)/1024 \\
T_{13} &= x^{13} = (42U_0 + 90U_1 + 210U_2 + 35U_3 + 90U_4 + U_5)/1024 \\
T_{14} &= x^{14} = (14U_0 + 28U_1 + 42U_2 + 21U_3)/256 \\
T_{15} &= x^{15} = (14U_0 + 14U_1 + 6U_2 + U_3)/128 \\
T_{16} &= x^{16} = (5U_0 + 9U_1 + 35U_2 + 90U_3 + 120U_4 + U_5)/1024 \\
T_{17} &= x^{17} = (42U_0 + 90U_1 + 210U_2 + 35U_3 + 90U_4 + U_5)/1024 \\

U_0 &= 1 \\
U_1 &= 2x \\
U_2 &= 4x^2 - 1 \\
U_3 &= 8x^3 - 4x \\
U_4 &= 16x^4 - 12x^2 + 1 \\
U_5 &= 32x^5 - 32x^3 + 6x \\
U_6 &= 64x^6 - 80x^4 + 24x^2 - 1 \\
U_7 &= 128x^7 - 192x^5 + 80x^3 - 8x \\
U_8 &= 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1 \\
U_9 &= 512x^9 - 1024x^7 + 672x^5 - 160x^3 + 10x \\
U_{10} &= 1024x^{10} - 2304x^8 + 1792x^6 - 560x^4 + 60x^2 - 1 \\

x^{10} &= (30240H_0 + 75600H_1 + 25200H_2 + 2520H_3 + 90H_4 + H_{10})/1024 \\
x^{9} &= (15120H_0 + 100800H_1 + 512H_2 + 72H_3 + H_{10})/512 \\
x^{8} &= (480H_0 + 3360H_1 + 840H_2 + 56H_3 + H_{10})/256 \\
x^{7} &= (840H_0 + 420H_1 + 42H_2 + H_{10})/128 \\
x^{6} &= (120H_0 + 30H_1 + H_{10})/64 \\
x^{5} &= (60H_1 + 20H_2 + H_{10})/32 \\
x^{4} &= (12H_2 + H_{10})/16 \\
x^{3} &= (6H_1 + H_{10})/8 \\
x^{2} &= (2H_2 + H_{10})/4 \\
x &= (H_{10})/2
\end{align*}
6.10.9 TABLE OF JACOBI POLYNOMIALS

Notation: \((m)_n = m(m + 1) \ldots (m + n - 1)\).

\(P_0^{(\alpha, \beta)}(x) = 1.\)

\(P_1^{(\alpha, \beta)}(x) = \frac{1}{2} \left(2(\alpha + 1) + (\alpha + \beta + 2)(x - 1)\right).\)

\(P_2^{(\alpha, \beta)}(x) = \frac{1}{8} \left(4(\alpha + 1) + 4(\alpha + \beta + 3)(\alpha + 2)(x - 1) + (\alpha + \beta + 3)(x - 1)^3\right).\)

\(P_3^{(\alpha, \beta)}(x) = \frac{1}{48} \left(8(\alpha + 1) + 12(\alpha + \beta + 4)(\alpha + 2)(x - 1) + 6(\alpha + \beta + 4)(\alpha + 3)(x - 1)^2 + (\alpha + \beta + 4)(x - 1)^3\right).\)

\(P_4^{(\alpha, \beta)}(x) = \frac{1}{384} \left(16(\alpha + 1) + 32(\alpha + \beta + 5)(\alpha + 2)(x - 1) + 24(\alpha + \beta + 5)(\alpha + 3)(x - 1)^2 + 8(\alpha + \beta + 5)(\alpha + 4)(x - 1)^3 + (\alpha + \beta + 5)(x - 1)^4\right).\)

6.10.10 SPHERICAL HARMONICS

The spherical harmonics are defined by

\[ Y_{lm}(\theta, \phi) = \sqrt{\frac{2l + 1}{4\pi}} \frac{(l - m)!}{(l + m)!} P_l^m(\cos \theta) e^{im\phi}. \] (6.10.3)

They satisfy

\[ Y_{l-1,m}(\theta, \phi) = (-1)^m Y_{l,m}^*(\theta, \phi), \] (6.10.4)

\[ Y_{l0}(\theta, \phi) = \sqrt{\frac{2l + 1}{4\pi}} P_l(\cos \theta), \] (6.10.5)

\[ Y_{lm} \left( \frac{\pi}{2}, \phi \right) = \begin{cases} \sqrt{\frac{(2l + 1)!}{4\pi}} \left(-1\right)^{\frac{l + m}{2}} \frac{(l + m)!}{(l - m)!}, & \frac{l + m}{2} \text{ integral}, \\ 0, & \frac{l + m}{2} \text{ not integral}. \end{cases} \] (6.10.6)

The normalization and orthogonality conditions are

\[ \int_0^{2\pi} d\phi \int_0^\pi \sin \theta \, d\theta \, Y_{lm}^* \left( \theta, \phi \right) Y_{lm'} \left( \theta, \phi \right) = \delta_{ll'} \delta_{mm'}. \] (6.10.7)

and

\[ \int_0^{2\pi} d\phi \int_0^\pi \sin \theta \, d\theta \, Y_{l_1m_2}^* \left( \theta, \phi \right) Y_{l_2m_3} \left( \theta, \phi \right) Y_{l_3m_4} \left( \theta, \phi \right), \]

\[ = \sqrt{\frac{(2l_2 + 1)(2l_3 + 1)}{4\pi(2l_1 + 1)}} \left( \begin{array}{c} l_1 \\ m_2 \end{array} \right) \left( \begin{array}{c} l_3 \\ m_3 \end{array} \right) \left( \begin{array}{c} l_1 \\ 0 \end{array} \right) \left( \begin{array}{c} l_3 \\ 0 \end{array} \right), \] (6.10.8)
where the terms on the right hand side are Clebsch–Gordan coefficients (see page 527).

Because of the (distributional) completeness relation,

$$
\sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\theta, \phi)Y_{lm}^\ast(\theta', \phi') = \delta(\phi - \phi')\delta(\cos \theta - \cos \theta'),
$$

(6.10.9)

an arbitrary function $g(\theta, \phi)$ can be expanded in spherical harmonics as

$$
g(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} Y_{lm}(\theta, \phi), \quad A_{lm} = \int Y_{lm}^\ast(\theta, \phi)g(\theta, \phi)\,d\Omega.
$$

(6.10.10)

In spherical coordinates,

$$
\nabla^2 [f(r)Y_{lm}(\theta, \phi)] = \left[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{df}{dr} \right) - l(l+1)\frac{f(r)}{r^2} \right] Y_{lm}(\theta, \phi).
$$

(6.10.11)

### 6.10.11 TABLE OF SPHERICAL HARMONICS

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<td>$Y_{00} = \frac{1}{\sqrt{4\pi}}$</td>
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| 1    | \[
|      | \begin{aligned}
|      | Y_{11} &= -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}, \\
|      | Y_{10} &= \sqrt{\frac{3}{4\pi}} \cos \theta,
|      | \end{aligned}
| 2    | \[
|      | \begin{aligned}
|      | Y_{22} &= \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{2i\phi}, \\
|      | Y_{21} &= -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi}, \\
|      | Y_{20} &= \frac{1}{2}\sqrt{\frac{5}{4\pi}} (3 \cos^2 \theta - 1),
|      | \end{aligned}
| 3    | \[
|      | \begin{aligned}
|      | Y_{33} &= -\sqrt{\frac{105}{4\pi}} \sin^3 \theta e^{3i\phi}, \\
|      | Y_{32} &= \sqrt{\frac{105}{2\pi}} \sin^2 \theta \cos \theta e^{2i\phi}, \\
|      | Y_{31} &= -\sqrt{\frac{21}{4\pi}} \sin \theta (5 \cos^2 \theta - 1) e^{i\phi}, \\
|      | Y_{30} &= \frac{1}{2}\sqrt{\frac{7}{4\pi}} (5 \cos^3 \theta - 3 \cos \theta),
|      | \end{aligned}
|
6.11 GAMMA FUNCTION

\[ \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt, \quad z = x + iy, \quad x > 0. \]

Graphs of \( \Gamma(x) \) and \( 1/\Gamma(x) \), \( x \) real.\(^1\)

6.11.1 RECURSION FORMULA

\[ \Gamma(z + 1) = z \Gamma(z). \]

The relation \( \Gamma(z) = \Gamma(z + 1)/z \) can be used to define the gamma function in the left half plane, \( z \neq 0, -1, -2, \ldots \).

6.11.2 SINGULAR POINTS

The gamma function has simple poles at \( z = -n \), (for \( n = 0, 1, 2, \ldots \)), with the respective residues \( (-1)^n/n! \); that is,

\[ \lim_{z \to -n} (z + n)\Gamma(z) = \frac{(-1)^n}{n!}. \]

\(^1\)From Temme, N.M., Special Functions: An Introduction to the Classical Functions of Mathematical Physics, John Wiley & Sons, New York, 1996. With permission.
6.11.3 SPECIAL VALUES

\[ \Gamma(n + 1) = n! \quad \text{if } n = 0, 1, 2, \ldots, \text{where } 0! = 1, \]
\[ \Gamma(1) = 1, \quad \Gamma(2) = 1, \quad \Gamma(3) = 2, \quad \Gamma(\frac{1}{2}) = \sqrt{\pi}, \]
\[ \Gamma(m + \frac{1}{2}) = \frac{1 \cdot 3 \cdot 5 \cdots (2m - 1)}{2^m} \sqrt{\pi}, \quad m = 1, 2, 3, \ldots, \]
\[ \Gamma(-m + \frac{1}{2}) = \frac{(-1)^m 2^m}{1 \cdot 3 \cdot 5 \cdots (2m - 1)} \sqrt{\pi}, \quad m = 1, 2, 3, \ldots. \]

\[ \Gamma(\frac{1}{4}) = 3.6256099082, \quad \Gamma(\frac{1}{2}) = 1.772453859, \quad \Gamma(\frac{3}{2}) = 1.3541179394, \]
\[ \Gamma(\frac{3}{4}) = 1.2254167024, \quad \Gamma(\frac{5}{4}) = 0.8862269254. \]

6.11.4 DEFINITION BY PRODUCTS

\[ \Gamma(z) = \lim_{n \to \infty} \frac{n! n^z}{z(z + 1) \cdots (z + n)}, \]
\[ \frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left[ (1 + z/n) e^{-z/n} \right], \quad \gamma \text{ is Euler’s constant}. \]

6.11.5 OTHER INTEGRALS

\[ \Gamma(z) \cos \frac{1}{2} \pi z = \int_0^\infty t^{z-1} \cos t \, dt, \quad 0 < \text{Re } z < 1, \]
\[ \Gamma(z) \sin \frac{1}{2} \pi z = \int_0^\infty t^{z-1} \sin t \, dt, \quad -1 < \text{Re } z < 1. \]

6.11.6 PROPERTIES

\[ \Gamma'(1) = \int_0^\infty \ln t \, e^{-t} \, dt = -\gamma. \]

Multiplication formula:

\[ \Gamma(2z) = \pi^{-1/2} 2^{2z-1} \Gamma(z) \Gamma(z + \frac{1}{2}). \]
Reflection formulas:
\[
\begin{align*}
\Gamma(z) \Gamma(1-z) &= \frac{\pi}{\sin \pi z}, \\
\Gamma(1/2 + z) \Gamma(1/2 - z) &= \frac{\pi}{\cos \pi z}, \\
\Gamma(z - n) &= (-1)^n \frac{\Gamma(z)}{\Gamma(n + 1 - z)} = (-1)^n \frac{\pi}{\sin \pi z \Gamma(n + 1 - z)}.
\end{align*}
\]

6.11.7 ASYMPTOTIC EXPANSION
For \( z \to \infty, \ |\arg z| < \pi \):
\[
\begin{align*}
\Gamma(z) &\sim \sqrt{2\pi/\pi z} e^{-z} \left[ 1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} + \cdots \right], \\
\ln \Gamma(z) &\sim \ln \left( \sqrt{2\pi/\pi z} e^{-z} \right) + \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)z^{2n-1}}, \\
&\sim \ln \left( \sqrt{2\pi/\pi z} e^{-z} \right) + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{120z^5} - \frac{1}{1680z^7} + \cdots,
\end{align*}
\]
where \( B_n \) are the Bernoulli numbers. If we let \( z = n \) a large positive integer, then a useful approximation for \( n! \) is given by Stirling’s formula,
\[
n! \sim \sqrt{2\pi n} n^n e^{-n}, \quad n \to \infty.
\]

6.11.8 LOGARITHMIC DERIVATIVE
Logarithmic derivative of the gamma function
\[
\psi(z) = \frac{d}{dz} \ln \Gamma(z) = -\gamma + \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{z+n} \right), \quad z \neq 0, -1, -2, \ldots.
\]

6.11.9 SPECIAL VALUES
\[
\psi(1) = -\gamma, \quad \psi(1/2) = -\gamma - 2 \ln 2.
\]

6.11.10 ASYMPTOTIC EXPANSION
For \( z \to \infty, \ |\arg z| < \pi \):
\[
\begin{align*}
\psi(z) &\sim \ln z - \frac{1}{2z} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2n \pi^{2n} z^{2n}}, \\
&\sim \ln z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots.
\end{align*}
\]
6.11.11 NUMERICAL VALUES

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6.12 BETA FUNCTION

\[ B(p, q) = \int_0^1 t^{p-1}(1-t)^{q-1} \, dt, \quad \text{Re} \, p > 0, \quad \text{Re} \, q > 0. \]
6.12.1 RELATION WITH GAMMA FUNCTION

\[ B(p, q) = \frac{\Gamma(p) \Gamma(q)}{\Gamma(p + q)}. \]

6.12.2 PROPERTIES

\[ B(p, q) = B(q, p), \]
\[ B(p, q + 1) = \frac{q}{p} B(p + 1, q) = \frac{q}{p + q} B(p, q), \]
\[ B(p, q) B(p + q, r) = \frac{\Gamma(p) \Gamma(q) \Gamma(r)}{\Gamma(p + q + r)}. \]

6.12.3 OTHER INTEGRALS

(In all cases Re \( p > 0 \), Re \( q > 0 \).)

\[ B(p, q) = 2 \int_0^{\pi/2} \sin^{2p-1} \theta \cos^{2q-1} \theta \, d\theta, \]
\[ = \int_0^\infty \frac{t^{p-1}}{(t+1)^{p+q}} \, dt, \]
\[ = \int_0^\infty e^{-pt} (1 - e^{-t})^{q-1} \, dt, \]
\[ = r^q (r + 1)^p \int_0^1 \frac{t^{p-1}(1 - t)^{q-1}}{(r + t)^{p+q}} \, dt, \quad r > 0. \]

6.13 ERROR FUNCTIONS AND FRESNEL INTEGRALS

\[ \text{erf} \, x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt, \]
\[ \text{erfc} \, x = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} \, dt. \]

The function \( \text{erfc} \, x \) is known as the complementary error function.
6.13.1 SERIES EXPANSIONS

\[
\text{erf} \, x = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n+1}}{(2n + 1) \, n!} = \frac{2}{\sqrt{\pi}} \left( x - \frac{x^3}{3} + \frac{1}{2! \, 5} \frac{x^5}{3! \, 7} - \cdots \right),
\]

\[
= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{3}{2}) e^{-x^2}}{\Gamma(n + \frac{3}{2})} x^{2n+1} = \frac{2}{\sqrt{\pi}} e^{-x^2} \left( x + \frac{2}{3} x^3 + \frac{4}{15} x^5 + \cdots \right).
\]

6.13.2 PROPERTIES

\[
\text{erf} \, x + \text{erfc} \, x = 1, \quad \text{erf}(-x) = -\text{erf} \, x, \quad \text{erfc}(-x) = 2 - \text{erfc} \, x.
\]

6.13.3 RELATIONSHIP WITH NORMAL PROBABILITY FUNCTION

\[
\int_{0}^{x} f(t) \, dt = \frac{1}{2} \text{erf} \, (x/\sqrt{2}), \quad f(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} t^2}.
\]

6.13.4 SPECIAL VALUES

\[
\text{erf}(\pm \infty) = \pm 1, \quad \text{erfc}(-\infty) = 2, \quad \text{erfc} \, \infty = 0,
\]

\[
\text{erf} \, x_0 = \text{erfc} \, x_0 = \frac{1}{2} \quad \text{if} \quad x_0 = 0.476936 \ldots.
\]

6.13.5 ASYMPTOTIC EXPANSION

For \( z \to \infty, \ |\arg z| < \frac{3}{4} \pi, \)

\[
\text{erfc} \, z \sim \frac{2}{\sqrt{\pi}} \frac{e^{-z^2}}{2z} \sum_{n=0}^{\infty} \frac{(-1)^n \, (2n)!}{n!(2z)^{2n}},
\]

\[
\sim \frac{2}{\sqrt{\pi}} \frac{e^{-z^2}}{2z} \left( 1 - \frac{1}{2z^2} + \frac{3}{4z^4} - \frac{15}{8z^6} + \cdots \right).
\]

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6.13.6 OTHER FUNCTIONS

Plasma dispersion function

\[ w(z) = e^{-z^2} \text{erfc}(-iz), \]
\[ = \frac{1}{\pi i} \int_{-\infty}^{\infty} e^{-t^2} \frac{1}{1 - iz} \, dt, \quad \text{Im} \, z > 0, \]
\[ = 2 e^{-z^2} - w(-z), \]
\[ = \sum_{n=0}^{\infty} \frac{(iz)^n}{\Gamma(n/2 + 1)}. \]

Dawson’s integral

\[ F(x) = e^{-x^2} \int_{0}^{x} e^{t^2} \, dt = -\frac{1}{2} \sqrt{\pi} e^{-x^2} \text{erf} \, ix. \]

Fresnel integrals

\[ C(z) = \sqrt{\frac{2}{\pi}} \int_{0}^{z} \cos t^2 \, dt, \quad S(z) = \sqrt{\frac{2}{\pi}} \int_{0}^{z} \sin t^2 \, dt. \]

Relations

\[ C(z) + i S(z) = \frac{1 + i}{2} \text{erf} \left( \frac{1 - i}{\sqrt{2}} z \right). \]

Limits

\[ \lim_{z \to \infty} C(z) = \frac{1}{2}, \quad \lim_{z \to \infty} S(z) = \frac{1}{2}. \]
Representations

\[ C(z) = \frac{1}{2} + f(z) \sin(z^2) - g(z) \cos(z^2), \]
\[ S(z) = \frac{1}{2} - f(z) \cos(z^2) - g(z) \sin(z^2), \]

where

\[ f(z) = \frac{1}{\pi \sqrt{2}} \int_0^\infty \frac{e^{-z^2 t}}{\sqrt{t(t^2 + 1)}} \, dt, \quad g(z) = \frac{1}{\pi \sqrt{2}} \int_0^\infty \frac{\sqrt{t} e^{-z^2 t}}{(t^2 + 1)} \, dt. \]

And for \( z \to \infty, |\arg z| < \frac{1}{2} \pi, \)

\[ f(z) \sim \frac{1}{\pi \sqrt{2}} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(2n + 1/2)}{z^{2n+1/2}}, \]
\[ \sim \frac{1}{\sqrt{2\pi} z} \left[ 1 - \frac{3}{4} z^2 + \frac{105}{16} z^4 - \ldots \right], \]
\[ g(z) \sim \frac{1}{\pi \sqrt{2}} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(2n + 3/2)}{z^{2n+3/2}}, \]
\[ \sim \frac{1}{2z \sqrt{2\pi} z} \left[ 1 - \frac{15}{4} z^2 + \frac{945}{16} z^4 - \ldots \right]. \]

6.13.7 PROPERTIES

\[ \sqrt{\frac{2}{\pi}} \int_x^\infty e^{i t z} \, dt = [g(z) + i f(z)] e^{i t^2}. \]

Cornu’s spiral, formed from Fresnel functions, is the set \( \{x, y, t\} \) where \( x = C(t), y = S(t), t \geq 0. \)

\[ ^2 \text{From Temme, N.M., Special Functions: An Introduction to the Classical Functions of Mathematical Physics, John Wiley & Sons, New York, 1996. With permission.} \]
### 6.13.8 NUMERICAL VALUES

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6.14 SINE, COSINE, AND EXPONENTIAL INTEGRALS

6.14.1 SINE AND COSINE INTEGRALS

\[ Si(z) = \int_0^z \frac{\sin t}{t} \, dt, \quad Ci(z) = \gamma + \ln z + \int_0^z \frac{\cos t - 1}{t} \, dt, \]

where \( \gamma \) is Euler’s constant.

Sine and cosine integrals \( Si(x) \) and \( Ci(x) \), \( 0 \leq x \leq 8 \).

6.14.2 ALTERNATIVE DEFINITIONS

\[ Si(z) = \frac{1}{2\pi} - \int_z^\infty \frac{\sin t}{t} \, dt, \quad Ci(z) = -\int_z^\infty \frac{\cos t}{t} \, dt. \]

6.14.3 LIMITS

\[ \lim_{z \to \infty} Si(z) = \frac{1}{2\pi}, \quad \lim_{z \to \infty} Ci(z) = 0. \]

\[ ^3 \text{From Temme, N.M., } \textit{Special Functions: An Introduction to the Classical Functions of Mathematical Physics}, \text{ John Wiley & Sons, New York, 1996. With permission.} \]
6.14.4 REPRESENTATIONS

\[ Si(z) = -f(z) \cos z - g(z) \sin z + \frac{1}{2} \pi, \]
\[ Ci(z) = +f(z) \sin z - g(z) \cos z, \]

where
\[
    f(z) = \int_0^\infty \frac{e^{-zt}}{t^2 + 1} \, dt, \quad g(z) = \int_0^\infty \frac{te^{-zt}}{t^2 + 1} \, dt.
\]

6.14.5 ASYMPTOTIC EXPANSION

For \( z \to \infty, \ |\arg z| < \pi \),

\[ f(z) \sim \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{z^{2n}}, \quad g(z) \sim \frac{1}{z^2} \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)!}{z^{2n}}. \]

6.14.6 EXPONENTIAL INTEGRALS

\[ E_n(z) = \int_1^\infty \frac{e^{-zt}}{t^n} \, dt, \quad \text{Re } z > 0, \quad n = 1, 2, \ldots . \]

\[ = \frac{z^{n-1} e^{-z}}{\Gamma(n)} \int_0^{\infty} \frac{e^{-zt}}{t^n} \, dt, \quad \text{Re } z > 0. \]

6.14.7 SPECIAL CASE

\[ E_1(z) = \int_1^\infty \frac{e^{-t}}{t} \, dt, \quad |\arg z| < \pi. \]

This function is also written as \(-\text{Ei}(-z)\). For real values of \( z = x \),

\[ \text{Ei}(x) = \int_0^x \frac{e^t}{t} \, dt, \]

where for \( x > 0 \) the integral should be interpreted as a Cauchy principal value integral.

6.14.8 LOGARITHMIC INTEGRAL

\[ \text{li}(x) = \int_0^x \frac{dt}{\ln t} = \text{Ei}(\ln x), \]

where for \( x > 1 \) the integral should be interpreted as a Cauchy principal value integral.
6.14.9 REPRESENTATIONS

\[ E_1(z) = -\gamma - \ln z + \int_0^z \frac{1 - e^{-t}}{t} \, dt, \]

\[ E_1\left(ze^{\pm \pi i}\right) = -\gamma - \ln z - \text{Ci}(z) + i \left[ -\frac{1}{2} \pi + \text{Si}(z) \right]. \]

6.14.10 NUMERICAL VALUES

\( \text{Si}(x), \ \text{Ci}(x), \ e^x E_1(x), \ e^{-x} \text{Ei}(x) \), for \( 0 \leq x \leq 5 \)

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6.15 POLYLOGARITHMS

\[ \text{Li}_1(z) = \int_0^z \frac{dt}{1-t} = -\ln(1-z), \quad \text{logarithm}, \]
\[ \text{Li}_2(z) = \int_0^z \frac{\text{Li}_1(t)}{t} \, dt = -\int_0^z \frac{\ln(1-t)}{t} \, dt, \quad \text{dilogarithm}, \]
\[ \text{Li}_n(z) = \int_0^z \frac{\text{Li}_{n-1}(t)}{t} \, dt, \quad n \geq 2, \quad \text{polylogarithm}. \]

6.15.1 ALTERNATIVE DEFINITION

For any complex \( \nu \)

\[ \text{Li}_\nu(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^\nu}, \quad |z| < 1. \]

6.15.2 SINGULAR POINTS

\( z = 1 \) is a singular point of \( \text{Li}_\nu(z) \).

6.15.3 INTEGRAL

\[ \text{Li}_\nu(z) = \frac{z}{\Gamma(\nu)} \int_0^\infty t^{\nu-1} \frac{e^t - z}{e^t} \, dt, \quad \text{Re } \nu > 0, \quad z \notin [1, \infty). \]

6.15.4 GENERATING FUNCTION

\[ \sum_{n=2}^{\infty} w^{n-1} \text{Li}_n(z) = z \int_0^\infty \frac{e^{wt} - 1}{e^t - z} \, dt, \quad z \notin [1, \infty). \]

The series converges for \( |w| < 1 \), the integral is defined for \( \text{Re } w < 1 \).

6.15.5 SPECIAL VALUES

\( \text{Li}_2(1) = \frac{1}{6} \pi^2, \quad \text{Li}_2(-1) = -\frac{1}{12} \pi^2, \quad \text{Li}_2(1/2) = \frac{1}{12} \pi^2 - \frac{1}{2} (\ln 2)^2, \)
\( \text{Li}_\nu(1) = \zeta(\nu), \quad \text{Re } \nu > 1 \quad (\text{Riemann zeta function}). \)
6.15.6 FUNCTIONAL EQUATIONS FOR DILOGARITHMS

\[ \text{Li}_2(z) + \text{Li}_2(1 - z) = \frac{1}{6} \pi^2 - \ln z \ln(1 - z), \]
\[ \frac{1}{2} \text{Li}_2(x^2) = \text{Li}_2(x) + \text{Li}_2(-x), \]
\[ \text{Li}_2(-1/x) + \text{Li}_2(-x) = -\frac{1}{6} \pi^2 - \frac{1}{2} (\ln x)^2, \]
\[ 2 \text{Li}_2(x) + 2 \text{Li}_2(y) + 2 \text{Li}_2(z) = \]
\[ \text{Li}_2(-xy/z) + \text{Li}_2(-yz/x) + \text{Li}_2(-zx/y), \]

where \(1/x + 1/y + 1/z = 1\).

6.16 HYPERGEOMETRIC FUNCTIONS

Recall the geometric series and binomial expansion (|z| < 1),

\[ (1 - z)^{-1} = \sum_{n=0}^{\infty} z^n, \quad (1 - z)^{-a} = \sum_{n=0}^{\infty} \binom{-a}{n} (-z)^n = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n. \]

where the shifted factorial, \((a)_n\), is defined in Section 1.2.6.

6.16.1 DEFINITION OF THE \(F\)-FUNCTION

Gauss hypergeometric function

\[ F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \]
\[ = 1 + \frac{ab}{c} z + \frac{a(a+1)b(b+1)}{c(c+1)2!} z^2 + \ldots, \quad |z| < 1, \]
\[ = F(a, b; c; z) = F(b, a; c; z), \]

where \(a, b\) and \(c\) may assume all complex values, \(c \neq 0, -1, -2, \ldots\).

6.16.2 POLYNOMIAL CASE

For \(m = 0, 1, 2, \ldots\)

\[ F(-m, b; c; z) = \sum_{n=0}^{m} \frac{(-m)_n (b)_n}{(c)_n n!} z^n = \sum_{n=0}^{m} (-1)^n \binom{m}{n} \frac{(b)_n}{(c)_n} z^n. \]
6.16.3 SPECIAL CASES

\[
F (a, b; z) = (1 - z)^{-a},
\]
\[
F (1, 1; 2; z) = -\frac{\ln(1 - z)}{z},
\]
\[
F \left( \frac{1}{2}, 1; \frac{3}{2}; z^2 \right) = \frac{1}{2z} \ln\left( \frac{1 + z}{1 - z} \right),
\]
\[
F \left( \frac{1}{2}, 1; \frac{3}{2}; z^2 \right) = \frac{\tan^{-1} z}{z},
\]
\[
F \left( \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2 \right) = \frac{\sin^{-1} z}{z},
\]
\[
F \left( \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2 \right) = \frac{\ln(z + \sqrt{1 + z^2})}{z}.
\]

6.16.4 SPECIAL VALUES

When \( \text{Re} (c - a - b) > 0 \),

\[
F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}.
\]

6.16.5 INTEGRAL

When \( \text{Re} c > \text{Re} b > 0 \),

\[
F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1-tz)^{-a} \, dt.
\]

6.16.6 FUNCTIONAL RELATIONSHIPS

\[
F(a, b; c; z) = (1 - z)^{-a} F(a, c - b; c; \frac{z}{z - 1}),
\]
\[
= (1 - z)^{-b} F(c - a, b; c; \frac{z}{z - 1}),
\]
\[
= (1 - z)^{c - a - b} F(c - a, c - b; c; z).
\]

6.16.7 DIFFERENTIAL EQUATION

\[
z(1 - z)F'' + [(c - (a + b + 1)z]F' - abF = 0,
\]

with (regular) singular points \( z = 0, 1, \infty \).
6.16.8 PROPERTIES

\[ \frac{d}{dz} F(a, b; c; z) = \frac{ab}{c} F(a + 1, b + 1; c + 1; z), \]
\[ \frac{d^n}{dz^n} F(a, b; c; z) = \frac{(a)_n (b)_n}{(c)_n} F(a + n, b + n; c + n; z). \]

6.16.9 RECURSION FORMULAE

Notation:  
\( F(a, b; c; z) \) are \( F(a + 1, b; c; z) \), \( F(a - 1, b; c; z) \), respectively, etc.

\[ (c - a)F(a-) + (2a - c - az + bz)F + a(z - 1)F(a+) = 0, \]
\[ c(c - 1)(z - 1)F(c-) + c[c - 1 - (2c - a - b - 1)z]F + (c - a)(c - b)zF(c+) = 0, \]
\[ c[a + (b - c)z]F - ac(1 - z)F(a+) + (c - a)(c - b)zF(c+) = 0, \]
\[ c(1 - z)F - cF(a-) + (c - b)zF(c+) = 0, \]
\[ (b - a)F + aF(a+) - bF(b+) = 0, \]
\[ (c - a - b)F + a(1 - z)F(a+) - (c - b)F(b-) = 0, \]
\[ (c - a - 1)F + aF(a+) - (c - 1)F(c-) = 0, \]
\[ (b - a)(1 - z)F - (c - a)F(a-) + (c - b)F(b-) = 0, \]
\[ [a - 1 + (b + 1 - c)z]F + (c - a)F(a-) - (c - 1)(1 - z)F(c-) = 0. \]

6.17 LEGENDRE FUNCTIONS

6.17.1 DIFFERENTIAL EQUATION

The Legendre differential equation is,

\[ (1 - z^2)w'' - 2zw' + v(v + 1)w = 0. \]

The solutions \( P_v(z), Q_v(z) \) can be given in terms of Gaussian hypergeometric functions.

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Legendre functions \( P_n(x), n = 1, 2, 3 \) (left) and \( Q_n(x), n = 0, 1, 2, 3 \), on the interval \([-1, 1]\).\(^4\)

### 6.17.2 Definition

\[
P_\nu(z) = F (-\nu, \nu + 1; 1; \frac{1}{2} - \frac{1}{2}z),
\]
\[
Q_\nu(z) = \frac{\sqrt{\pi} \Gamma(\nu + 1)}{\Gamma(\nu + \frac{3}{2}) (2z)^{\nu+1}} F \left( \frac{1}{2} \nu + 1, \frac{1}{2} \nu + \frac{1}{2}; \nu + \frac{3}{2}; z^{-2} \right).
\]

The \( Q \)–function is not defined if \( \nu = -1, -2, \ldots \).

### 6.17.3 Polynomial Case

Legendre polynomial \( \nu = n = 0, 1, 2, \ldots \),

\[
P_n(x) = F (-n, n + 1; 1; \frac{1}{2} - \frac{1}{2}x)
= \sum_{k=0}^{m} (-1)^k (2n - 2k)! 2^k k! (n - k)! (n - 2k)! x^{n-2k}, 
\]
m = \begin{cases} 
\frac{1}{2} n, & \text{if } n \text{ even,} \\
\frac{1}{2} (n - 1), & \text{if } n \text{ odd.}
\end{cases}

### 6.17.4 Singular Points

\( P_\nu(z) \) has a singular point at \( z = -1 \) and is analytic in the remaining part of the complex \( z \)–plane, with a branch cut along \((-\infty, -1]\). \( Q_\nu(z) \) has singular points at \( z = \pm 1 \) and is analytic in the remaining part of the complex \( z \)–plane, with a branch cut along \((-\infty, +1]\).

6.17.5 RELATIONSHIPS

\[ P_{-v-1}(z) = P_v(z), \]
\[ Q_{-v-1}(z) = Q_v(z) - \pi \cot \nu \pi P_v(z). \]

6.17.6 SPECIAL CASE

The function \( Q_n(z) \). We distinguish two cases

\[ Q_n(x), \quad x \in (-1, 1), \quad Q_n(z), \quad z \notin [-1, 1]. \]

\[ Q_0(z) = \frac{1}{2} \ln \frac{z + 1}{z - 1}, \quad Q_1(z) = \frac{1}{2} z \ln \frac{z + 1}{z - 1} - 1, \]
\[ Q_n(z) = P_n(z)Q_0(z) - \sum_{k=0}^{n-1} \frac{(2k + 1)(1 - (-1)^{n+k})}{(n+k+1)(n-k)} P_k(z), \]
\[ Q_0(x) = \frac{1}{2} \ln \frac{1+x}{1-x}, \quad Q_1(x) = \frac{1}{2} x \ln \frac{1+x}{1-x} - 1, \]
\[ Q_n(x) = P_n(x)Q_0(x) - \sum_{k=0}^{n-1} \frac{(2k + 1)(1 - (-1)^{n+k})}{(n+k+1)(n-k)} P_k(x). \]

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<td>5</td>
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<td>( P_5(x)Q_0(x) - \frac{63}{6} x^4 + \frac{49}{8} x^2 - \frac{8}{15} )</td>
</tr>
</tbody>
</table>

Legendre polynomials \( P_n(x) \) and functions \( Q_n(x), x \in (-1, 1). \)

6.17.7 RECURSION RELATIONSHIPS

\[ (v+1)P_{v+1}(z) = (2v+1)zP_v(z) - vP_{v-1}(z), \]
\[ (2v+1)P_v(z) = P'_{v+1}(z) - P'_{v-1}(z), \]
\[ (v+1)P_v(z) = P'_{v+1}(z) - zP'_v(z), \]
\[ vP_v(z) = zP'_v(z) - P'_{v-1}(z), \]
\[ (1-z^2)P'_v(z) = vP_{v-1}(z) - vzP_v(z). \]

The functions \( Q_v(z) \) satisfy the same relations.
6.17.8 INTEGRALS

\[ P_\nu(\cosh \alpha) = \frac{2}{\pi} \int_0^\alpha \frac{\cosh(\nu + \frac{1}{2})\theta}{\sqrt{2 \cosh \alpha - 2 \cosh \theta}} d\theta, \]
\[ = \frac{1}{\pi} \int_\alpha^0 \frac{\cosh(\nu + \frac{1}{2})\theta}{\sqrt{2 \cosh \alpha - 2 \cosh \theta}} d\theta, \]
\[ = \frac{1}{\pi} \int_0^\pi \frac{d\psi}{(\cosh \alpha + \sinh \alpha \cos \psi)^{\nu+1}}, \]
\[ = \frac{1}{\pi} \int_0^\pi (\cosh \alpha + \sinh \alpha \cos \psi)^\nu d\psi. \]

\[ R_\nu(\cos \beta) = \frac{2}{\pi} \int_0^\beta \frac{\cos(\nu + \frac{1}{2})\theta}{\sqrt{2 \cos \theta - 2 \cos \beta}} d\theta, \]
\[ = \frac{1}{\pi} \int_\beta^0 \frac{\cos(\nu + \frac{1}{2})\theta}{\sqrt{2 \cos \theta - 2 \cos \beta}} d\theta, \]
\[ = \frac{1}{\pi} \int_0^\pi \frac{d\psi}{(\cos \beta + i \sin \beta \cos \psi)^{\nu+1}}, \]
\[ = \frac{1}{\pi} \int_0^\pi (\cos \beta + i \sin \beta \cos \psi)^\nu d\psi. \]

\[ Q_\nu(z) = 2^{\nu-1} \int_{-1}^1 \frac{(1 - t^2)^\nu}{(z - t)^{\nu+1}} dt, \quad \text{Re } \nu > -1, \ |\arg z| < \pi, \ z \notin [-1, 1], \]
\[ = \int_0^\infty \left[ z + \sqrt{z^2 - 1} \cos \phi \right]^{\nu-1} d\phi, \]
\[ = \int_\alpha^\infty \frac{e^{-(\nu+1/2)\theta}}{\sqrt{2 \cosh \theta - 2 \cosh \alpha}} d\theta, \quad z = \cosh \alpha. \]

6.17.9 DIFFERENTIAL EQUATION

The associated Legendre differential equation is,

\[ (1 - z^2)y'' - 2zy' + \left[ \nu(\nu + 1) - \frac{\mu^2}{1 - z^2} \right] y = 0. \]

The solutions \( P_\mu(\nu), Q_\mu(\nu), \) the associated Legendre functions, can be given in terms of Gauss hypergeometric functions. We only consider integer values of \( \mu, \nu, \) and replace them with \( m, n, \) respectively. Then the associated differential equation follows from the Legendre differential equation after it has been differentiated \( m \) times.
6.17.10 RELATIONSHIPS BETWEEN THE ASSOCIATED AND ORDINARY LEGENDRE FUNCTIONS

\[ P_n^m(z) = (1 - z^2)^{\frac{1}{2} m} \int_{-1}^{1} P_n(z) \, dz, \quad P_n^{-m}(z) = (n - m)! \frac{n + m)!}{(n + m)!} P_n^m(z), \]

\[ Q_n^m(z) = (1 - z^2)^{\frac{1}{2} m} \int_{-1}^{1} Q_n(z) \, dz, \quad Q_n^{-m}(z) = (n - m)! \frac{n + m)!}{(n + m)!} Q_n^m(z), \]

\[ P_n^{-m}(z) = (1 - z^2)^{-\frac{1}{2} m} \int_{1}^{z} \cdots \int_{1}^{z} P_n(z) \, dz^m, \]

\[ Q_n^{-m}(z) = (-1)^m(1 - z^2)^{-\frac{1}{2} m} \int_{1}^{\infty} \cdots \int_{1}^{\infty} Q_n(z) \, dz^m, \]

\[ P_n^{m-1}(z) = P_n^{m}(z). \]

6.17.11 ORTHOGONALITY RELATIONSHIP

Let \( n \geq m \).

\[ \int_{-1}^{1} P_n^m(x) P_k^m(x) \, dx = \begin{cases} 
0, & \text{if } k \neq n, \\
\frac{2}{2n + 1} \frac{(n + m)!}{(n - m)!}, & \text{if } k = n.
\end{cases} \]

6.17.12 RECURSION RELATIONSHIPS

\[ P_n^{m+1}(z) + \frac{2mz}{\sqrt{z^2 - 1}} P_n^m(z) = (n - m + 1)(n + m) P_n^{m-1}(z), \]

\[ (z^2 - 1) \frac{d P_n^m(z)}{dz} = mz P_n^m(z) + \sqrt{z^2 - 1} P_n^{m+1}(z), \]

\[ (2n + 1)z P_n^m(z) = (n - m + 1) P_{n+1}^m(z) + (n + m) P_{n-1}^m(z), \]

\[ (z^2 - 1) \frac{d P_n^m(z)}{dz} = (n - m + 1) P_{n+1}^m(z) - (n + 1)z P_n^m(z), \]

\[ P_n^{m-1}(z) - P_n^{m+1}(z) = -(2n + 1) \sqrt{z^2 - 1} P_n^{m-1}(z). \]

The functions \( Q_n^m(z) \) satisfy the same relations.

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6.18 BESSEL FUNCTIONS

6.18.1 DIFFERENTIAL EQUATION

The Bessel differential equation,

\[ z^2 y'' + z y' + (z^2 - \nu^2) y = 0. \]

The solutions are denoted with

\[ J_\nu(z), \quad Y_\nu(z) \]

(the ordinary Bessel functions)

and

\[ H^{(1)}_\nu(z), \quad H^{(2)}_\nu(z) \]

(the Hankel functions).

Further solutions are

\[ J_{-\nu}(z), \quad Y_{-\nu}(z), \quad H^{(1)}_{-\nu}(z), \quad H^{(2)}_{-\nu}(z). \]

When \( \nu \) is an integer,

\[ J_{-n}(z) = (-1)^n J_n(z), \quad n = 0, 1, 2, \ldots. \]

Bessel functions \( J_0(x), J_1(x), Y_0(x), Y_1(x), 0 \leq x \leq 12. \)

6.18.2 SINGULAR POINTS

The Bessel differential equation has a regular singularity at \( z = 0 \) and an irregular singularity at \( z = \infty \).
6.18.3 RELATIONSHIPS

\[ H_0^{(1)}(z) = J_0(z) + iY_0(z), \quad H_0^{(2)}(z) = J_0(z) - iY_0(z). \]

Neumann function: If \( \nu \neq 0, \pm 1, \pm 2, \ldots \)

\[ Y_\nu(z) = \frac{\cos \nu \pi J_\nu(z)}{\sin \nu \pi} - J_{-\nu}(z). \]

When \( \nu = n \) (integer) then the limit \( \nu \to n \) should be taken in the right-hand side of

this equation. Complete solutions to Bessel’s equation may be written as

\[ c_1 J_\nu(z) + c_2 J_{-\nu}(z), \quad \text{if } \nu \text{ is not an integer}, \]
\[ c_1 J_\nu(z) + c_2 Y_\nu(z), \quad \text{for any value of } \nu, \]
\[ c_1 H_1^{(1)}(z) + c_2 H_2^{(2)}(z), \quad \text{for any value of } \nu. \]

6.18.4 SERIES EXPANSIONS

For any complex \( z \),

\[ J_\nu(z) = (1/2)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n (1/2)^{2n}}{\Gamma(n + \nu + 1)n!}, \]
\[ J_0(z) = 1 - (1/2)^2 + \frac{1}{2! 2!} (1/2)^4 - \frac{1}{3! 3!} (1/2)^6 + \ldots, \]
\[ J_1(z) = 1/2 \left[ 1 - \frac{1}{1! 2!} (1/2)^2 + \frac{1}{2! 3!} (1/2)^4 - \frac{1}{3! 4!} (1/2)^6 + \ldots \right], \]
\[ Y_\nu(z) = \frac{2}{\pi} J_\nu(z) \ln(1/2) - \frac{(1/2)^{-\nu}}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} (1/2)^{2k} - \]
\[ \frac{(1/2)^{\nu}}{\pi} \sum_{k=0}^{\infty} \psi(k+1) + \psi(n+k+1) \frac{(-1)^k (1/2)^{2k}}{k! (n+k)!}, \]

where \( \psi \) is the logarithmic derivative of the gamma function.

6.18.5 RECURRENCE RELATIONSHIPS

\[ C_{\nu-1}(z) + C_{\nu+1}(z) = \frac{2\nu}{z} C_\nu(z), \]
\[ C_{\nu-1}(z) - C_{\nu+1}(z) = 2C'_\nu(z), \]
\[ C'_\nu(z) = C_{\nu-1}(z) - \frac{\nu}{z} C_\nu(z), \]
\[ C'_\nu(z) = -C_{\nu+1}(z) + \frac{\nu}{z} C_\nu(z), \]

where \( C_\nu(z) \) denotes one of the functions \( J_\nu(z), Y_\nu(z), H_1^{(1)}(z), H_2^{(2)}(z) \).
6.18.6 BEHAVIOR AS $z \to 0$

Let $\Re \nu > 0$.

$$J_\nu(z) \sim \left(\frac{1}{2}z\right)^\nu \Gamma(\nu + 1), \quad Y_\nu(z) \sim -\frac{1}{\pi} \Gamma(\nu) \left(\frac{2}{z}\right)^\nu,$$

$$H_\nu^{(1)}(z) \sim \frac{1}{\pi i} \Gamma(\nu) \left(\frac{2}{z}\right)^\nu, \quad H_\nu^{(2)}(z) \sim -\frac{1}{\pi i} \Gamma(\nu) \left(\frac{2}{z}\right)^\nu.$$

The same relations hold as $\Re \nu \to \infty$, with $z$ fixed.

6.18.7 INTEGRALS

Let $\Re z > 0$ and $\nu$ be any complex number.

$$J_\nu(z) = \frac{1}{\pi} \int_0^\pi \cos(\nu \theta - z \sin \theta) d\theta - \frac{\sin \nu \pi}{\pi} \int_0^\infty e^{-\nu t - z \sinh t} dt,$$

$$Y_\nu(z) = \frac{1}{\pi} \int_0^\pi \sin(z \sin \theta - \nu \theta) d\theta - \int_0^\infty (e^{\nu t} + e^{-\nu t} \cos \nu \pi) e^{-z \sinh t} dt.$$

When $\nu = n$ (integer), the second integral in the first relation disappears.

6.18.8 FOURIER EXPANSION

For any complex $z$,

$$e^{-iz \sin t} = \sum_{n=-\infty}^{\infty} e^{-int} J_n(z),$$

with Parseval relation

$$\sum_{n=-\infty}^{\infty} J_n^2(z) = 1.$$

6.18.9 AUXILIARY FUNCTIONS

Let $\chi = z - (1/2 \nu + 1/4)\pi$ and define

$$P(v, z) = \sqrt{\pi z/2} \left[ J_\nu(z) \cos \chi + Y_\nu(z) \sin \chi \right],$$

$$Q(v, z) = \sqrt{\pi z/2} \left[ -J_\nu(z) \sin \chi + Y_\nu(z) \cos \chi \right].$$

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6.18.10 INVERSE RELATIONSHIPS

\[ J_\nu(z) = \sqrt{\frac{2}{\pi z}} \left[ P(\nu, z) \cos \chi - Q(\nu, z) \sin \chi \right], \]

\[ Y_\nu(z) = \sqrt{\frac{2}{\pi z}} \left[ P(\nu, z) \sin \chi + Q(\nu, z) \cos \chi \right], \]

For the Hankel functions,

\[ H^{(1)}_\nu(z) = \sqrt{\frac{2}{\pi z}} \left[ P(\nu, z) + iQ(\nu, z) \right] e^{i\chi}, \]

\[ H^{(2)}_\nu(z) = \sqrt{\frac{2}{\pi z}} \left[ P(\nu, z) - iQ(\nu, z) \right] e^{-i\chi}. \]

The functions \( P(\nu, z), Q(\nu, z) \) are the slowly varying components in the asymptotic expansions of the oscillatory Bessel and Hankel functions.

6.18.11 ASYMPTOTIC EXPANSIONS

Let \( (\alpha, n) \) be defined by

\[ (\alpha, n) = \frac{2^{-2n}}{n!} [(4\alpha^2 - 1)(4\alpha^2 - 3^2) \cdots [4\alpha^2 - (2n - 1)^2]], \]

\[ = \frac{\Gamma(\frac{1}{2} + \alpha + n)}{n! \Gamma(\frac{1}{2} + \alpha - n)}, \quad n = 0, 1, 2, \ldots, \]

\[ = \left( -1 \right)^n \frac{\cos(\pi \alpha)}{\pi n!} \Gamma(\frac{1}{2} + \alpha + n)\Gamma(\frac{1}{2} - \alpha + n), \]

with recursion

\[ (\alpha, n + 1) = -\frac{(n + \frac{1}{2})^2 - \alpha^2}{n + 1} (\alpha, n), \quad n = 1, 2, 3, \ldots, \quad (\alpha, 0) = 1. \]

Then, for \( z \to \infty \),

\[ P(v, z) \sim \sum_{n=0}^{\infty} (-1)^n \frac{(v, 2n)}{(2z)^n}, \quad Q(v, z) \sim \sum_{n=0}^{\infty} (-1)^n \frac{(v, 2n + 1)}{(2z)^{n+1}}. \]

With \( \mu = 4\nu^2 \),

\[ P(v, z) \sim 1 - \frac{(\mu - 1)(\mu - 9)}{2! (8z)^2} + \frac{(\mu - 1)(\mu - 9)(\mu - 25)(\mu - 49)}{4! (8z)^4} - \ldots, \]

\[ Q(v, z) \sim \frac{\mu - 1}{8z} - \frac{(\mu - 1)(\mu - 9)(\mu - 25)}{3! (8z)^3} + \ldots. \]

For large positive values of \( x \),

\[ J_\nu(x) = \sqrt{\frac{2}{\pi x}} \left[ \cos \left( x - \frac{1}{2} \nu \pi - \frac{1}{4} \pi \right) + \mathcal{O}(x^{-1}) \right], \]

\[ Y_\nu(x) = \sqrt{\frac{2}{\pi x}} \left[ \sin \left( x - \frac{1}{2} \nu \pi - \frac{1}{4} \pi \right) + \mathcal{O}(x^{-1}) \right]. \]

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6.18.12 ZEROS OF BESSEL FUNCTIONS

For $\nu \geq 0$, the zeros $j_{\nu,k}, y_{\nu,k}$ of $J_\nu(x), Y_\nu(x)$ can be arranged as sequences

$$0 < j_{\nu,1} < j_{\nu,2} < \cdots < j_{\nu,n} < \cdots, \quad \lim_{n\to\infty} j_{\nu,n} = \infty,$$

$$0 < y_{\nu,1} < y_{\nu,2} < \cdots < y_{\nu,n} < \cdots, \quad \lim_{n\to\infty} y_{\nu,n} = \infty.$$

Between two consecutive positive zeros of $J_\nu(x)$, there is exactly one zero of $J_{\nu+1}(x)$. Conversely, between two consecutive positive zeros of $J_{\nu+1}(x)$, there is exactly one zero of $J_\nu(x)$. The same holds for the zeros of $Y_\nu(z)$. Moreover, between each pair of consecutive positive zeros of $J_\nu(x)$, there is exactly one zero of $Y_\nu(x)$, and conversely.

6.18.13 ASYMPTOTICS OF THE ZEROS

When $\nu$ is fixed, $s \gg \nu$, and $\mu = 4\nu^2$,

$$j_{\nu,s} \sim \alpha - \frac{\mu - 1}{8\alpha} \left[ 1 - \frac{4(7\mu^2 - 31)}{3(8\alpha)^2} - \frac{32(83\mu^2 - 982\mu + 3779)}{15(8\alpha)^4} + \ldots \right]$$

where $\alpha = (s + \frac{1}{2}v - \frac{1}{3})\pi$; $y_{\nu,s}$ has the same asymptotic expansion with $\alpha = (s + \frac{1}{2}v - \frac{1}{3})\pi$.

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Positive zeros $j_{\nu,n}, y_{\nu,n}$ of Bessel functions $J_\nu(x), Y_\nu(x), \nu = 0, 1$. ©1996 CRC Press LLC
6.18.14 HALF ORDER BESSEL FUNCTIONS

For integer values of \( n \), let

\[
\begin{align*}
 j_n(z) &= \sqrt{\pi/(2z)} \, J_{n+\frac{1}{2}}(z), \\
 y_n(z) &= \sqrt{\pi/(2z)} \, Y_{n+\frac{1}{2}}(z).
\end{align*}
\]

Then

\[
\begin{align*}
 j_0(z) &= y_{-1}(z) = \sin z/z, \\
 y_0(z) &= j_{-1}(z) = -\cos z/z,
\end{align*}
\]

and, for \( n = 0, 1, 2, \ldots \),

\[
\begin{align*}
 j_n(z) &= (-z)^n \left[ \frac{1}{z} \frac{d}{dz} \right]^n \sin z, \\
 y_n(z) &= -(-z)^n \left[ \frac{1}{z} \frac{d}{dz} \right]^n \cos z.
\end{align*}
\]

Recursion relationships

The functions \( j_n(z), y_n(z) \) both satisfy

\[
\begin{align*}
 z[f_{n-1}(z) + f_{n+1}(z)] &= (2n + 1) f_n(z), \\
 nf_{n-1}(z) - (n+1) f_{n+1}(z) &= (2n + 1) f'_n(z).
\end{align*}
\]

Differential equation

\[
z^2 f'' + 2zf' + [z^2 - n(n + 1)] f = 0.
\]

6.18.15 MODIFIED BESSEL FUNCTIONS

Differential equation

\[
z^2 y'' + zy' - (z^2 + \nu^2) y = 0.
\]

With solutions \( I_\nu(z), K_\nu(z) \),

\[
\begin{align*}
 I_\nu(z) &= \left( \frac{z}{2} \right)^\nu \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{\Gamma(n + \nu + 1) \, n!}, \\
 K_\nu(z) &= \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu \pi},
\end{align*}
\]

where the right-hand side should be determined by a limiting process when \( \nu \) assumes integer values. When \( n = 0, 1, 2, \ldots \),

\[
K_n(z) = (-1)^{n+1} I_n(z) \ln \frac{z}{2} + \frac{1}{2} \left( \frac{z}{2} \right)^n \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left( \frac{-z^2}{4} \right)^k
\]

\[
+ \frac{(-1)^n}{2} \left( \frac{z}{2} \right)^n \sum_{k=0}^{\infty} \left[ \psi(k+1) + \psi(n+k+1) \right] \frac{(z/2)^{2k}}{k! \, (n+k)!}.
\]

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Relations with the ordinary Bessel functions

\[ I_\nu(z) = e^{-\frac{1}{2}\nu\pi i} J_\nu \left( z e^{\frac{1}{2}\pi i} \right), \quad -\pi < \arg z \leq \frac{1}{2}\pi, \]

\[ I_\nu(z) = e^{\frac{1}{2}\nu\pi i} J_\nu \left( z e^{-\frac{1}{2}\pi i} \right), \quad \frac{1}{2}\pi < \arg z \leq \pi, \]

\[ K_\nu(z) = \frac{1}{2\pi} i e^{\frac{1}{2}\nu\pi i} H_{\nu}^{(1)} \left( z e^{\frac{1}{2}\pi i} \right), \quad -\pi < \arg z \leq \frac{1}{2}\pi, \]

\[ K_\nu(z) = -\frac{1}{2\pi} i e^{-\frac{1}{2}\nu\pi i} H_{\nu}^{(2)} \left( z e^{-\frac{1}{2}\pi i} \right), \quad -\frac{1}{2}\pi < \arg z \leq \pi, \]

\[ Y_\nu \left( z e^{\frac{1}{2}\pi i} \right) = e^{\frac{1}{2}(\nu+1)\pi i} I_\nu(z) - \frac{2}{\pi} e^{-\frac{1}{2}\nu\pi i} K_\nu(z), \quad -\pi < \arg z \leq \frac{1}{2}\pi. \]

For \( n = 0, 1, 2, \ldots \),

\[ I_n(z) = i^{-n} J_n(iz), \quad Y_n(iz) = i^{n+1} I_n(z) - \frac{2}{\pi} i^{-n} K_n(z), \]

\[ I_{-n}(z) = I_n(z), \quad K_{-n}(z) = K_n(z), \quad \text{for any } \nu. \]

Recursion relationships

\[ I_{\nu-1}(z) - I_{\nu+1}(z) = \frac{2\nu}{z} I_\nu(z), \quad K_{\nu+1}(z) - K_{\nu-1}(z) = \frac{2\nu}{z} K_\nu(z) \]

\[ I_{\nu-1}(z) + I_{\nu+1}(z) = 2I_\nu'(z), \quad K_{\nu-1}(z) + K_{\nu+1}(z) = -2K_\nu'(z). \]

Integrals

\[ I_\nu(z) = \frac{1}{\pi} \int_0^\pi e^{z\cos \theta} \cos (\nu \theta) \, d\theta - \frac{\sin \nu \pi}{\pi} \int_0^\infty e^{-zt - z \cosh t} \, dt, \]

\[ K_\nu(z) = \int_0^\infty e^{-zt - z \cosh t} \, dt. \]

When \( \nu = n \) (integer), the second integral in the first relation disappears.
### 6.18.16 NUMERICAL VALUES

<table>
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<th>( J_1(x) )</th>
<th>( Y_0(x) )</th>
<th>( Y_1(x) )</th>
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6.19 ELLIPTIC INTEGRALS

6.19.1 DEFINITIONS

Any integral of the type \( \int R(x, y) \, dx \), where \( R(x, y) \) is a rational function of \( x \) and \( y \), with

\[
y^2 = a_0 x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4, \quad |a_0| + |a_1| > 0,
\]

<table>
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<td>0.16397227</td>
<td>0.54780756</td>
<td>0.6002738</td>
</tr>
</tbody>
</table>

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a polynomial of the third or fourth degree in \( x \), is called an elliptic integral.

\[
F(\phi, k) = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}},
\]
\[
= \int_0^x \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}}, \quad x = \sin \phi, \quad k^2 < 1.
\]

\[
E(\phi, k) = \int_0^\phi \sqrt{1 - k^2 \sin^2 \theta} \, d\theta,
\]
\[
= \int_0^x \frac{\sqrt{1 - K^2 t^2}}{\sqrt{1 - t^2}} \, dt, \quad x = \sin \phi, \quad k^2 < 1.
\]

\[
\Pi(n; \phi, k) = \int_0^\phi \frac{1}{1 + n \sin^2 \theta \sqrt{1 - k^2 \sin^2 \theta}} \, d\theta
\]
\[
= \int_0^x \frac{1}{1 + n t^2 \sqrt{(1 - t^2)(1 - k^2 t^2)}} \, dt, \quad x = \sin \phi, \quad k^2 < 1.
\]

If \( n > 1 \), this integral should be interpreted as a Cauchy principal value integral.

### 6.19.2 COMPLETE ELLIPTIC INTEGRAL OF THE FIRST AND SECOND KIND

In terms of the Gauss hypergeometric function,

\[
K = F\left(\frac{1}{2} \pi, k\right) = \frac{1}{2} \pi F\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right),
\]
\[
E = E\left(\frac{1}{2} \pi, k\right) = \frac{1}{2} \pi F\left(-\frac{1}{2}, \frac{1}{2}; 1; k^2\right).
\]

### 6.19.3 COMPLEMENTARY INTEGRALS

In these expressions, primes do not mean mean derivatives.

\[
K' = F\left(k', \frac{1}{2} \pi\right), \quad E' = E\left(k', \frac{1}{2} \pi\right), \quad k' = \sqrt{1 - k^2}.
\]

\( k \) is called the modulus, \( k' \) is called the complementary modulus.

The Legendre relation is,

\[
K E' + E K' - K K' = \frac{1}{2} \pi.
\]
The complete elliptic integrals $E$ and $K$, $0 \leq k \leq 1$.\(^5\)

6.19.4 EXTENSION OF THE RANGE OF $\phi$

\[
F(\pi, k) = 2K, \quad E(\pi, k) = 2E,
\]

and, for $m = 0, 1, 2, \ldots$,

\[
\begin{align*}
F(\phi + m\pi, k) &= mF(\pi, k) + F(\phi, k) = 2mK + F(\phi, k), \\
E(\phi + m\pi, k) &= mE(\pi, k) + E(\phi, k) = 2mE + E(\phi, k).
\end{align*}
\]

6.19.5 DEFINITION

Elliptic functions are the inverses of elliptic integrals. If $u = F(\phi, k)$ (the elliptic integral of the first kind), that is,

\[
u = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \int_0^x \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}}, \quad x = \sin \phi, \quad k^2 < 1,
\]

then $x(u)$, the inverse function, is an elliptic function. The inverse relation is written as

\[
x = \sin \phi = sn(u, k).
\]

$\phi$ is called the amplitude of $u$ and denoted by $am u$. Two other functions are defined by

\[
\begin{align*}
\text{cn}(u, k) &= \cos \phi, \quad \text{dn}(u, k) = \sqrt{1 - k^2 \sin^2 \phi} = \sqrt{1 - k^2 sn^2(u, k)} = \Delta(\phi).
\end{align*}
\]

Analogous definitions of these functions are

\[
u = \int_1^{\text{cn}(u, k)} \frac{dt}{\sqrt{(1 - t^2)(k^2 t^2 + k^2)}} \quad \text{and} \quad \int_1^{\text{dn}(u, k)} \frac{dt}{\sqrt{(1 - t^2)(t^2 - k^2)}}.
\]

6.19.6 PROPERTIES

\[
\begin{align*}
\text{sn}^2(u, k) + \text{cn}^2(u, k) &= 1, \\
\text{dn}^2(u, k) + k^2 \text{sn}^2(u, k) &= 1, \\
\text{dn}^2(u, k) - k^2 \text{cn}^2(u, k) &= 1 - k^2 = k'^2.
\end{align*}
\]
6.19.7 Periods of the Elliptic Functions

The elliptic functions are **doubly periodic functions** with respect to the variable \( u \).

The periods of \( \text{sn}(u, k) \) are \( 4K \) and \( 2iK' \),
\( \text{cn}(u, k) \) are \( 4K \) and \( 2K + 2iK' \),
\( \text{dn}(u, k) \) are \( 2K \) and \( 4iK' \).

6.19.8 Series Expansions

\[
\begin{align*}
\text{sn}(u, k) &= u - (1 + k^2) \frac{u^3}{3!} + (1 + 14k^2 + k^4) \frac{u^5}{5!} - (1 + 135k^2 + 135k^4 + k^6) \frac{u^7}{7!} + \ldots, \\
\text{cn}(u, k) &= 1 - \frac{u^2}{2!} + (1 + 4k^2) \frac{u^4}{4!} - (1 + 44k^2 + 16k^4) \frac{u^6}{6!} + \ldots, \\
\text{dn}(u, k) &= 1 - k^2 \frac{u^2}{2!} + k^2(4 + k^2) \frac{u^4}{4!} - k^2(16 + 44k^2 + k^4) \frac{u^6}{6!} + \ldots.
\end{align*}
\]

Let the *nome* \( q \) be defined by \( q = e^{-\pi K / K'} \) and \( v = \pi u / (2K) \). Then

\[
\begin{align*}
\text{sn}(u, k) &= \frac{2\pi}{kK} \sum_{n=0}^{\infty} \frac{q^{n+\frac{1}{2}}}{1 - q^{2n+1}} \sin[(2n + 1)v], \\
\text{cn}(u, k) &= \frac{2\pi}{kK} \sum_{n=0}^{\infty} \frac{q^{n+\frac{1}{2}}}{1 + q^{2n+1}} \cos[(2n + 1)v], \\
\text{dn}(u, k) &= \frac{\pi}{2K} + \frac{2\pi}{K} \sum_{n=1}^{\infty} \frac{q^n}{1 + q^{2n}} \cos(2nv).
\end{align*}
\]
6.20 CLEBSCH–GORDAN COEFFICIENTS

\[
\begin{aligned}
\binom{j_1}{m_1} \frac{j_2}{m_2} & \binom{j}{m} = \delta_{m,m_1+m_2} \sqrt{(j_1+j_2-j)!(j+j_1-j_2)!(j+j_2-j_1)!(2j+1)} \frac{(j_1+j_2+1)!}{(j_1+j_1+j_2)!} \\
& \times \sum_k (-1)^k \sqrt{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(j+m)!(j-m)!} \\
& \times k!(j_1+j_2-j-k)!(j_1-m_1-k)!(j_2+m_2-k)!(j-j_2+m_1+k)!(j-j_1-m_2+k)!.
\end{aligned}
\]

Conditions:

- Each of \{j_1, j_2, j, m_1, m_2, m\} may be an integer, or half an integer. Additionally:
  - \(j > 0, j_1 > 0, j_2 > 0\) and \(j + j_1 + j_2\) is an integer.
  - \(j_1 + j_2 - j \geq 0\).
  - \(j_1 - j_2 + j \geq 0\).
  - \(-j_1 + j_2 + j \geq 0\).
  - \(|m_1| \leq j_1, |m_2| \leq j_2, |m| \leq j\).
  - \(\binom{j_1}{m_1} \frac{j_2}{m_2} \binom{j}{m} = 0\) if \(m_1 + m_2 \neq m\).

Special values:

- \(\binom{j_1}{m_1} \frac{0}{m} \binom{j}{m} = \delta_{m,j} \delta_{m_1,m}\).
- \(\binom{j_1}{0} \frac{j_2}{0} \binom{j}{0} = 0\) when \(j_1 + j_2 + j\) is an odd integer.
- \(\binom{j_1}{m_1} \frac{j_1}{m_1} \binom{j}{m} = 0\) when \(2j_1 + j\) is an odd integer.

Symmetry relations: all of the following are equal to \(\binom{j_1}{m_1} \frac{j_2}{m_2} \binom{j}{m}\):

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\begin{itemize}
  \item \[
    \begin{pmatrix}
      j_2 & j_1 \\
      -m_2 & -m_1
    \end{pmatrix}
  \]
  \item \[
    (-1)^{j_1+j_2-j}
    \begin{pmatrix}
      j_2 & j_1 \\
      m_1 & m_2
    \end{pmatrix}
  \]
  \item \[
    (-1)^{j_1+j_2-j}
    \begin{pmatrix}
      j_1 & j_2 \\
      -m_1 & -m_2
    \end{pmatrix}
  \]
  \item \[
    \sqrt{\frac{j_2+1}{2j_2+1}} (-1)^{j_2+m_2}
    \begin{pmatrix}
      j & j_2 \\
      -m & m_2
    \end{pmatrix}
  \]
  \item \[
    \sqrt{\frac{j_2+1}{2j_2+1}} (-1)^{j_2+m_1+j-m}
    \begin{pmatrix}
      j & j_2 \\
      m & -m_2
    \end{pmatrix}
  \]
  \item \[
    \sqrt{\frac{j_2+1}{2j_2+1}} (-1)^{j_2+m_1}
    \begin{pmatrix}
      j_1 & j_2 \\
      m_1 & -m_2
    \end{pmatrix}
  \]
  \item \[
    \sqrt{\frac{j_2+1}{2j_2+1}} (-1)^{j_2-m_1}
    \begin{pmatrix}
      j_1 & j_2 \\
      m & m_2
    \end{pmatrix}
  \]
\end{itemize}

By use of the symmetry relations, Clebsch–Gordan coefficients may be put in
the standard form \( j_1 \leq j_2 \leq j \) and \( m \geq 0 \).

| \( m_2 \) | \( m \) | \( j_1 \) | \( j \) | \( \begin{pmatrix}
      j_1 & j_2 \\
      m_1 & m_2
    \end{pmatrix}
  \) |
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<td>$m$</td>
<td>$j_1$</td>
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|       | 1   | 0     | 1   | -0.707105 |
| $-1/2$| 1/2 | 1     | 1   | 0.750000  
|       | 0   | 1/2   | 1   | 0.353553  
|       | 1/2 | 0     | 1   | -0.353553 |
|       | 1/2 | 1/2   | 1   | -0.750000 
|       | 0   | 1     | 1   | 0.707105  
|       | 1   | 1     | 1   | -0.707105 |
| $-1/2$| 0   | 1/2   | 3/2 | 0.707105  
|       | 0   | 0     | 1/2 | 0.866025  
|       | 1/2 | 0     | 1/2 | 0.707105  
|       | 0   | 1/2   | 3/2 | 0.816496  
|       | 1/2 | 1/2   | 3/2 | 0.577349  
|       | 1/2 | 1     | 1/2 | 0.912873  
|       | 1   | 1     | 1/2 | 0.790570  
|       | 1   | 3/2   | 1/2 | 1.000000  
|       | -1  | 0     | 1    | 0.408249  
|       | 0   | 0     | 1    | 0.816496  
|       | 1   | 0     | 1    | 0.408249  
| $-1/2$| 1/2 | 1    | 2    | 0.559018  
|       | 0   | 1/2   | 1    | 0.790570  
|       | 1/2 | 1/2   | 1    | 0.790570  
|       | 1   | 1/2   | 1    | 0.559018  
|       | 0   | 1     | 1    | 0.707105  
|       | 1   | 1     | 1    | 0.707105  
|       | 1/2 | 3/2   | 1    | 0.853916  
|       | 1   | 3/2   | 1    | 0.853916  
|       | 1   | 2     | 1    | 1.000000  |

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6.21 INTEGRAL TRANSFORMS: PRELIMINARIES

- \( I = (a, b) \) is an interval, where \(-\infty \leq a < b \leq \infty\).
- \( L^1(I) \) is the set of all absolutely integrable functions on \( I \). In particular, \( L^1(\mathbb{R}) \) is the set of all absolutely integrable functions on the real line \( \mathbb{R} \).
- \( L^2(I) \) is the set of all square integrable functions on \( I \).
- If \( f \) is integrable over every finite closed subinterval of \( I \), but not necessarily on \( I \) itself, we say that \( f \) is locally integrable on \( I \). For example, the function \( f(x) = \frac{1}{x} \) is not integrable on the interval \( I = (0, 1) \), yet it is locally integrable on it.
- A function \( f(x) \), defined on a closed interval \([a, b]\), is said to be of bounded variation if there is a positive number \( M \) so that, for any partition \( a = x_0 < x_1 < x_2 < \cdots, x_n = b \), the following relation holds:
  \[
  \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq M.
  \]
- If \( f \) has a derivative \( f' \) at every point of \([a, b]\), then by the mean value theorem, for any \( a \leq x < y \leq b \), we have \( f(x) - f(y) = f'(z)(x - y) \), for some \( x < z < y \). If \( f' \) is bounded, then \( f \) is of bounded variation.
- The left limit of a function \( f(x) \) at a point \( t \) (if it exists) will be denoted by \( \lim_{x \to t^-} f(x) = f(t-) \), and likewise the right limit at \( t \) will be denoted by \( \lim_{x \to t^+} f(x) = f(t+) \).

6.22 FOURIER INTEGRAL TRANSFORM

The origin of the Fourier integral transformation can be traced to Fourier’s celebrated work on the Analytical Theory of Heat, which appeared in 1822. Fourier’s major finding was to show that an “arbitrary” function defined on a finite interval could be expanded in a trigonometric series (series of sinusoidal functions). In an attempt to extend his results to functions defined on the infinite interval \((0, \infty)\), Fourier introduced what is now known as the Fourier integral transform.

The Fourier integral transform of a function \( f(t) \) is defined by

\[
\mathcal{F}(f)(\omega) = \hat{f}(\omega) = F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{it\omega} dt,
\]

whenever the integral exists.

There is no universal agreement on the definition of the Fourier integral transform. Some authors take the kernel of the transformation as \( e^{-it\omega} \), so that the kernel of the
Inverse transformation is \( e^{it\omega} \). In either case, if we define the Fourier transform as

\[
\hat{f}(\omega) = a \int_{-\infty}^{\infty} f(t) e^{\pm it\omega} dt,
\]

then its inverse is

\[
f(t) = b \int_{-\infty}^{\infty} \hat{f}(\omega) e^{\mp it\omega} d\omega,
\]

(6.22.2)

for some constants \( a \) and \( b \), with \( ab = 1/2\pi \). Again there is no agreement on the choice of the constants; sometimes one of them is taken as 1 so that the other is \( 1/(2\pi) \). For the sake of symmetry, we choose \( a = b = 1/\sqrt{2\pi} \). The functions \( f \) and \( \hat{f} \) are called a Fourier transform pair.

Another definition that is popular in the engineering literature is the one in which the kernel of the transform is taken as \( e^{2\pi it\omega} \) (or \( e^{-2\pi it\omega} \)) so that the kernel of the inverse transform is \( e^{-2\pi it\omega} \) (or \( e^{2\pi it\omega} \)). The main advantage of this definition is that the constants \( a \) and \( b \) disappear and the Fourier transform pair becomes

\[
\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{\pm 2\pi it\omega} dt \quad \text{and} \quad f(t) = \int_{-\infty}^{\infty} \hat{f}(\omega) e^{\mp 2\pi it\omega} d\omega.
\]

(6.22.3)

The Fourier cosine and sine coefficients of \( f(t) \) are defined by

\[
a(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \omega t \, dt \quad \text{and} \quad b(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin \omega t \, dt.
\]

(6.22.4)

The Fourier cosine and sine coefficients are related to the Fourier cosine and sine integral transforms. For example, if \( f \) is even, then \( a(\omega) = \sqrt{2/\pi} F_c(\omega) \) and, if \( f \) is odd, \( b(\omega) = \sqrt{2/\pi} F_s(\omega) \) (see Section 6.22.7).

Two other integrals related to the Fourier integral transform are Fourier’s repeated integral and the allied integral. Fourier’s repeated integral, \( S(f, t) \), of \( f(t) \) is defined by

\[
S(f, t) = \frac{1}{\pi} \int_{0}^{\infty} (a(\omega) \cos \omega t + b(\omega) \sin \omega t) \, d\omega,
\]

\[
= \int_{0}^{\infty} d\omega \int_{-\infty}^{\infty} f(x) \cos \omega (t-x) \, dx.
\]

(6.22.5)

The allied Fourier integral, \( \tilde{S}(f, t) \), of \( f \) is defined by

\[
\tilde{S}(f, t) = \frac{1}{\pi} \int_{0}^{\infty} (b(\omega) \cos \omega t - a(\omega) \sin \omega t) \, d\omega,
\]

\[
= \int_{0}^{\infty} d\omega \int_{-\infty}^{\infty} f(x) \sin \omega (x-t) \, dx.
\]

(6.22.6)

### 6.22.1 Existence

For the Fourier integral transform to exist, it is sufficient that \( f \) be absolutely integrable on \(( -\infty, \infty )\), i.e., \( f \in L^1(\mathbb{R}) \).
THEOREM 6.22.1 (Riemann–Lebesgue lemma)

If \( f \in L^1(\mathbb{R}) \), then its Fourier transform \( \hat{f}(\omega) \) is defined everywhere, uniformly continuous, and tends to zero as \( \omega \to \pm \infty \).

The uniform continuity follows from the relationship

\[
|\hat{f}(\omega + h) - \hat{f}(\omega)| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(t)||e^{iht} - 1| \, dt,
\]

and the tendency toward zero at \( \pm \infty \) is a consequence of the Riemann–Lebesgue lemma.

THEOREM 6.22.2 (Riemann–Lebesgue lemma)

Let \( f \in L^1(I) \), where \( I = (a, b) \) is finite or infinite and let \( \omega \) be a real variable. Let \( a \leq a' < b' \leq b \) and \( \hat{f}_\omega(\lambda, a', b') = \int_{a'}^{b'} f(t)e^{i\lambda \omega t} \, dt \). Then \( \lim_{\omega \to \pm \infty} \hat{f}_\omega(\lambda, a', b') = 0 \), and the convergence is uniform in \( a' \) and \( b' \). In particular, \( \lim_{\omega \to \pm \infty} \int_{-\infty}^{\infty} f(t)e^{i\omega t} \, dt = 0 \).

6.22.2 PROPERTIES

1. Linearity: The Fourier transform is linear,

\[
\mathcal{F}[af(t) + bg(t)](\omega) = a\mathcal{F}[f(t)](\omega) + b\mathcal{F}[g(t)](\omega) = a\hat{f}(\omega) + b\hat{g}(\omega),
\]

where \( a \) and \( b \) are complex numbers.

2. Translation: \( \mathcal{F}[f(t - b)](\omega) = e^{ib\omega}\hat{f}(\omega) \).

3. Dilation (scaling): \( \mathcal{F}[f(at)](\omega) = \frac{1}{a}\hat{f}\left(\frac{\omega}{a}\right), \quad a \neq 0 \).

4. Translation and dilation:

\[
\mathcal{F}[f(at - b)](\omega) = \frac{1}{a}e^{ib\omega/a}\hat{f}\left(\frac{\omega}{a}\right), \quad a > 0.
\]

5. Complex conjugation: \( \mathcal{F}[f(t)](\omega) = \overline{\hat{f}(-\omega)} \).

6. Modulation: \( \mathcal{F}[e^{iatf(t)}](\omega) = \hat{f}(\omega + a) \), and

\[
\mathcal{F}[e^{iatf(bt)}](\omega) = \frac{1}{b}\hat{f}\left(\frac{\omega + a}{b}\right), \quad b > 0. \tag{6.22.7}
\]

7. Differentiation: If \( f^{(k)} \in L^1(\mathbb{R}) \), for \( k = 0, 1, 2, \cdots, n \) and \( \lim_{|t| \to \infty} f^{(k)}(t) = 0 \) for \( k = 0, 1, 2, \cdots, n - 1 \), then

\[
\mathcal{F}[f^{(n)}(t)](\omega) = (-i\omega)^n \hat{f}(\omega). \tag{6.22.8}
\]

8. Integration: Let \( f \in L^1(\mathbb{R}) \), and define \( g(x) = \int_{-\infty}^{x} f(t) \, dt \). If \( g \in L^1(\mathbb{R}) \), then \( \hat{g}(\omega) = -\hat{f}(\omega)/(i\omega) \).

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9. Multiplication by polynomials: If \( t^k f(t) \in L^1(\mathbb{R}) \), then
\[
\mathcal{F}[t^k f(t)](\omega) = \frac{1}{(i)^k} \hat{f}^{(k)}(\omega),
\]
and hence,
\[
\mathcal{F}\left[\sum_{k=0}^{n} a_k t^k \right] f(t)](\omega) = \sum_{k=0}^{n} \frac{a_k}{(i)^k} \hat{f}^{(k)}(\omega).
\]

10. Convolution: The convolution operation, \( \ast \), associated with the Fourier transform is defined as
\[
h(t) = (f \ast g)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)g(t-x)dx,
\]
where \( f \) and \( g \) are defined over the whole real line.

**Theorem 6.22.3**

If \( f \) and \( g \) belong to \( L^1(\mathbb{R}) \), then so does \( h \). Moreover, \( \hat{h}(\omega) = \hat{f}(\omega)\hat{g}(\omega) \). If \( \hat{f} \) and \( \hat{g} \) belong to \( L^1(\mathbb{R}) \), then
\[
(\hat{f} \ast \hat{g})(\omega) = (\hat{f} \ast \hat{g})(\omega).
\]

11. Parseval’s relation: If \( F \) and \( G \) are the Fourier transforms of \( f \) and \( g \) respectively, then Parseval’s relation is
\[
\int_{-\infty}^{\infty} F(\omega)G(\omega) d\omega = \int_{-\infty}^{\infty} f(t)g(-t) dt.
\]
Replacing \( G \) by \( \bar{G} \) (so that \( g(-t) \) is replaced by \( \bar{g}(t) \)) results in a more convenient form of Parseval’s relation
\[
\int_{-\infty}^{\infty} F(\omega)\bar{G}(\omega) d\omega = \int_{-\infty}^{\infty} f(t)\bar{g}(t) dt.
\]
In particular, for \( f = g \),
\[
\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |f(t)|^2 dt.
\]

### 6.22.3 Inversion Formula

Many of the theorems on the inversion of the Fourier transform are based on Dini’s condition which can be stated as follows:

If \( f \in L^1(\mathbb{R}) \), then a necessary and sufficient condition that
\[
S(f, x) = \lim_{\lambda \to \infty} \frac{1}{\pi} \int_{-\infty}^{\infty} f(t)\frac{\sin \lambda(x-t)}{(x-t)} dt = a
\]
is that
\[
\lim_{\lambda \to \infty} \int_0^\delta (f(x+y) + f(x-y) - 2a) \frac{\sin \lambda y}{y} dy = 0,
\]
for any fixed \(\delta > 0\).

By the Riemann–Lebesgue lemma, this condition is satisfied if
\[
\int_0^\delta \frac{|f(x+y) + f(x-y) - 2a|}{y} dy < \infty,
\]
for some \(\delta > 0\). In particular, condition (6.22.15) holds for \(a = f(x)\), if \(f\) is differentiable at \(x\), and for \(a = \frac{[f(x+0) + f(x-0)]}{2}\), if \(f\) is of bounded variation in a neighborhood of \(x\).

THEOREM 6.22.4 (Inversion theorem)
Let \(f\) be a locally integrable function, of bounded variation in a neighborhood of the point \(x\). If \(f\) satisfies either one of the following conditions:

1. \(f(t) \in L^1(\mathbb{R})\), or
2. \(f(t)/(1 + |t|) \in L^1(\mathbb{R})\), and the integral \(\int_{-\infty}^\infty f(t) e^{i\omega t} dt\) converges uniformly on every finite interval of \(\omega\),

then
\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \hat{f}(t) e^{-ix\omega} dt = \lim_{\lambda \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-\lambda}^\lambda f(t) e^{-ix\omega} dt \]

is equal to \([f(x+0) + f(x-0)]/2\) whenever the expression has meaning, to \(f(x)\) whenever \(f(x)\) is continuous at \(x\), and to \(f(x)\) almost everywhere. If \(f\) is continuous and of bounded variation in the interval \((a,b)\), then the convergence is uniform in any interval interior to \((a,b)\).

6.22.4 POISSON SUMMATION FORMULA

The Poisson summation formula may be written in the form
\[
\frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} f\left(t + \frac{k\pi}{\sigma}\right) = \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \hat{f}(2k\sigma) e^{-2ik\omega}, \quad \sigma > 0,
\]
provided that the two series converge. A sufficient condition for the validity of Equation (6.22.16) is that \(f = O(1 + |t|)^{-\alpha}\) as \(|t| \to \infty\), and \(\hat{f} = O\left((1 + |\omega|)^{-\alpha}\right)\) as \(|\omega| \to \infty\) for some \(\alpha > 1\).

Another version of the Poisson summation formula is
\[
\sum_{k=-\infty}^{\infty} \hat{f}(\omega + k\sigma) \bar{g}(\omega + k\sigma) = \frac{1}{\sigma} \sum_{k=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) \bar{g}\left(t - \frac{2\pi k}{\sigma}\right) dt\right) e^{2\pi i k \omega / \sigma}.
\]

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6.22.5 SHANNON'S SAMPLING THEOREM

If $f$ is a function band-limited to $[-\sigma, \sigma]$, i.e.,

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} F(\omega) e^{i\omega t} d\omega,$$

with $F \in L^2(-\sigma, \sigma)$, then it can be reconstructed from its sample values at the points $t_k = (k\pi)/\sigma, k = 0, \pm 1, \pm 2, \cdots$, via the formula

$$f(t) = \sum_{k=-\infty}^{\infty} f(t_k) \frac{\sin \sigma(t - t_k)}{\sigma(t - t_k)}, \quad (6.22.18)$$

with the series absolutely and uniformly convergent on compact sets.

The series on the right-hand side of Equation (6.22.18) can be written as

$$\sin \sigma t \sum_{k=-\infty}^{\infty} (-1)^k \frac{f(t_k)}{(\sigma t - k\pi)},$$

which is a special case of a Cardinal series (these series have the form $\sin \sigma t \sum_{k=-\infty}^{\infty} C_k \frac{(-1)^k}{(\sigma t - k\pi)}$).

6.22.6 UNCERTAINTY PRINCIPLE

Let $T$ and $W$ be two real numbers defined by

$$T^2 = \frac{1}{E} \int_{-\infty}^{\infty} t^2 |f(t)|^2 dt \quad \text{and} \quad W^2 = \frac{1}{E} \int_{-\infty}^{\infty} \omega^2 |\hat{f}(\omega)|^2 d\omega,$$

where

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega.$$

Assuming that $f$ is differentiable and $\lim_{|t| \to \infty} tf^2(t) = 0$, then $2TW \geq 1$, or

$$\left( \int_{-\infty}^{\infty} t^2 |f(t)|^2 dt \right)^{1/2} \left( \int_{-\infty}^{\infty} \omega^2 |\hat{f}(\omega)|^2 d\omega \right)^{1/2} \geq \frac{1}{2} \int_{-\infty}^{\infty} |f(t)|^2 dt.$$

This means that $f$ and $\hat{f}$ cannot both be very small. Another related property of the Fourier transform is that, if either one of the functions $f$ or $\hat{f}$ vanishes outside some finite interval, then the other one must trail on to infinity. In other words, they can not both vanish outside any finite interval.

6.22.7 FOURIER SINE AND COSINE TRANSFORMS

The Fourier cosine transform, $F_c(\omega)$, and the Fourier sine transform, $F_s(\omega)$, of $f(t)$ are defined as

$$F_c(\omega) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \cos \omega t \, dt \quad \text{and} \quad F_s(\omega) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \sin \omega t \, dt. \quad (6.22.19)$$

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The inverse transforms have the same functional form:

\[ f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\omega) \cos \omega t \, d\omega = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(\omega) \sin \omega t \, d\omega. \] (6.22.20)

If \( f \) is even, i.e., \( f(t) = f(-t) \), then \( F(\omega) = F_c(\omega) \), and if \( f \) is odd, i.e., \( f(t) = -f(-t) \), then \( F(\omega) = iF_s(\omega) \).

### 6.23 DISCRETE FOURIER TRANSFORM (DFT)

The discrete Fourier transform of the sequence \( \{a_n\}_{n=0}^{N-1} \), where \( N \geq 1 \), is a sequence \( \{A_m\}_{m=0}^{N-1} \), defined by

\[ A_m = \sum_{n=0}^{N-1} a_n (W_N)^{mn}, \quad \text{for } m = 0, 1, \ldots, N-1, \] (6.23.1)

where \( W_N = e^{2\pi i/N} \). Note that \( \sum_{m=0}^{N-1} W_N^{m(k-n)} = N \delta_{kn} \). For example, the DFT of the sequence \( \{1, 0, 1, 1\} \) is \( \{3, -i, 1, i\} \).

The inversion formula is

\[ a_n = \frac{1}{N} \sum_{m=0}^{N-1} A_m W_N^{-mn}, \quad n = 0, 1, \ldots, N-1. \] (6.23.2)

Equations (6.23.1) and (6.23.2) are called a discrete Fourier transform (DFT) pair of order \( N \). The factor \( 1/N \) and the negative sign in the exponent of \( W_N \) that appear in Equation (6.23.2) are sometimes introduced in Equation (6.23.1) instead. We use the notation

\[ \mathcal{F}_N[\{a_n\}] = A_m \quad \mathcal{F}_N^{-1}[\{A_m\}] = a_n, \]

to indicate that the discrete Fourier transform of order \( N \) of the sequence \( \{a_n\} \) is \( \{A_m\} \) and that the inverse transform of \( \{A_m\} \) is \( \{a_n\} \).

Because \( W_N^{(m+N)n} = W_N^{mn} \), Equations (6.23.1) and (6.23.2) can be used to extend the sequences \( \{a_n\}_{n=0}^{N-1} \) and \( \{A_m\}_{m=0}^{N-1} \), as periodic sequences with period \( N \). This means that \( A_{m+N} = A_m \), and \( a_{n+N} = a_n \). Using this, the summation limits, 0 and \( N-1 \), can be replaced with \( n_1 \) and \( n_1 + N - 1 \), respectively, where \( n_1 \) is any integer. In the special case where \( n_1 = -M \) and \( N = 2M + 1 \), Equations (6.23.1) and (6.23.2) become

\[ A_m = \sum_{n=-M}^{M} a_n W_N^{mn}, \quad \text{for } m = -M, -M+1, \ldots, M-1, M, \] (6.23.3)

and

\[ a_n = \frac{1}{2M+1} \sum_{m=-M}^{M} A_m W_N^{-mn}, \quad \text{for } n = -M, -M+1, \ldots, M-1, M. \] (6.23.4)
6.23.1 PROPERTIES

1. **Linearity:** The discrete Fourier transform is linear, that is

   \[ \mathcal{F}_N[\alpha(a_n) + \beta(b_n)] = \alpha A_m + \beta B_m, \]

   for any complex numbers \( \alpha \) and \( \beta \), where the sum of two sequences is defined as \( (a_n) + (b_n) = (a_n + b_n) \).

2. **Translation:** \( \mathcal{F}_N[(a_n - k)] = W_N^k A_m \), or \( e^{2\pi i nk/N} A_m = \sum_{m=0}^{N-1} a_n W_N^{mn} \).

3. **Modulation:** \( \mathcal{F}_N[W_N a_n] = A_m + k \), or \( A_m + k = \sum_{m=0}^{N-1} \bar{a}_n W_N^{mn} \).

4. **Complex Conjugation:** \( \mathcal{F}_N[\bar{a}_n] = \bar{A}_m \), or \( \bar{A}_m = \sum_{m=0}^{N-1} \bar{a}_n W_N^{mn} \).

5. **Symmetry:** \( \mathcal{F}_N[a_n] = \bar{A}_m \), or \( \bar{A}_m = \sum_{m=0}^{N-1} a_n W_N^{mn} \).

6. **Convolution:** The convolution of the sequences \( \{a_n\}_{n=0}^{N-1} \) and \( \{b_n\}_{n=0}^{N-1} \) is the sequence \( \{c_n\}_{n=0}^{N-1} \) given by

   \[ c_n = \sum_{k=0}^{N-1} a_k b_{n-k}. \]  

   (6.23.5)

   The convolution relation of the DFT is \( \mathcal{F}_N[c_n] = \mathcal{F}_N[a_n] \mathcal{F}_N[b_n] \), or \( C_m = A_m B_m \). A consequence of this and Equation (6.23.2), is the relation

   \[ \sum_{k=0}^{N-1} a_k b_{n-k} = \frac{1}{N} \sum_{m=0}^{N-1} A_m B_m W_N^{-mn}. \]  

   (6.23.6)

7. **Parseval’s relation:**

   \[ \sum_{n=0}^{N-1} a_n \bar{a}_n = \frac{1}{N} \sum_{m=0}^{N-1} A_m D_m. \]  

   (6.23.7)

   In particular,

   \[ \sum_{n=0}^{N-1} |a_n|^2 = \frac{1}{N} \sum_{m=0}^{N-1} |A_m|^2. \]  

   (6.23.8)

   In (4) and (5), the fact that \( \bar{W}_N = W_N^{-1} \) has been used. A sequence \( \{a_n\} \) is said to be even if \( \{a_{-n}\} = \{a_n\} \) and is said to be odd if \( \{a_{-n}\} = \{-a_n\} \). The following are consequences of (4) and (5):

   1. If \( \{a_n\} \) is a sequence of real numbers, i.e., \( \bar{a}_n = a_n \), then \( \bar{A}_m = A_{-m} \).
   2. \( \{a_n\} \) is real and even if and only if \( \{A_m\} \) is real and even.
   3. \( \{a_n\} \) is real and odd if and only if \( \{A_m\} \) is pure imaginary and odd.
6.24 FAST FOURIER TRANSFORM (FFT)

To determine $A_m$ for each $m = 0, 1, \ldots, M - 1$ (using Equation (6.23.1)), $M - 1$ multiplications are required. Hence the total number of multiplications required to determine all the $A_m$‘s is $(M - 1)^2$. This number can be reduced by using decimation.

Assuming $M$ is even, we put $M = 2N$ and write

$$F_{2^N}[a_n] = A_m. \quad (6.24.1)$$

Now split $\{a_n\}$ into two sequences, one consisting of terms with even subscripts ($b_n = a_{2n}$) and one with odd subscripts ($c_n = a_{2n+1}$). Then

$$A_m = B_m + W_{2N}^m C_m. \quad (6.24.2)$$

For the evaluation of $B_m$ and $C_m$, the total number of multiplications required is $2(N - 1)^2$. To determine $A_m$ from Equation (6.24.2), we must calculate the product $W_{2N}^m C_m$ for each fixed $m$. Therefore, the total number of multiplications required to determine $A_m$ from Equation (6.24.2) is $2(N - 1)^2 + 2N - 1 = 2N^2 - 2N + 1$.

But had we determined $A_m$ from (6.24.1), we would have performed $(2N - 1)^2 = 4N^2 - 4N + 1$ multiplications. Thus, splitting the sequence $\{a_n\}$ into two sequences and then applying the discrete Fourier transform, reduces the number of multiplications required to evaluate $A_m$ approximately by a factor of 2.

If $N$ is even, this process can be repeated. Split $\{b_n\}$ and $\{c_n\}$ into four sequences, each of length $N/2$. Then $B_m$ and $C_m$ are determined in terms of four discrete Fourier transforms, each of order $N/2$. This process can be continued $k - 1$ times if $M = 2^k$ for some positive integer $k$.

If we denote the required number of multiplications for the discrete Fourier transform of order $N = 2^k$ by $F(N)$, then $F(2N) = 2F(N) + N$ and $F(2) = 1$, which leads to $F(N) = \frac{N}{2} \log_2 N$.

6.25 MULTIDIMENSIONAL FOURIER TRANSFORMS

If $x = (x_1, x_2, \ldots, x_n)$ and $u = (u_1, u_2, \ldots, u_n)$, then

1. Fourier transform

$$F(u) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-i(x \cdot u)} \, dx.$$  

2. Inverse Fourier transform

$$f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} F(u) e^{i(x \cdot u)} \, du.$$  

3. Parseval’s relation

$$\int_{\mathbb{R}^n} f(x) \overline{g(x)} \, dx = (2\pi)^{-n} \int_{\mathbb{R}^n} F(u) \overline{G(u)} \, du.$$  

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6.26 LAPLACE TRANSFORM

The Laplace transformation dates back to the work of the French mathematician, Pierre Simon Marquis de Laplace (1749–1827), who used it in his work on probability theory in the 1780’s.

The Laplace transform of a function \( f(t) \) is defined as

\[
F(s) = (\mathcal{L} f)(s) = \int_0^\infty f(t)e^{-st} \, dt, \tag{6.26.1}
\]

whenever the integral exists for at least one value of \( s \). The transform variable, \( s \), can be taken as a complex number. We say that \( f \) is Laplace transformable or the Laplace transformation is applicable to \( f \) if \( (\mathcal{L} f) \) exists for at least one value of \( s \).

The integral on the right-hand side of Equation (6.26.1) is called the Laplace integral of \( f \).

6.26.1 EXISTENCE AND DOMAIN OF CONVERGENCE

Sufficient conditions for the existence of the Laplace transform are

1. \( f \) is a locally integrable function, i.e., \( \int_0^a |f(t)| \, dt < \infty \), for any \( a > 0 \).
2. \( f \) is of (real) exponential type, i.e., for some constants \( M, t_0 > 0 \) and real \( \gamma \), \( f \) satisfies

\[
|f(t)| \leq Me^{\gamma t}, \quad \text{for all } t \geq t_0. \tag{6.26.2}
\]

If \( f \) is a locally integrable function on \([0, \infty)\) and of (real) exponential type \( \gamma \), then the Laplace integral of \( f \), \( \int_0^\infty f(t)e^{-st} \, dt \), converges absolutely for \( \Re s > \gamma \) and uniformly for \( \Re s \geq \gamma_1 > \gamma \). Consequently, \( F(s) \) is analytic in the half-plane \( \Omega = \{ s \in \mathbb{C} : \Re s > \gamma \} \). It can be shown that if \( F(s) \) exists for some \( s_0 \), then it also exists for any \( s \) for which \( \Re s > \Re s_0 \). The actual domain of existence of the Laplace transform may be larger than the one given above. For example, the function \( f(t) = \cos \omega t \) is of real exponential type zero, but \( F(s) \) exists for \( \Re s > -1 \).

If \( f(t) \) is a locally integrable function on \([0, \infty)\), not of exponential type, and

\[
\int_0^\infty f(t)e^{-st} \, dt \tag{6.26.3}
\]

cconverges for some complex number \( s_0 \), then the Laplace integral

\[
\int_0^\infty f(t)e^{-st} \, dt \tag{6.26.4}
\]

cconverges in the region \( \Re s > \Re s_0 \) and converges uniformly in the region \( |\arg(s - s_0)| \leq \theta' < \frac{\pi}{2} \). Moreover, if Equation (6.26.3) diverges, then so does Equation (6.26.4) for \( \Re s < \Re s_0 \).
6.26.2 PROPERTIES

1. Linearity: $\mathcal{L}(\alpha f + \beta g) = \alpha \mathcal{L}(f) + \beta \mathcal{L}(g) = \alpha F + \beta G$, for any constants $\alpha$ and $\beta$.

2. Dilation: $[\mathcal{L}(f(at))] (s) = \frac{1}{a} F \left( \frac{s}{a} \right)$, for $a > 0$.

3. Multiplication by Exponential Functions: $[\mathcal{L} \left( e^{-at}f(t) \right)] (s) = F(s - a)$.

4. Translation: $[\mathcal{L}(f(t-a)H(t-a))] (s) = e^{-as}F(s)$ for $a > 0$. This can be put in the form $[\mathcal{L}(f(t)H(t-a))] (s) = e^{-as} [\mathcal{L}(f(t+a))] (s)$, where $H$ is the Heaviside function. Examples:
   (a) If $g(t) = \begin{cases} 0, & 0 \leq t \leq a, \\ (t-a)^v, & a < t, \end{cases}$ then $g(t) = f(t-a)H(t-a)$ where $f(t) = t^v (Re \nu > -1)$. Since $\mathcal{L}(t^v) = \Gamma(v+1)/s^{v+1}$, it follows that $(\mathcal{L}g)(s) = e^{-as} \Gamma(v+1)/s^{v+1}$, for $Re s > 0$.
   (b) If 
   $$g(t) = \begin{cases} t, & 0 \leq t \leq a, \\ 0, & a < t, \end{cases}$$ 
   we may write $g(t) = t[H(t) - H(t-a)] = tH(t) - (t-a)H(t-a) - aH(t-a)$. Thus by properties (1) and (4),
   $$G(s) = \frac{1}{s^2} - \frac{1}{s^2} e^{-as} - \frac{a}{s} e^{-as}.$$

5. Differentiation of the transformed function: If $f$ is a differentiable function of exponential type, $\lim_{t \to 0^+} f(t)$ exists, and $f'$ is locally integrable on $[0, \infty)$, then the Laplace transform of $f'$ exists, and 
   $$(\mathcal{L} f') (s) = sF(s) - f(0). \quad (6.26.5)$$ 
   Note that although $f$ is assumed to be of exponential type, $f'$ need not be. For example, $f(t) = \sin e^{it}$, but $f'(t) = 2t e^{it} \cos e^{it}$.

6. Differentiation of higher orders: Let $f$ be an $n$ differentiable function so that $f^{(k)}$ (for $k = 0, 1, \ldots, n-1$) are of exponential type with the additional assumption that $\lim_{t \to 0^+} f^{(k)}(t) = f^{(k)}(0^+)$ exists. If $f^{(n)}$ is locally integrable on $[0, \infty)$, then its Laplace transform exists, and
   $$[\mathcal{L} \left( f^{(n)} \right)] (s) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0). \quad (6.26.6)$$
7. **Integration:** If \( g(t) = \int_0^t f(x) \, dx \), then (if the transforms exist) \( G(s) = F(s)/s \).

Repeated applications of this rule result in

\[
\left[ \mathcal{L} \left( f^{(-n)} \right) \right](s) = \frac{1}{s^n} F(s),
\]

(6.26.7)

where \( f^{(-n)} \) is the \( n \)th antiderivative of \( f \) defined by \( f^{(-n)}(t) = \int_0^t dt_n \int_0^{t_{n-1}} \ldots \int_0^{t_1} f(t) \, dt \). **Section 6.26.1** shows that the Laplace transform is an analytic function in a half-plane. Hence it has derivatives of all orders at any point in that half-plane. The next property shows that we can evaluate these derivatives by direct differentiation.

8. **Multiplication by powers of \( t \):** Let \( f \) be a locally integrable function whose Laplace integral converges absolutely and uniformly for \( \Re s > \sigma \). Then \( F \) is analytic in \( \Re s > \sigma \), and (for \( n = 0, 1, 2, \ldots \), with \( \Re s > \sigma \))

\[
\left[ \mathcal{L} \left( t^n f(t) \right) \right](s) = \left( -\frac{d}{ds} \right)^n F(s),
\]

(6.26.8)

\[
\left[ \mathcal{L} \left( \left( \frac{d}{dt} \right)^n f(t) \right) \right](s) = \left( \frac{d}{ds} \right)^n F(s),
\]

where \( \left( \frac{d}{dt} \right)^n \) is the operator \( \left( \frac{d}{dt} \right) \) applied \( n \) times.

9. **Division by powers of \( t \):** If \( f \) is a locally integrable function of exponential type so that \( f(t)/t \) is a Laplace transformable function, then

\[
\left[ \mathcal{L} \left( \frac{f(t)}{t} \right) \right](s) = \int_s^\infty F(u) \, du,
\]

(6.26.9)

or, more generally,

\[
\left[ \mathcal{L} \left( \frac{f(t)}{t^n} \right) \right](s) = \int_s^\infty \cdots \int_s^\infty F(s) \, (ds)^n
\]

(6.26.10)

is the \( n \)th repeated integral. It follows from properties (7) and (9) that

\[
\left[ \mathcal{L} \left( \int_0^t \frac{f(x)}{x} \, dx \right) \right](s) = \frac{1}{s} \int_s^\infty F(u) \, du.
\]

(6.26.11)

10. **Periodic functions:** Let \( f \) be a locally integrable function that is periodic with period \( T \). Then

\[
[\mathcal{L}(f)](s) = \frac{1}{(1 - e^{-sT})} \int_0^T f(t)e^{-st} \, dt.
\]

(6.26.12)

11. **Hardy’s theorem:** If \( f(t) = \sum_{n=0}^\infty c_n t^n \) for \( t \geq 0 \) and \( \sum_{n=0}^\infty \frac{c_n s^n}{s_0^n} \) converges for some \( s_0 > 0 \), then \( [\mathcal{L}(f)](s) = \sum_{n=0}^\infty \frac{c_n s^n}{s_0^n} \) for \( \Re s > s_0 \).
6.26.3 INVERSION FORMULAE

**Inversion by integration**

If \( f(t) \) is a locally integrable function on \([0, \infty)\) such that

1. \( f \) is of bounded variation in a neighborhood of a point \( t_0 \geq 0 \) (a right-hand neighborhood if \( t_0 = 0 \)),
2. The Laplace integral of \( f \) converges absolutely on the line \( \text{Re } s = c \),

then

\[
\lim_{T \to \infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(s) e^{st} ds = \begin{cases} 
0, & \text{if } t_0 < 0, \\
\frac{f(0+)}{2} & \text{if } t_0 = 0, \\
\frac{[f(t_0+) + f(t_0-)]}{2} & \text{if } t_0 > 0.
\end{cases}
\]

In particular, if \( f \) is differentiable on \((0, \infty)\) and satisfies the above conditions, then

\[
\lim_{T \to \infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(s) e^{st} ds = f(t), \quad 0 < t < \infty.
\]

The integral here is taken to be a Cauchy principal value since, in general, this integral may be divergent. For example, if \( f(t) = 1 \), then \( F(s) = \frac{1}{s} \) and, for \( c = 1 \) and \( t = 0 \), the integral \( \int_{1-i\infty}^{1+i\infty} \frac{1}{s} ds \) diverges.

**Inversion by partial fractions**

Suppose that \( F \) is a rational function \( F(x) = \frac{P(s)}{Q(s)} \) in which the degree of the denominator \( Q \) is greater than that of the numerator \( P \). For instance, let \( F \) be represented in its most reduced form where \( P \) and \( Q \) have no common zeros, and assume that \( Q \) has only simple zeros at \( a_1, \ldots, a_n \), then

\[
f(t) = \mathcal{L}^{-1} (F(s)) (t) = \mathcal{L}^{-1} \left( \frac{P(s)}{Q(s)} \right) (t) = \sum_{k=1}^{n} \frac{P(a_k)}{Q'(a_k)} e^{-a_k t}.
\]

For example, if \( P(s) = s - 5 \) and \( Q(s) = s^2 + 6s + 13 \), then \( a_1 = -3 + 2i \), \( a_2 = -3 - 2i \), and it follows that

\[
f(t) = \mathcal{L}^{-1} \left( \frac{s - 5}{s^2 + 6s + 13} \right) = \frac{(2i - 8)}{4i} e^{(-3+2i)t} + \frac{(2i + 8)}{4i} e^{(-3-2i)t}
\]

\[
= e^{-3t} (\cos 2t - 4 \sin 2t).
\]

6.26.4 CONVOLUTION

Let \( f(t) \) and \( g(t) \) be locally integrable functions on \([0, \infty)\), and assume that their Laplace integrals converge absolutely in some half-plane \( \text{Re } s > \alpha \). Then the convo-
The convolution operation, \( \ast \), associated with the Laplace transform, is defined by
\[
h(t) = (f \ast g)(t) = \int_0^t f(x)g(t-x)dx. \tag{6.26.14}
\]
The convolution of \( f \) and \( g \) is a locally integrable function on \([0, \infty)\) that is continuous if either \( f \) or \( g \) is continuous. Additionally, it has a Laplace transform given by
\[
H(s) = (\mathcal{L}h)(s) = F(s)G(s), \tag{6.26.15}
\]
where \((\mathcal{L}f)(s) = F(s)\) and \((\mathcal{L}g)(s) = G(s)\).

### 6.27 Z-TRANSFORM

The Z-transform of a sequence \( \{f(n)\}_{-\infty}^{\infty} \) is defined by
\[
Z[f(n)] = F(z) = \sum_{n=-\infty}^{\infty} f(n)z^{-n}, \tag{6.27.1}
\]
for all complex numbers \( z \) for which the series converges.

The series converges at least in a ring of the form \( 0 \leq r_1 < z < r_2 \leq \infty \), whose radii, \( r_1 \) and \( r_2 \), depend on the behavior of \( f(n) \) at \( \pm \infty \):
\[
r_1 = \limsup_{n \to \infty} \sqrt[n]{|f(n)|}, \quad r_2 = \liminf_{n \to \infty} \frac{1}{\sqrt[n]{|f(-n)|}}. \tag{6.27.2}
\]

If there is more than one sequence involved, we may denote \( r_1 \) and \( r_2 \) by \( r_1(f) \) and \( r_2(f) \) respectively. The function \( F(z) \) is analytic in this ring, but it may be possible to continue it analytically beyond the boundaries of the ring. If \( f(n) = 0 \) for \( n < 0 \), then \( r_2 = \infty \), and if \( f(n) = 0 \) for \( n \geq 0 \), then \( r_1 = 0 \).

Let \( z = re^{i\theta} \). Then the Z-transform evaluated at \( r = 1 \) is the Fourier transform of the sequence \( \{f(n)\}_{-\infty}^{\infty} \),
\[
\sum_{n=-\infty}^{\infty} f(n)e^{-in\theta}. \tag{6.27.3}
\]

Examples:

1. Let \( a \) be a complex number and define \( f(n) = a^n \), for \( n \geq 0 \), and zero otherwise, then
\[
Z[f(n)] = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{z}{z - a}, \quad |z| > |a|. \tag{6.27.4}
\]
2. If \( f(n) = na^n \), for \( n \geq 0 \), and zero otherwise, then
\[
Z[f(n)] = \sum_{n=0}^{\infty} na^n z^{-n} = \frac{az}{(z-a)^2}, \quad |z| > |a|.
\]

3. Let \( u(n) = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0 \end{cases} \), then
\[
Z[u(n)] = \frac{z}{z-1}.
\]

4. Let \( \delta(n) = \begin{cases} 1, & n = 0, \\ 0, & \text{otherwise} \end{cases} \), then
\[
Z[\delta(n-k)] = z^{-k} \text{ for } k = 0, \pm 1, \pm 2, \ldots.
\]

#### 6.27.1 PROPERTIES

Let the region of convergence of the \( Z \)-transform of the sequence \( \{f(n)\} \) be denoted by \( D_f \).

1. Linearity:
\[
Z[af(n)+bg(n)] = aZ[f(n)] + bZ[g(n)] = aF(z) + bG(z), \quad z \in D_f \cap D_g.
\]

The region \( D_f \cap D_g \) contains the ring \( r_1 < |z| < r_2 \), where
\( r_1 = \max\{r_1(f), r_1(g)\} \) and \( r_2 = \min\{r_2(f), r_2(g)\} \).

2. Translation:
\[
Z[f(n-k)] = z^{-k}F(z).
\]

3. Multiplication by exponentials:
\[
Z[(anf(n))] = F(z/a) \text{ when } ar_1 < |z| < ar_2.
\]

4. Multiplication by powers of \( n \):
For \( k = 0, 1, 2, \ldots \) and \( z \in D_f \),
\[
Z[(nf^k(n))] = (-1)^k \left( z \frac{d}{dz} \right)^k F(z). \tag{6.27.5}
\]

5. Conjugation:
\[
Z[\bar{f}(-n)] = \bar{F} \left( \frac{1}{\bar{z}} \right).
\]

6. Initial and final values:
If \( f(n) = 0 \) for \( n < 0 \), then \( \lim_{z \to \infty} F(z) = f(0) \) and, conversely, if \( F(z) \) is defined for \( r_1 < |z| \) and for some integer \( m \), \( \lim_{z \to \infty} z^m F(z) = A \), then \( f(m) = A \) and \( f(n) = 0 \), for \( n < m \).

7. Parseval’s relation:
Let \( F(z) \) and \( G(z) \) be the \( Z \)-transforms of \( \{f(n)\} \) and \( \{g(n)\} \), respectively. Then
\[
\sum_{n=-\infty}^{\infty} f(n)\bar{g}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{i\omega})\bar{G}(e^{i\omega}) \, d\omega. \tag{6.27.6}
\]

In particular,
\[
\sum_{n=-\infty}^{\infty} |f(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(e^{i\omega})|^2 \, d\omega. \tag{6.27.7}
\]
6.27.2 INVERSION FORMULA

Consider the sequences

\[ f(n) = u(n) = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0 \end{cases}, \quad \text{and} \quad g(n) = -u(-n - 1) = \begin{cases} -1, & n < 0, \\ 0, & n \geq 0. \end{cases} \]

Note that \( F(z) = \frac{z}{z - 1} \) for \(|z| > 1\), and \( G(z) = \frac{z}{z - 1} \) for \(|z| < 1\). Hence, the inverse Z-transform of the function \( z/(z - 1) \) is not unique. In general, the inverse Z-transform is not unique, unless its region of convergence is specified.

1. **Inversion by using series representation:**
   If \( F(z) \) is given by its series
   \[ F(z) = \sum_{n=-\infty}^{\infty} a_n z^{-n}, \quad r_1 < z < r_2, \]
   then its inverse Z-transform is unique and equals \( \{f(n) = a_n\} \) for all \( n \).

2. **Inversion by using complex integration:**
   If \( F(z) \) is given in a closed form as an algebraic expression and its domain of analyticity is known, then its inverse Z-transform can be obtained by using the relationship
   \[ f(n) = \frac{1}{2\pi i} \oint_{\gamma} F(z) z^{n-1} dz, \tag{6.27.8} \]
   where \( \gamma \) is a closed contour surrounding the origin in the domain of analyticity of \( F(z) \).

3. **Inversion by using Fourier series:**
   If the domain of analyticity of \( F \) contains the unit circle, \(|r| = 1\), and if \( F \) is single valued therein, then \( F(e^{i\theta}) \) is a periodic function with period \( 2\pi \), and, consequently, it can be expanded in a Fourier series. The coefficients of the series form the inverse Z-transform of \( F \) and they are given explicitly by
   \[ f(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{i\theta}) e^{in\theta} d\theta. \tag{6.27.9} \]

4. **Inversion by using partial fractions:**
   Dividing Equation (6.27.4) by \( z \) and differentiating both sides with respect to \( z \), results in
   \[ Z^{-1} \left[(z - a)^{-k}\right] = \left(\frac{n - 1}{n - k}\right) a^{n-k} u(n - k), \tag{6.27.10} \]
   for \( k = 1, 2, \cdots \) and \(|z| > |a| > 0\). Moreover, from the example on page 544,
   \[ Z^{-1} \left[z^{-k}\right] = \delta(n - k). \tag{6.27.11} \]
Let \( F(z) \) be a rational function of the form
\[
F(z) = \frac{P(z)}{Q(z)} = \frac{a_N z^N + \cdots + a_1 z + a_0}{b_M z^M + \cdots + b_1 z + b_0}.
\]

- Consider the case \( N < M \). The denominator \( Q(z) \) can be factored over the field of complex numbers as
\[
Q(z) = c(z - z_1)^{k_1} \cdots (z - z_m)^{k_m},
\]
where \( c \) is a constant and \( k_1, \ldots, k_m \) are positive integers satisfying \( k_1 + \cdots + k_m = M \). Hence, \( F \) can be written in the form
\[
F(z) = \sum_{i=1}^{m} \sum_{j=1}^{k_i} A_{i,j} \frac{1}{(z - z_i)^j}, \quad (6.27.12)
\]
where
\[
A_{i,j} = \lim_{z \to z_i} \frac{d^{j-1}}{dz^{j-1}} (z - z_i)^j F(z). \quad (6.27.13)
\]
The inverse \( Z \)-transform of the fractional decomposition Equation (6.27.12) in the region that is exterior to the smallest circle containing all the zeros of \( Q(z) \) can be obtained by using Equation (6.27.10).

- Consider the case \( N \geq M \). We must divide until \( F \) can be reduced to the form
\[
F(z) = H(z) + \frac{R(z)}{Q(z)}
\]
where the remainder polynomial, \( R(z) \), has degree less than or equal to \( M - 1 \), and the quotient, \( H(z) \), is a polynomial of degree, at most, \( N - M \). The inverse \( Z \)-transform of the quotient polynomial can be obtained by using Equation (6.27.11) and that of \( R(z)/Q(z) \) can be obtained as in the case \( N < M \).

Example: to find the inverse \( Z \)-transform of the function,
\[
F(z) = \frac{z^4 + 5}{(z - 1)^2(z - 2)}, \quad |z| > 2,
\]
the partial fraction expansion,
\[
\frac{z^4 + 5}{(z - 1)^2(z - 2)} = z + 4 - \frac{1}{(z - 1)^2} - \frac{5}{(z - 1)} + \frac{16}{(z - 2)},
\]
is created. With the aid of Equation (6.27.10) and Equation (6.27.11),
\[
Z^{-1}[F(z)] = \delta(n+1) + 4\delta(n) - (n-1)u(n-2) - 5u(n-1) + 16 \cdot 2^{n-1}u(n-1),
\]
or \( f(n) = -n - 4 + 16 \cdot 2^{n-1}, \) for \( n \geq 2 \), with the initial values \( f(-1) = 1, f(0) = 4, \) and \( f(1) = 11 \).
6.27.3 CONVOLUTION AND PRODUCT

The convolution of two sequences, \( \{f(n)\}_{-\infty}^{\infty} \) and \( \{g(n)\}_{-\infty}^{\infty} \), is a sequence \( \{h(n)\}_{-\infty}^{\infty} \) defined by \( h(n) = \sum_{k=-\infty}^{\infty} f(k)g(n-k) \). The Z-transform of the convolution of two sequences is the product of their Z-transforms,

\[
Z[h(n)] = Z[f(n)]Z[g(n)],
\]

for \( z \in D_f \cap D_g, \) or \( H(z) = F(z)G(z) \).

The Z-transform of the product of two sequences is given by

\[
Z[f(n)g(n)] = \frac{1}{2\pi i} \oint_{\gamma} F(\omega)G\left(\frac{z}{\omega}\right) \frac{d\omega}{\omega}, \tag{6.27.14}
\]

where \( \gamma \) is a closed contour surrounding the origin in the domain of convergence of \( F(\omega) \) and \( G(z/\omega) \).

6.28 HILBERT TRANSFORM

The Hilbert transform of \( f \) is defined as

\[
(\mathcal{H}f)(x) = \hat{f}(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{t-x} \, dt = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x+t)}{t} \, dt \tag{6.28.1}
\]

where the integral is a Cauchy principal value.

Since the definition is given in terms of a singular integral, it is sometimes impractical to use. An alternative definition is given below. First, let \( f \) be an integrable function, and define \( a(t) \) and \( b(t) \) by

\[
a(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos tx \, dx, \quad b(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin tx \, dx. \tag{6.28.2}
\]

Consider the function \( F(z) \), defined by the Fourier integral

\[
F(z) = \int_{0}^{\infty} (a(t) - ib(t))e^{izt} \, dt = U(z) + i\tilde{U}(z), \tag{6.28.3}
\]

where \( z = x + iy \). The real and imaginary parts of \( F \) are

\[
U(z) = \int_{0}^{\infty} (a(t) \cos xt + b(t) \sin xt)e^{-zt} \, dt, \quad \text{and}
\]

\[
\tilde{U}(z) = \int_{0}^{\infty} (a(t) \sin xt - b(t) \cos xt)e^{-zt} \, dt.
\]
Formally,
\[ \lim_{y \to 0} U(z) = f(x) = \int_0^\infty (a(t) \cos xt + b(t) \sin xt) \, dt, \]  
(6.28.4)

and
\[ \lim_{y \to 0} \tilde{U}(z) = -\tilde{f}(x) = \int_0^\infty (a(t) \sin xt - b(t) \cos xt) \, dt, \]  
(6.28.5)

The Hilbert transform of a function \( f \), given by Equation (6.28.4), is defined as the function \( \tilde{f} \) given by Equation (6.28.5).

### 6.28.1 EXISTENCE

If \( f \in L^1(\mathbb{R}) \), then its Hilbert transform \( (Hf)(x) \) exists for almost all \( x \). For \( f \in L^p(\mathbb{R}), p > 1 \), there is the following stronger result:

**THEOREM 6.28.1**

Let \( f \in L^p(\mathbb{R}), p > 1 \). Then \( (Hf)(x) \) exists for almost all \( x \) and defines a function that also belongs to \( L^p(\mathbb{R}) \) with
\[
\int_{-\infty}^{\infty} |(Hf)(x)|^p \, dx \leq C_p \int_{-\infty}^{\infty} |f(x)|^p \, dx.
\]

In the special case of \( p = 2 \), we have
\[
\int_{-\infty}^{\infty} |(Hf)(x)|^2 \, dx = \int_{-\infty}^{\infty} |f(x)|^2 \, dx. \quad (6.28.6)
\]

The theorem is not valid if \( p = 1 \) because, although it is true that \( (Hf)(x) \) is defined almost everywhere, it is not necessarily in \( L^1(\mathbb{R}) \). The function \( f(t) = (t \log 2t)^{-1} H(t) \) provides a counterexample.

### 6.28.2 PROPERTIES

1. **Translation:** The Hilbert transformation commutes with the translation operator 
   \( (Hf)(x+a) = H(f(t+a))(x) \).
2. **Dilation:** The Hilbert transformation also commutes with the dilation operator
   \[
   (Hf)(ax) = H(f(at))(x) \quad a > 0,
   \]
   but
   \[
   (Hf)(ax) = -H(f(at))(x) \quad \text{for} \quad a < 0.
   \]
3. **Multiplication by t**: $\mathcal{H}(tf(t))(x) = x(\mathcal{H}f)(x) + \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \, dt$.

4. **Differentiation**: $\mathcal{H}(f'(t))(x) = (\mathcal{H}f)'(x)$, provided that $f(t) = O(t)$ as $|t| \to \infty$.

5. **Orthogonality**: The Hilbert transform of $f \in L^2(\mathbb{R})$ is orthogonal to $f$ in the sense $\int_{-\infty}^{\infty} f(x)(\mathcal{H}f)(x) \, dx = 0$.

6. **Parity**: The Hilbert transform of an even function is odd and that of an odd function is even.

7. **Inversion formula**: If $(\mathcal{H}f)(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{t-x} \, dx$, then

   $f(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(\mathcal{H}f)(x)}{x-t} \, dx$, or, symbolically,

   $\mathcal{H}(\mathcal{H}f)(x) = -f(x), \quad (6.28.7)$

   that is, applying the Hilbert transform twice returns the negative of the original function. Moreover, if $f \in L^1(\mathbb{R})$ has a bounded derivative, then the allied integral (see Equation 6.22.6) equals $(\mathcal{H}f)(x)$.

### 6.28.3 RELATIONSHIP WITH THE FOURIER TRANSFORM

From Equations (6.28.3)-(6.28.5), we obtain

$$\lim_{\gamma \to 0} F(z) = F(x) = \int_0^\infty (a(t) - ib(t))e^{ixt} \, dt = f(x) - i(\mathcal{H}f)(x),$$

where $a(t)$ and $b(t)$ are given by Equation (6.28.2).

Let $g$ be a real-valued integrable function and consider its Fourier transform $\hat{g}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t)e^{ixt} \, dt$. If we denote the real and imaginary parts of $\hat{g}$ by $f$ and $\tilde{f}$, respectively, then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t) \cos xt \, dt, \quad \text{and} \quad \hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t) \sin xt \, dt.$$

Splitting $g$ into its even and odd parts, $g_e$ and $g_o$, respectively, we obtain

$$g_e(t) = \frac{g(t) + g(-t)}{2} \quad \text{and} \quad g_o(t) = \frac{g(t) - g(-t)}{2};$$

hence

$$f(x) = \sqrt{\frac{\pi}{2}} \int_0^\infty g_e(t) \cos xt \, dt, \quad \text{and} \quad \hat{f}(x) = \sqrt{\frac{\pi}{2}} \int_0^\infty g_o(t) \sin xt \, dt,$$

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or

\[ f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_e(t) e^{ixt} dt, \quad \text{and} \quad \tilde{f}(x) = -\frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_o(t) e^{ixt} dt. \]

This shows that, if the Fourier transform of the even part of a real-valued function represents a function \( f(x) \), then the Fourier transform of the odd part represents the Hilbert transform of \( f \) (up to multiplication by \( i \)).

**THEOREM 6.28.2**

*Let* \( f \in L^1(\mathbb{R}) \) *and assume that* \( \mathcal{H}f \) *is also in* \( L^1(\mathbb{R}) \). *Then*

\[ \mathcal{F}(\mathcal{H}f)(\omega) = -i \, \text{sgn}(\omega) \mathcal{F}(f)(\omega), \]  

(6.28.8)

*where* \( \mathcal{F} \) *denotes the Fourier transformation.* *Similarly, if* \( f \in L^2(\mathbb{R}) \), *then* \( \mathcal{H}f \) *is in* \( L^2(\mathbb{R}) \), *and Equation (6.28.8) remains valid.*

### 6.29 HANKEL TRANSFORM

The Hankel transform of order \( \nu \) of a function \( f(x) \) is defined as

\[ \mathcal{H}_\nu(f)(y) = F_\nu(y) = \int_{0}^{\infty} f(x) \sqrt{\frac{y}{x}} J_\nu(yx) dx, \]  

(6.29.1)

for \( y > 0 \) and \( \nu > -1/2 \), where \( J_\nu(z) \) is the Bessel function of the first kind of order \( \nu \).

The Hankel transforms of order 1/2 and −1/2 are equal to the Fourier sine and cosine transforms, respectively, because

\[ J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x. \]

As with the Fourier transform, there are many variations on the definition of the Hankel transform. Some authors define it as

\[ G_\nu(y) = \int_{0}^{\infty} x g(x) J_\nu(yx) dx; \]  

(6.29.2)

however, the two definitions are equivalent; we only need to replace \( f(x) \) by \( \sqrt{x} g(x) \) and \( F_\nu(y) \) by \( \sqrt{y} G_\nu(y) \).
6.29.1 PROPERTIES

1. **Existence**: Since $\sqrt{x}J_\nu(x)$ is bounded on the positive real axis, the Hankel transform of $f$ exists if $f \in L^1(0, \infty)$.

2. **Multiplication by $x^m$**:

   \[
   \mathcal{H}_\nu \left( x^m f(x) \right)(y) = \frac{1}{y} \frac{d}{dy} \left[ y^{\nu+m-1/2} F_{\nu+m}(y) \right].
   \]

3. **Division by $x$**:

   \[
   \mathcal{H}_\nu \left( \frac{2y}{x} f(x) \right)(y) = y \left[ F_{\nu-1}(y) + F_{\nu+1}(y) \right].
   \]
   and also

   \[
   \mathcal{H}_\nu \left( \frac{f(x)}{x} \right)(y) = y^{1/2-\nu} \int_0^y t^{\nu-1/2} F_{\nu-1}(t) \, dt.
   \]

4. **Differentiation**:

   \[
   \mathcal{H}_\nu \left( \frac{2v f'(x)}{x} \right)(y) = (v - 1/2) y F_{\nu+1}(y) - (v + 1/2) y F_{\nu-1}(y).
   \]

5. **Differentiation and multiplication by powers of $x$**:

   \[
   \mathcal{H}_\nu \left[ x^{1/2-v} \left( \frac{1}{x} \frac{d}{dx} \right)^m \left( x^{\nu+m-1/2} f(x) \right) \right](y) = y^m F_{\nu+m}(y).
   \]

6. **Parseval's relation**: Let $F_\nu$ and $G_\nu$ denote the Hankel transforms of order $\nu$ of $f$ and $g$, respectively. Then

   \[
   \int_0^\infty F_\nu(y) G_\nu(y) \, dy = \int_0^\infty f(x) g(x) \, dx. \quad (6.29.3)
   \]

   In particular,

   \[
   \int_0^\infty |F_\nu(y)|^2 \, dy = \int_0^\infty |f(x)|^2 \, dx. \quad (6.29.4)
   \]

7. **Inversion formula**: If $f$ is absolutely integrable on $(0, \infty)$ and of bounded variation in a neighborhood of point $x$, then

   \[
   \int_0^\infty F_\nu(y) \sqrt{y}J_\nu(yx) \, dy = \frac{f(x + 0) + f(x - 0)}{2},
   \]
   whenever the expression on the right-hand side of the equation has a meaning; the integral converges to $f(x)$ whenever $f$ is continuous at $x$. 

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### 6.30 TABLES OF TRANSFORMS

**Finite sine transforms**

\[ f_s(n) = \int_0^{\pi} F(x) \sin nx \, dx, \text{ for } n = 1, 2, \ldots \]

<table>
<thead>
<tr>
<th>No.</th>
<th>( f_s(n) )</th>
<th>( F(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((-1)^{n+1} f_s(n))</td>
<td>( F(\pi - x) )</td>
</tr>
<tr>
<td>2</td>
<td>(1/n)</td>
<td>(\pi - x/\pi)</td>
</tr>
<tr>
<td>3</td>
<td>((-1)^{n+1}/n)</td>
<td>(x/\pi)</td>
</tr>
<tr>
<td>4</td>
<td>(1 - (-1)^n/n)</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>(2 \frac{n\pi}{n^2 \sin \frac{n\pi}{2}})</td>
<td>(\begin{cases} x &amp; \text{when } 0 &lt; x &lt; \pi/2 \ \pi - x &amp; \text{when } \pi/2 &lt; x &lt; \pi \end{cases})</td>
</tr>
<tr>
<td>6</td>
<td>((-1)^{n+1}/n^3)</td>
<td>(x(\pi^2 - x^2)/6\pi)</td>
</tr>
<tr>
<td>7</td>
<td>(1 - (-1)^n/n^3)</td>
<td>(x(\pi - x)/2)</td>
</tr>
<tr>
<td>8</td>
<td>(\frac{\pi^2(-1)^{n-1}}{n} - 2[1 - (-1)^n]/n^3)</td>
<td>(x^2)</td>
</tr>
<tr>
<td>9</td>
<td>(\pi(-1)^n \left( \frac{6 - \pi^2}{n^3 - \pi n} \right))</td>
<td>(x^3)</td>
</tr>
<tr>
<td>10</td>
<td>(\frac{n}{n^2 + c^2} \left[ 1 - (-1)^n e^{cx} \right])</td>
<td>(e^{cx})</td>
</tr>
<tr>
<td>11</td>
<td>(\sinh c(\pi - x))</td>
<td>(\frac{\sinh c\pi}{\sinh c})</td>
</tr>
<tr>
<td>12</td>
<td>(\frac{n}{n^2 - k^2}) with (k \neq 0, 1, 2, \ldots)</td>
<td>(\sin k(\pi - x)/\sin k\pi)</td>
</tr>
<tr>
<td>13</td>
<td>(\begin{cases} \pi/2 &amp; \text{when } n = m \ 0 &amp; \text{when } n \neq m, m = 1, 2, \ldots \end{cases})</td>
<td>(\sin mx)</td>
</tr>
<tr>
<td>14</td>
<td>(\frac{n}{n^2 - k^2} \left[ 1 - (-1)^n \cos k\pi \right]) with (k \neq 1, 2, \ldots)</td>
<td>(\cos kx)</td>
</tr>
<tr>
<td>15</td>
<td>(\frac{n}{n^2 - m^2} \left[ 1 - (-1)^{n+m} \right]) when (n \neq m = 1, 2, \ldots) (0 when (n = m))</td>
<td>(\cos mx)</td>
</tr>
<tr>
<td>16</td>
<td>(\frac{n}{(n^2 - k^2)^2}) with (k \neq 0, 1, 2, \ldots)</td>
<td>(\begin{cases} \pi \sin kx &amp; \text{when } n = m \ \frac{\pi \sin kx - x \cos k(\pi - x)}{2k \sin^2 k\pi} &amp; \text{when } n \neq m \end{cases})</td>
</tr>
<tr>
<td>17</td>
<td>(b^n/n) with (</td>
<td>b</td>
</tr>
<tr>
<td>18</td>
<td>(\frac{1 - (-1)^n}{n} b^n) with (</td>
<td>b</td>
</tr>
</tbody>
</table>
Finite cosine transforms

\[ f_c(n) = \int_0^\pi F(x) \cos nx \, dx, \text{ for } n = 0, 1, 2, \ldots \]

<table>
<thead>
<tr>
<th>No.</th>
<th>( f_c(n) )</th>
<th>( F(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((-1)^n f_c(n))</td>
<td>(F(\pi - x))</td>
</tr>
<tr>
<td>2</td>
<td>[\begin{array}{c} \pi \quad n = 0 \ 0 \quad n = 1, 2, \ldots \end{array}]</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>[\begin{array}{c} 0 \quad n = 0 \ -\frac{2}{n} \sin \frac{n\pi}{2} \quad n = 1, 2, \ldots \end{array}]</td>
<td>[\begin{array}{l} 1 \quad \text{for } 0 &lt; x &lt; \pi/2 \ -1 \quad \text{for } \pi/2 &lt; x &lt; \pi \end{array}]</td>
</tr>
<tr>
<td>4</td>
<td>[\begin{array}{c} \frac{\pi^2}{2} \quad n = 0 \ (-1)^n - \frac{1}{n^2} \quad n = 1, 2, \ldots \end{array}]</td>
<td>(x)</td>
</tr>
<tr>
<td>5</td>
<td>[\begin{array}{c} \frac{\pi^2}{6} \quad n = 0 \ (-1)^n/n^2 \quad n = 1, 2, \ldots \end{array}]</td>
<td>(x^2/2\pi)</td>
</tr>
<tr>
<td>6</td>
<td>[\begin{array}{c} 0 \quad n = 0 \ 1/n^2 \quad n = 1, 2, \ldots \end{array}]</td>
<td>(\frac{(x - \pi)^2 - \pi}{2\pi} - \frac{\pi}{6})</td>
</tr>
<tr>
<td>7</td>
<td>[\begin{array}{c} \frac{\pi^4}{4} \quad n = 0 \ 3\pi^2 \frac{(-1)^n}{n^2} \quad n = 1, 2, \ldots \ -\frac{1}{6} - \frac{(-1)^n}{n^4} \quad n = 1, 2, \ldots \end{array}]</td>
<td>(x^3)</td>
</tr>
<tr>
<td>8</td>
<td>(\frac{(-1)^n e^\pi - 1}{n^2 + c^2})</td>
<td>(\frac{1}{c} e^{\pi c})</td>
</tr>
<tr>
<td>9</td>
<td>(\frac{1}{n^2 + c^2})</td>
<td>(\cosh c(\pi - x))</td>
</tr>
<tr>
<td>10</td>
<td>(\frac{k}{n^2 - k^2} [(-1)^n \cos \pi k - 1])</td>
<td>(\sin kx)</td>
</tr>
<tr>
<td>11</td>
<td>[\begin{array}{c} \frac{0}{n^2 - m^2} \quad m = 1, 2, \ldots \ (-1)^{n+m} - 1 \quad m \neq 1, 2, \ldots \end{array}]</td>
<td>(\frac{1}{m} \sin mx)</td>
</tr>
<tr>
<td>12</td>
<td>(\frac{1}{n^2 - k^2})</td>
<td>(-\frac{\cos k(\pi - x)}{k \sin k\pi})</td>
</tr>
<tr>
<td>13</td>
<td>(\pi/2 \quad \text{when } n = m) &lt;br/&gt; (0 \quad \text{when } n \neq m)</td>
<td>(\cos mx) ((m = 1, 2, \ldots))</td>
</tr>
</tbody>
</table>

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### Fourier sine transforms

\[ F(\omega) = F_s(f)(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(\omega x) \, dx, \quad \omega > 0 \]

<table>
<thead>
<tr>
<th>No.</th>
<th>( f(x) )</th>
<th>( F(\omega) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \begin{cases} 1 &amp; 0 &lt; x &lt; a \ 0 &amp; x &gt; a \end{cases} )</td>
<td>( \sqrt{\frac{2}{\pi}} \frac{1-\cos(\omega a)}{\omega} )</td>
</tr>
<tr>
<td>2</td>
<td>( x^{p-1} ) ( (0 &lt; p &lt; 1) )</td>
<td>( \sqrt{\frac{2}{\pi}} \frac{\Gamma(p)}{\omega^p} \sin \frac{p\pi}{2} )</td>
</tr>
<tr>
<td>3</td>
<td>( \begin{cases} \sin x &amp; 0 &lt; x &lt; a \ 0 &amp; x &gt; a \end{cases} )</td>
<td>( \frac{1}{\sqrt{2\pi}} \left( \sin[\alpha(1-\omega)] + \sin[\alpha(1+\omega)] \right) )</td>
</tr>
<tr>
<td>4</td>
<td>( e^{-x} )</td>
<td>( \sqrt{\frac{2}{\pi}} \frac{1}{1+\omega^2} )</td>
</tr>
<tr>
<td>5</td>
<td>( xe^{-x^{2}/2} )</td>
<td>( \omega e^{-\omega^2/2} )</td>
</tr>
</tbody>
</table>
| 6   | \( \cos \frac{x^2}{2} \) | \( \sqrt{2} \left[ \sin \frac{\omega^2}{2} C \left( \frac{\omega^2}{2} \right) \right. \)  
                   | \( \cos \frac{\omega^2}{2} S \left( \frac{\omega^2}{2} \right) \) |
| 7   | \( \sin \frac{x^2}{2} \) | \( \sqrt{2} \left[ \cos \frac{\omega^2}{2} C \left( \frac{\omega^2}{2} \right) \right. \)  
                   | \( \cos \frac{\omega^2}{2} S \left( \frac{\omega^2}{2} \right) \) |

### Fourier cosine transforms

\[ F(\omega) = F_c(f)(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(\omega x) \, dx, \quad \omega > 0. \]

<table>
<thead>
<tr>
<th>No.</th>
<th>( f(x) )</th>
<th>( F(\omega) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \begin{cases} 1 &amp; 0 &lt; x &lt; a \ 0 &amp; x &gt; a \end{cases} )</td>
<td>( \sqrt{\frac{2}{\pi}} \frac{\sin(\omega a)}{\omega} )</td>
</tr>
<tr>
<td>2</td>
<td>( x^{p-1} ) ( (0 &lt; p &lt; 1) )</td>
<td>( \sqrt{\frac{2}{\pi}} \frac{\Gamma(p)}{\omega^p} \cos \frac{p\pi}{2} )</td>
</tr>
<tr>
<td>3</td>
<td>( \begin{cases} \cos x &amp; 0 &lt; x &lt; a \ 0 &amp; x &gt; a \end{cases} )</td>
<td>( \frac{1}{\sqrt{2\pi}} \left( \sin[\alpha(1-\omega)] + \sin[\alpha(1+\omega)] \right) )</td>
</tr>
<tr>
<td>4</td>
<td>( e^{-x} )</td>
<td>( \sqrt{\frac{2}{\pi}} \frac{1}{1+\omega^2} )</td>
</tr>
<tr>
<td>5</td>
<td>( e^{-x^{2}/2} )</td>
<td>( e^{-\omega^2/2} )</td>
</tr>
<tr>
<td>6</td>
<td>( \cos \frac{x^2}{2} )</td>
<td>( \cos \frac{\omega^2}{2} - \frac{\omega}{2} )</td>
</tr>
<tr>
<td>7</td>
<td>( \sin \frac{x^2}{2} )</td>
<td>( \cos \frac{\omega^2}{2} + \frac{\omega}{2} )</td>
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</table>

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Fourier transforms: functional relations

\[ F(\omega) = \mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{i\omega x} \, dx, \quad \omega > 0. \]

<table>
<thead>
<tr>
<th>No.</th>
<th>( f(x) )</th>
<th>( F(\omega) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( ag(x) + bh(x) )</td>
<td>( aG(\omega) + bH(\omega) )</td>
</tr>
<tr>
<td>2</td>
<td>( f(ax) ) ( a \neq 0 ), ( \text{Im} , a = 0 )</td>
<td>( \frac{1}{</td>
</tr>
<tr>
<td>3</td>
<td>( f(-x) )</td>
<td>( F(-\omega) )</td>
</tr>
<tr>
<td>4</td>
<td>( \overline{f(x)} )</td>
<td>( \overline{F(-\omega)} )</td>
</tr>
<tr>
<td>5</td>
<td>( f(x - \tau) ) ( \text{Im} , \tau = 0 )</td>
<td>( e^{-i\omega \tau} F(\omega) )</td>
</tr>
<tr>
<td>6</td>
<td>( e^{i\Omega x} f(x) ) ( \text{Im} , \Omega = 0 )</td>
<td>( F(\omega - \Omega) )</td>
</tr>
<tr>
<td>7</td>
<td>( F(x) )</td>
<td>( 2\pi f(-\omega) )</td>
</tr>
<tr>
<td>8</td>
<td>( \frac{\partial^n}{dx^n} f(x) )</td>
<td>( (i\omega)^n F(\omega) )</td>
</tr>
<tr>
<td>9</td>
<td>( (-i\chi)^n f(x) )</td>
<td>( \frac{d^n}{d\chi^n} F(\omega) )</td>
</tr>
<tr>
<td>10</td>
<td>( \frac{\partial^2}{\partial a^2} f(x, a) )</td>
<td>( \frac{\partial^2}{\partial a^2} F(\omega, a) )</td>
</tr>
</tbody>
</table>
Fourier transforms

\[ F(\omega) = F(f)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\omega x} \, dx, \quad \omega > 0. \]

<table>
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<th>No.</th>
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<th>( F(\omega) )</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>( \delta(x) )</td>
<td>( 1/\sqrt{2\pi} )</td>
</tr>
<tr>
<td>2</td>
<td>( \delta(x - \tau) )</td>
<td>( e^{i\omega \tau} / \sqrt{2\pi} )</td>
</tr>
<tr>
<td>3</td>
<td>( \delta^{(n)}(x) )</td>
<td>( (-i\omega)^n / \sqrt{2\pi} )</td>
</tr>
<tr>
<td>4</td>
<td>( H(x) = \begin{cases} 1 &amp; x &gt; 0 \ 0 &amp; x &lt; 0 \end{cases} )</td>
<td>( -\frac{1}{i\omega \sqrt{2\pi}} + \sqrt{\frac{\pi}{2}} \delta(\omega) )</td>
</tr>
<tr>
<td>5</td>
<td>( \text{sgn}(x) = \begin{cases} 1 &amp; x &gt; 0 \ -1 &amp; x &lt; 0 \end{cases} )</td>
<td>( \sqrt{\frac{2}{\pi}} \frac{1}{i\omega} )</td>
</tr>
<tr>
<td>6</td>
<td>( \begin{cases} 1 &amp;</td>
<td>x</td>
</tr>
<tr>
<td>7</td>
<td>( e^{i\Omega x} \begin{cases} 1 &amp;</td>
<td>x</td>
</tr>
<tr>
<td>8</td>
<td>( e^{-a</td>
<td>x</td>
</tr>
<tr>
<td>9</td>
<td>( \frac{\sin \Omega x}{x} )</td>
<td>( \sqrt{\frac{2}{\pi}} [H(\Omega - \omega) - H(-\Omega - \omega)] )</td>
</tr>
</tbody>
</table>
Fourier transforms

\[ F(\omega) = F(f)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\omega x} \, dx, \quad \omega > 0. \]

<table>
<thead>
<tr>
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<th>( f(x) )</th>
<th>( F(\omega) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>( \sin ax/x )</td>
<td>( \sqrt{\frac{\pi}{2}} )</td>
</tr>
<tr>
<td>11</td>
<td>( \begin{cases} e^{iax} &amp; p &lt; x &lt; q \ 0 &amp; x &lt; p, x &gt; q \end{cases} )</td>
<td>( \frac{i}{\sqrt{2\pi}} e^{ip(\omega+a)} - e^{iq(\omega+a)}/\omega + a )</td>
</tr>
<tr>
<td>12</td>
<td>( \begin{cases} e^{-cx+iax} &amp; x &gt; 0 \ 0 &amp; x &lt; 0 \end{cases} ) ( (c &gt; 0) )</td>
<td>( \frac{i}{\sqrt{2\pi}} (\omega + a + ic) )</td>
</tr>
<tr>
<td>13</td>
<td>( e^{-px^2} ) ( \text{Re } p &gt; 0 )</td>
<td>( \frac{1}{\sqrt{2\pi}} e^{-a^2/4p} )</td>
</tr>
<tr>
<td>14</td>
<td>( \cos px^2 )</td>
<td>( \frac{1}{\sqrt{2\pi}} \cos \left( \frac{\omega^2}{2p} - \frac{\pi}{4} \right) )</td>
</tr>
<tr>
<td>15</td>
<td>( \sin px^2 )</td>
<td>( \frac{1}{\sqrt{2\pi}} \cos \left( \frac{\omega^2}{2p} + \frac{\pi}{4} \right) )</td>
</tr>
<tr>
<td>16</td>
<td>(</td>
<td>x</td>
</tr>
<tr>
<td>17</td>
<td>( e^{-a</td>
<td>\omega</td>
</tr>
<tr>
<td>18</td>
<td>( \frac{\cosh ax}{\cosh \pi x} ) ( (-\pi &lt; a &lt; \pi) )</td>
<td>( \frac{\sqrt{2}}{\pi} \cos \frac{\omega}{2} \cosh \frac{\omega}{a} \cos \frac{\omega}{a + \cosh \omega} )</td>
</tr>
<tr>
<td>19</td>
<td>( \frac{\sin ax}{\sinh \pi x} ) ( (-\pi &lt; a &lt; \pi) )</td>
<td>( \frac{1}{\sqrt{2\pi}} \sin a \cos a + \cosh \omega )</td>
</tr>
<tr>
<td>20</td>
<td>( \begin{cases} \frac{1}{\sqrt{a^2+x^2}} &amp;</td>
<td>x</td>
</tr>
<tr>
<td>21</td>
<td>( \frac{\sin[b\sqrt{a^2+x^2}]}{\sqrt{a^2+x^2}} )</td>
<td>( \frac{1}{\sqrt{2\pi}} J_0(a\sqrt{b^2 - \omega^2}) )</td>
</tr>
<tr>
<td>22</td>
<td>( P_n(x) ) (</td>
<td>x</td>
</tr>
<tr>
<td>23</td>
<td>( \begin{cases} \frac{\cos[b\sqrt{a^2-x^2}]}{\sqrt{a^2-x^2}} &amp;</td>
<td>x</td>
</tr>
<tr>
<td>24</td>
<td>( \begin{cases} \frac{\cosh[b\sqrt{a^2-x^2}]}{\sqrt{a^2-x^2}} &amp;</td>
<td>x</td>
</tr>
</tbody>
</table>
### Multidimensional Fourier transforms

<table>
<thead>
<tr>
<th>No.</th>
<th>$f(x)$</th>
<th>$F(u)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>In $n$-dimensions</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$f(ax)$</td>
<td>$</td>
</tr>
<tr>
<td>2</td>
<td>$f(x-a)$</td>
<td>$e^{-ia \cdot u} F(u)$</td>
</tr>
<tr>
<td>3</td>
<td>$e^{ia \cdot x} f(x)$</td>
<td>$F(v-a)$</td>
</tr>
<tr>
<td>4</td>
<td>$F(x)$</td>
<td>$(2\pi)^n f(-u)$</td>
</tr>
</tbody>
</table>

**Two dimensions:** let $x = (x, y)$ and $u = (u, v)$.

<table>
<thead>
<tr>
<th>No.</th>
<th>$f(ax, by)$</th>
<th>$\frac{1}{\pi^{1/2}} F\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$f(x-a, y-b)$</td>
<td>$e^{i(ax-by)} F(u, v)$</td>
</tr>
<tr>
<td>6</td>
<td>$e^{(ax-by)} f(x, y)$</td>
<td>$F(u-a, v-b)$</td>
</tr>
<tr>
<td>7</td>
<td>$F(x, y)$</td>
<td>$(2\pi)^2 F(-u, -v)$</td>
</tr>
<tr>
<td>8</td>
<td>$\delta(x-a) \delta(y-b)$</td>
<td>$\frac{1}{2\pi} e^{-i(ax-by)}$</td>
</tr>
<tr>
<td>9</td>
<td>$e^{-x^2/4a^2-y^2/4b^2}$</td>
<td>$2\sqrt{ab} e^{-ax^2-by^2}$</td>
</tr>
<tr>
<td>10</td>
<td>$1 \begin{cases} 1 &amp;</td>
<td>x</td>
</tr>
<tr>
<td>11</td>
<td>$\frac{2 \sin au \sin bv}{\pi uv}$</td>
<td>$\delta(v)$</td>
</tr>
<tr>
<td>12</td>
<td>$1 \begin{cases} 1 &amp;</td>
<td>x</td>
</tr>
<tr>
<td>13</td>
<td>$1 \begin{cases} 1 &amp; x^2+y^2 &lt; a^2 \ 0 &amp; \text{otherwise (circle)} \end{cases}$</td>
<td>$a J_1(a \sqrt{u^2+v^2}) \sqrt{u^2+v^2}$</td>
</tr>
<tr>
<td>14</td>
<td>$\delta(x-a) \delta(y-b) \delta(z-c)$</td>
<td>$\frac{1}{(2\pi)^{3/2}} e^{-i(ax+by+cz)}$</td>
</tr>
<tr>
<td>15</td>
<td>$e^{-x^2/4a^2-y^2/4b^2-z^2/4c}$</td>
<td>$2^{3/2} \sqrt{abc} e^{-au^2-bv^2-cw^2}$</td>
</tr>
<tr>
<td>16</td>
<td>$1 \begin{cases} 1 &amp;</td>
<td>x</td>
</tr>
<tr>
<td>17</td>
<td>$1 \begin{cases} 1 &amp; x^2+y^2+z^2 &lt; a^2 \ 0 &amp; \text{otherwise (ball)} \end{cases}$</td>
<td>$\frac{\sin au \sin bv \sin cw}{\sqrt{2\pi \rho^3}} \rho^2 = u^2 + v^2 + w^2$</td>
</tr>
</tbody>
</table>

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Laplace transforms: functional relations

\[ F(s) = \mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st} \, dt. \]

<table>
<thead>
<tr>
<th>No.</th>
<th>( f(t) )</th>
<th>( F(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( af(t) + bg(t) )</td>
<td>( aF(s) + bG(s) )</td>
</tr>
<tr>
<td>2</td>
<td>( f'(t) )</td>
<td>( sF(s) - F(0+) )</td>
</tr>
<tr>
<td>3</td>
<td>( f''(t) )</td>
<td>( s^2F(s) - sF(0+) - F'(0+) )</td>
</tr>
<tr>
<td>4</td>
<td>( f^{(n)}(t) )</td>
<td>( s^nF(s) - \sum_{k=0}^{n-1} s^{n-1-k}F^{(k)}(0+) )</td>
</tr>
<tr>
<td>5</td>
<td>( \int_0^t f(\tau) , d\tau )</td>
<td>( \frac{1}{s}F(s) )</td>
</tr>
<tr>
<td>6</td>
<td>( \int_0^t \int_0^\tau f(u) , du , d\tau )</td>
<td>( \frac{1}{s^2}F(s) )</td>
</tr>
<tr>
<td>7</td>
<td>( \int_0^t f_1(t-\tau)f_2(\tau) , d\tau = f_1 * f_2 )</td>
<td>( F_1(s)F_2(s) )</td>
</tr>
<tr>
<td>8</td>
<td>( tf(t) )</td>
<td>( -F'(s) )</td>
</tr>
<tr>
<td>9</td>
<td>( t^n f(t) )</td>
<td>( (-1)^n F^{(n)}(s) )</td>
</tr>
<tr>
<td>10</td>
<td>( \frac{1}{\tau}f(t) )</td>
<td>( \int_\tau^\infty F(z) , dz )</td>
</tr>
<tr>
<td>11</td>
<td>( e^{at}f(t) )</td>
<td>( F(s-a) )</td>
</tr>
<tr>
<td>12</td>
<td>( f(t+b) ) with ( f(t) = 0 ) for ( t &lt; 0 )</td>
<td>( e^{-bt}F(s) )</td>
</tr>
<tr>
<td>13</td>
<td>( \frac{1}{\tau}f\left(\frac{1}{\tau}\right) )</td>
<td>( F(cs) )</td>
</tr>
<tr>
<td>14</td>
<td>( \frac{1}{\tau}e^{bi/c}f\left(\frac{1}{\tau}\right) )</td>
<td>( F(cs-b) )</td>
</tr>
<tr>
<td>15</td>
<td>( f(t+a) = f(t) )</td>
<td>( \int_0^a e^{-zt} f(t) , dt / 1 - e^{-as} )</td>
</tr>
<tr>
<td>16</td>
<td>( f(t+a) = -f(t) )</td>
<td>( \int_0^a e^{-zt} f(t) , dt / 1 + e^{-as} )</td>
</tr>
<tr>
<td>17</td>
<td>( \sum_{k=1}^n \frac{p(a_k)}{q(a_k)} e^{at} ) with ( q(t) = (t-a_1) \cdots (t-a_n) )</td>
<td>( \frac{p(s)}{q(s)} )</td>
</tr>
<tr>
<td>18</td>
<td>( e^{at} \sum_{k=1}^n \frac{\phi(n-k)(a)}{(n-k)!} t^{k-1} )</td>
<td>( \phi(s)/(s-a)^n )</td>
</tr>
</tbody>
</table>
Laplace transforms

\[ F(s) = L(f)(s) = \int_0^\infty f(t)e^{-st} \, dt. \]

<table>
<thead>
<tr>
<th>No.</th>
<th>( F(s) )</th>
<th>( f(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>( \delta(t) ), delta function</td>
</tr>
<tr>
<td>2</td>
<td>( 1/s )</td>
<td>( H(t) ), unit step function</td>
</tr>
<tr>
<td>3</td>
<td>( 1/s^2 )</td>
<td>( t )</td>
</tr>
<tr>
<td>4</td>
<td>( 1/s^n ) for ( n = 1, 2, \ldots )</td>
<td>( \frac{t^{n-1}}{(n-1)!} )</td>
</tr>
<tr>
<td>5</td>
<td>( 1/\sqrt{s} )</td>
<td>( 1/\sqrt{\pi t} )</td>
</tr>
<tr>
<td>6</td>
<td>( s^{-3/2} )</td>
<td>( 2\sqrt{1/\pi} )</td>
</tr>
<tr>
<td>7</td>
<td>( s^{-(n+1)/2} ) for ( n = 1, 2, \ldots )</td>
<td>( n^{\nu+1/2}/\sqrt{\pi(2\nu-1)!} )</td>
</tr>
<tr>
<td>8</td>
<td>( \frac{\Gamma(k)}{s^k} ) for ( (k &gt; 0) )</td>
<td>( t^{k-1} )</td>
</tr>
<tr>
<td>9</td>
<td>( \frac{1}{s-a} )</td>
<td>( e^{at} )</td>
</tr>
<tr>
<td>10</td>
<td>( \frac{1}{(s-a)^2} )</td>
<td>( te^{at} )</td>
</tr>
<tr>
<td>11</td>
<td>( \frac{1}{(s-a)^n} ) for ( n = 1, 2, \ldots )</td>
<td>( \frac{1}{(n-1)!} t^{n-1} e^{at} )</td>
</tr>
<tr>
<td>12</td>
<td>( \frac{\Gamma(k)}{(s-a)^k} ) for ( (k &gt; 0) )</td>
<td>( t^{k-1} e^{at} )</td>
</tr>
<tr>
<td>13</td>
<td>( \frac{s}{(s-a)(s-b)} ) for ( a \neq b )</td>
<td>( \frac{1}{a-b} (e^{at} - e^{bt}) )</td>
</tr>
<tr>
<td>14</td>
<td>( \frac{1}{(s-a)(s-b)} ) for ( a \neq b )</td>
<td>( \frac{1}{a-b} (ae^{at} - be^{bt}) )</td>
</tr>
<tr>
<td>15</td>
<td>( \frac{1}{(s-a)(s-b)(s-c)} ) for ( a, b, c ) distinct</td>
<td>( \frac{1}{c-b} (e^{at} + c) e^{ct} - (a-b) e^{bt} )</td>
</tr>
<tr>
<td>16</td>
<td>( \frac{1}{s^2+a^2} )</td>
<td>( \frac{1}{a} \sin at )</td>
</tr>
<tr>
<td>17</td>
<td>( \frac{t}{s^2+a^2} )</td>
<td>( \frac{1}{a} \sinh at )</td>
</tr>
<tr>
<td>18</td>
<td>( \frac{1}{s^2-a^2} )</td>
<td>( \frac{1}{a} \cosh at )</td>
</tr>
<tr>
<td>19</td>
<td>( \frac{1}{(s^2+a^2)^2} )</td>
<td>( \frac{1}{a^2} (1 - \cos at) )</td>
</tr>
<tr>
<td>20</td>
<td>( \frac{1}{(s^2+a^2)^3} )</td>
<td>( \frac{1}{a^3} (at - \sin at) )</td>
</tr>
<tr>
<td>21</td>
<td>( \frac{1}{s^4+at^2} )</td>
<td>( \frac{1}{a^2} (\sin at - at \cos at) )</td>
</tr>
<tr>
<td>22</td>
<td>( \frac{1}{(s^2+at^2)^2} )</td>
<td>( \frac{1}{a^2} \sin at )</td>
</tr>
<tr>
<td>23</td>
<td>( \frac{1}{s^4+at^2} )</td>
<td>( \frac{1}{a^2} \sin at )</td>
</tr>
<tr>
<td>24</td>
<td>( \frac{x^2}{(s^2+at^2)^3} )</td>
<td>( \frac{1}{a^2} (\sin at + at \cos at) )</td>
</tr>
</tbody>
</table>
Laplace transforms

\[ F(s) = \mathcal{L}(f)(s) = \int_{0}^{\infty} f(t)e^{-st} \, dt. \]

<table>
<thead>
<tr>
<th>No.</th>
<th>( F(s) )</th>
<th>( f(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>( \frac{s^2-a^2}{(s^2+a^2)^2} )</td>
<td>( t \cos at )</td>
</tr>
<tr>
<td>26</td>
<td>( \frac{s}{(s^2+a^2)(s^2+b^2)} )</td>
<td>( \frac{\cos at-\cos bt}{b^2-a^2} )</td>
</tr>
<tr>
<td>27</td>
<td>( \frac{1}{(s-a)^2+b^2} )</td>
<td>( \frac{1}{2}e^{at} \sin bt )</td>
</tr>
<tr>
<td>28</td>
<td>( \frac{s-a}{(s-a)^2+b^2} )</td>
<td>( e^{at} \cos bt )</td>
</tr>
<tr>
<td>29</td>
<td>( \frac{1}{(s-a)^2+b^2} )</td>
<td>( -e^{-at} \left( \sum_{n=1}^{\infty} \frac{2n-1}{n!} e^{-at} \sin bt \right) )</td>
</tr>
<tr>
<td>30</td>
<td>( \frac{s}{(s-a)^2+b^2} )</td>
<td>( e^{-at} \left{ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \frac{b}{(b^2-a^2)^n} \left[ n \sin bt - \frac{b}{(b^2-a^2)^n} \right] \right} )</td>
</tr>
<tr>
<td>31</td>
<td>( \frac{3a^2}{s^3+a^3} )</td>
<td>( -e^{-at/2} \left( \cos \frac{at\sqrt{3}}{2} - \sqrt{3} \sin \frac{at\sqrt{3}}{2} \right) )</td>
</tr>
<tr>
<td>32</td>
<td>( \frac{1}{s+a} )</td>
<td>( \frac{1}{2at} \sin at \sinh at )</td>
</tr>
<tr>
<td>33</td>
<td>( \frac{-a}{s+a} )</td>
<td>( \sin at \cosh at - \cos at \sinh at )</td>
</tr>
<tr>
<td>34</td>
<td>( \frac{1}{s-a} )</td>
<td>( \frac{1}{2a} (\sinh at - \sin at) )</td>
</tr>
<tr>
<td>35</td>
<td>( \frac{1}{s^2-a^2} )</td>
<td>( \frac{1}{2a} (\cosh at - \cos at) )</td>
</tr>
<tr>
<td>36</td>
<td>( \frac{4a^2}{(s^2+a^2)^2} )</td>
<td>( (1+a^2t^2) \sin at - \cos at )</td>
</tr>
<tr>
<td>37</td>
<td>( \frac{1}{s} \left( \frac{1}{s-a} \right)^n )</td>
<td>( L_n(t) = \frac{e^t}{n!} \frac{d^n}{dt^n} (t^n e^{-t}) )</td>
</tr>
<tr>
<td>38</td>
<td>( \frac{s}{(s-a)^3} )</td>
<td>( \frac{1}{\sqrt{2\pi}a^2} e^{at}(1+2at) )</td>
</tr>
<tr>
<td>39</td>
<td>( \sqrt{s-a} - \sqrt{s-b} )</td>
<td>( \frac{1}{2\sqrt{\pi}t} (e^{at} - e^{at}) )</td>
</tr>
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<td>40</td>
<td>( \frac{1}{\sqrt{s+a}} )</td>
<td>( \frac{1}{\sqrt{\pi}a} - ae^{a^2t^2} \text{erfc}(a\sqrt{t}) )</td>
</tr>
<tr>
<td>41</td>
<td>( \frac{1}{\sqrt{s-a}} )</td>
<td>( \frac{1}{\sqrt{\pi}a} + ae^{a^2t^2} \text{erf}(a\sqrt{t}) )</td>
</tr>
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<td>42</td>
<td>( \frac{1}{\sqrt{s+a}} )</td>
<td>( \frac{1}{\sqrt{\pi}a} - \frac{2a^2}{\sqrt{\pi}} e^{-a^2t^2} f_0 e^{\frac{a^2}{2}} e^{a^2t^2} d\tau )</td>
</tr>
<tr>
<td>43</td>
<td>( \frac{1}{\sqrt{s-a^2}} )</td>
<td>( \frac{1}{\sqrt{\pi}a} e^{a^2t^2} \text{erf}(a\sqrt{t}) )</td>
</tr>
<tr>
<td>44</td>
<td>( \frac{1}{\sqrt{s+a^2}} )</td>
<td>( \frac{2}{\sqrt{\pi}a} e^{-a^2t^2} f_0 e^{\frac{a^2}{2}} e^{a^2t^2} d\tau )</td>
</tr>
</tbody>
</table>
Laplace transforms

\[ F(s) = \mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st}\,dt. \]

<table>
<thead>
<tr>
<th>No.</th>
<th>( F(s) )</th>
<th>( f(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>45</td>
<td>( \frac{b^2 - a^2}{(s - a^2)(b + \sqrt{s})} )</td>
<td>( e^{at}[b - a, \text{erf}(a\sqrt{t})] - be^{bt}, \text{erfc}(b\sqrt{t}) )</td>
</tr>
<tr>
<td>46</td>
<td>( \frac{1}{\sqrt{s(a+b)}} )</td>
<td>( e^{at}, \text{erfc}(a\sqrt{t}) )</td>
</tr>
<tr>
<td>47</td>
<td>( \frac{1}{(s+a)\sqrt{s+b}} )</td>
<td>( \frac{1}{\sqrt{s-a}} e^{-at}, \text{erf}(\sqrt{b} - a\sqrt{t}) )</td>
</tr>
<tr>
<td>48</td>
<td>( \frac{b^2 - a^2}{\sqrt{s(a-b)}(a+b)} )</td>
<td>( e^{at}\left[ \frac{b}{a} \text{erf}(a\sqrt{t}) - 1 \right] + e^{bt}, \text{erfc}(b\sqrt{t}) )</td>
</tr>
<tr>
<td>49</td>
<td>( \frac{(1-s)^n}{\sqrt{1+s}} )</td>
<td>( \frac{n!}{(2n)!\sqrt{\pi}} H_{2n}(\sqrt{t}) )</td>
</tr>
<tr>
<td>50</td>
<td>( \frac{(1-s)^n}{\sqrt{1+s}} )</td>
<td>( -\frac{n!}{(2n+1)!\sqrt{\pi}} H_{2n+1}(\sqrt{t}) )</td>
</tr>
<tr>
<td>51</td>
<td>( \frac{s}{s+a} - 1 )</td>
<td>( ae^{-at}, [I_1(at) + I_0(at)] )</td>
</tr>
<tr>
<td>52</td>
<td>( \frac{1}{\sqrt{s+a}\sqrt{s+b}} )</td>
<td>( e^{-(a+b)t/2}I_0\left(\frac{a-b}{\sqrt{2}}t\right) )</td>
</tr>
<tr>
<td>53</td>
<td>( \frac{\Gamma(k)}{(s+a)^k(s+b)^{k}} ) (( k \geq 0 ))</td>
<td>( \sqrt{\pi} \left(\frac{1}{a-b}\right)^{k-1/2} e^{-(a+b)t/2} \times I_{k-1/2}\left(\frac{a-b}{\sqrt{2}}t\right) )</td>
</tr>
<tr>
<td>54</td>
<td>( \frac{1}{\sqrt{s+a}\sqrt{s+b}} )</td>
<td>( se^{-(a+b)t/2} \left[ I_0\left(\frac{a-b}{\sqrt{2}}t\right) + I_1\left(\frac{a-b}{\sqrt{2}}t\right) \right] )</td>
</tr>
<tr>
<td>55</td>
<td>( \frac{1}{\sqrt{s+a}\sqrt{s+b}\sqrt{s-a}} )</td>
<td>( \frac{1}{2}e^{-at}I_1(at) )</td>
</tr>
<tr>
<td>56</td>
<td>( \frac{1}{\sqrt{s+a}} )</td>
<td>( J_0(at) )</td>
</tr>
<tr>
<td>57</td>
<td>( \frac{(\sqrt{s^2+a^2}-s)^k}{\sqrt{s^2+a^2}} ) (( k &gt; -1 ))</td>
<td>( a^k J_k(at) )</td>
</tr>
<tr>
<td>58</td>
<td>( \frac{1}{(s+a)^k} ) (( k &gt; 0 ))</td>
<td>( \frac{1}{\sqrt{s+a}} \left(\frac{1}{2\pi}\right)^{1/2} J_{k-1/2}(at) )</td>
</tr>
<tr>
<td>59</td>
<td>( \frac{(s^2+a^2)}{s^2-a^2} ) (( k &gt; 0 ))</td>
<td>( \frac{k!}{\Gamma(k)} J_k(at) )</td>
</tr>
<tr>
<td>60</td>
<td>( \frac{(s-a^2)}{s^2-a^2} ) (( k &gt; -1 ))</td>
<td>( a^k I_k(at) )</td>
</tr>
<tr>
<td>61</td>
<td>( \frac{1}{(s^2-a^2)^k} ) (( k &gt; 0 ))</td>
<td>( \frac{1}{\sqrt{s+a}} \left(\frac{1}{2\pi}\right)^{k-1/2} I_{k-1/2}(at) )</td>
</tr>
</tbody>
</table>
Laplace transforms

\[ F(s) = \mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st} \, dt. \]

<table>
<thead>
<tr>
<th>No.</th>
<th>( F(s) )</th>
<th>( f(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>62</td>
<td>( \frac{e^{-ks}}{s} )</td>
<td>\begin{align*} 0 &amp; \text{ when } 0 &lt; t &lt; k \ 1 &amp; \text{ when } t &gt; k \end{align*}</td>
</tr>
<tr>
<td>63</td>
<td>( \frac{e^{-ks}}{s^2} )</td>
<td>\begin{align*} 0 &amp; \text{ when } 0 &lt; s &lt; k \ t - k &amp; \text{ when } t &gt; k \end{align*}</td>
</tr>
<tr>
<td>64</td>
<td>( \frac{e^{-ks}}{s^p} )</td>
<td>\begin{align*} 0 &amp; \text{ when } 0 &lt; t &lt; k \ \frac{(t-k)^{p-1}}{\Gamma(p)} &amp; \text{ when } t &gt; k \end{align*}</td>
</tr>
<tr>
<td>65</td>
<td>( \frac{1 - e^{-ks}}{s} )</td>
<td>\begin{align*} 1 &amp; \text{ when } 0 &lt; t &lt; k \ 0 &amp; \text{ when } t &gt; k \end{align*}</td>
</tr>
<tr>
<td>66</td>
<td>( \frac{a}{\sqrt{s + a\sqrt{s}}} \cosh \frac{\pi t}{2a} )</td>
<td>(</td>
</tr>
<tr>
<td>67</td>
<td>( \frac{1}{a} e^{-a/s} )</td>
<td>( J_0(2\sqrt{aT}) )</td>
</tr>
<tr>
<td>68</td>
<td>( \frac{1}{\sqrt{s}} e^{-a/s} )</td>
<td>( \frac{1}{\sqrt{\pi a}} \cos 2\sqrt{aT} )</td>
</tr>
<tr>
<td>69</td>
<td>( \frac{1}{\sqrt{s}} e^{a/s} )</td>
<td>( \frac{1}{\sqrt{\pi a}} \cosh 2\sqrt{aT} )</td>
</tr>
<tr>
<td>70</td>
<td>( \frac{1}{\sqrt{\pi a}} e^{-a/s} )</td>
<td>( \frac{1}{\sqrt{\pi a}} \sin 2\sqrt{aT} )</td>
</tr>
<tr>
<td>71</td>
<td>( \frac{1}{\sqrt{\pi a}} e^{a/s} )</td>
<td>( \frac{1}{\sqrt{\pi a}} \sinh 2\sqrt{aT} )</td>
</tr>
<tr>
<td>72</td>
<td>( \frac{1}{\sqrt{\pi}} e^{-a/s} )</td>
<td>( (\frac{1}{\sqrt{\pi}}) (k-1)^{1/2} J_{k-1}(2\sqrt{aT}) )</td>
</tr>
<tr>
<td>73</td>
<td>( \frac{1}{\sqrt{\pi}} e^{a/s} )</td>
<td>( (\frac{1}{\sqrt{\pi}}) (k-1)^{1/2} I_{k-1}(2\sqrt{aT}) )</td>
</tr>
<tr>
<td>74</td>
<td>( e^{-a\sqrt{s}} )</td>
<td>( \frac{a}{\sqrt{\pi t}} e^{-a^2/4t} )</td>
</tr>
<tr>
<td>75</td>
<td>( \frac{1}{\sqrt{s}} e^{-a\sqrt{s}} )</td>
<td>( \text{erfc} \left( \frac{a}{2\sqrt{t}} \right) )</td>
</tr>
<tr>
<td>76</td>
<td>( \frac{1}{\sqrt{s}} e^{-a\sqrt{s}} )</td>
<td>( \frac{1}{\sqrt{\pi t}} e^{-a^2/4t} )</td>
</tr>
<tr>
<td>77</td>
<td>( s^{-3/2} e^{-a\sqrt{s}} )</td>
<td>( t \sqrt{\frac{\pi}{4}} e^{-a^2/4t} - a \text{ erfc} \left( \frac{a}{2\sqrt{t}} \right) )</td>
</tr>
<tr>
<td>78</td>
<td>( \frac{e^{x_k\sqrt{s}}}{\sqrt{s(x + \sqrt{s})}} )</td>
<td>( e^{ak+a^2t} \text{ erfc} \left( a\sqrt{t} + \frac{k}{2\sqrt{t}} \right) )</td>
</tr>
</tbody>
</table>
Laplace transforms

\[ F(s) = \mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st} \, dt. \]

<table>
<thead>
<tr>
<th>No.</th>
<th>( F(s) )</th>
<th>( f(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>79</td>
<td>( \frac{e^{-k\sqrt{s^2+a^2}}}{\sqrt{s^2+a^2}} )</td>
<td>( (k \geq 0) \quad J_0(a\sqrt{t^2+2kt}) )</td>
</tr>
<tr>
<td>80</td>
<td>( \frac{1}{s} \log s )</td>
<td>( \Gamma'(1) - \log t )</td>
</tr>
<tr>
<td>81</td>
<td>( \frac{1}{s} \log s )</td>
<td>( (k &gt; 0) \quad t^{k-1} \left[ \frac{\Gamma'(k)}{\Gamma(k)} - \log t \right] )</td>
</tr>
<tr>
<td>82</td>
<td>( \frac{\log s}{s-a} )</td>
<td>( (a &gt; 0) \quad e^{at}[\log a - Ei(-at)] )</td>
</tr>
<tr>
<td>83</td>
<td>( \frac{\log s}{s^{2}+1} )</td>
<td>( \cos t , \text{Si}(t) - \sin t , \text{Ci}(t) )</td>
</tr>
<tr>
<td>84</td>
<td>( \frac{s \log s}{s^{2}+1} )</td>
<td>( -\sin t , \text{Si}(t) - \cos s , \text{Ci}(t) )</td>
</tr>
<tr>
<td>85</td>
<td>( \frac{1}{s} \log(1+as) )</td>
<td>( (a &gt; 0) \quad -Ei\left(-\frac{1}{a}\right) )</td>
</tr>
<tr>
<td>86</td>
<td>( \log \frac{s-a}{s-b} )</td>
<td>( \frac{1}{s}(e^{bt} - e^{at}) )</td>
</tr>
<tr>
<td>87</td>
<td>( \frac{1}{s} \log(1+a^2s^2) )</td>
<td>( -2 , \text{Ci}\left(-\frac{1}{a}\right) )</td>
</tr>
<tr>
<td>88</td>
<td>( \frac{1}{s} \log(s^2+a^2) )</td>
<td>( (a &gt; 0) \quad 2 \log a - 2 , \text{Ci}(at) )</td>
</tr>
<tr>
<td>89</td>
<td>( \frac{1}{s} \log(s^2+a^2) )</td>
<td>( (a &gt; 0) \quad \frac{2}{s}[at \log a + \sin at - at , \text{Ci}(at)] )</td>
</tr>
<tr>
<td>90</td>
<td>( \log \frac{s^2+a^2}{s^2} )</td>
<td>( \frac{2}{s}(1 - \cos at) )</td>
</tr>
<tr>
<td>91</td>
<td>( \log \frac{s^2-a^2}{s^2} )</td>
<td>( \frac{2}{s}(1 - \cosh at) )</td>
</tr>
<tr>
<td>92</td>
<td>( \tan^{-1} \frac{a}{s} )</td>
<td>( \frac{1}{s} \sin at )</td>
</tr>
<tr>
<td>93</td>
<td>( e^{k^2s^2} , \text{erfc}(ks) )</td>
<td>( (k &gt; 0) \quad \frac{1}{\sqrt{k}}e^{-t^2/4k^2} )</td>
</tr>
<tr>
<td>94</td>
<td>( \frac{1}{s} e^{k^2s^2} , \text{erfc}(ks) )</td>
<td>( (k &gt; 0) \quad \text{erf}\left(\frac{1}{\sqrt{k}}\right) )</td>
</tr>
<tr>
<td>95</td>
<td>( e^{ks} , \text{erfc}(\sqrt{k}s) )</td>
<td>( (k &gt; 0) \quad \frac{\sqrt{k}}{\pi \sqrt{k^2+1}} )</td>
</tr>
<tr>
<td>96</td>
<td>( \frac{1}{\sqrt{s}} e^{ks} , \text{erfc}(\sqrt{k}s) )</td>
<td>( (k &gt; 0) \quad \frac{1}{\sqrt{\pi(k^2+1)}} )</td>
</tr>
<tr>
<td>97</td>
<td>( \text{erf}\left(\frac{a}{\sqrt{s}}\right) )</td>
<td>( \frac{1}{\sqrt{t}} \sin(2a\sqrt{t}) )</td>
</tr>
<tr>
<td>98</td>
<td>( -e^{as} , \text{Ei}(-as) )</td>
<td>( (a &gt; 0) \quad \frac{1}{1+a} )</td>
</tr>
<tr>
<td>99</td>
<td>( \frac{1}{a} + s e^{as} , \text{Ei}(-as) )</td>
<td>( (a &gt; 0) \quad \frac{1}{1+a} )</td>
</tr>
<tr>
<td>100</td>
<td>( \left[ \frac{s}{\pi} - \text{Si}(s) \right] \cos s + \text{Ci}(s) \sin s )</td>
<td>( \frac{1}{s^{2}+1} )</td>
</tr>
<tr>
<td>101</td>
<td>( K_0(as) )</td>
<td>( \begin{cases} 0 &amp; \text{when } 0 &lt; t &lt; k \ \frac{1}{(t^2-k^2)^{1/2}} &amp; \text{when } t &gt; k \end{cases} )</td>
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</tbody>
</table>

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Hankel transforms

\[ \mathcal{H}_\nu(f)(y) = F_\nu(y) = \int_0^\infty f(x) \sqrt{xy} J_\nu(yx) \, dx, \quad y > 0. \]

<table>
<thead>
<tr>
<th>No.</th>
<th>( f(x) )</th>
<th>( F_\nu(y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>\begin{cases} x^{\nu+1/2}, &amp; 0 &lt; x &lt; 1 \ 0, &amp; 1 &lt; x \end{cases}</td>
<td>y^{-1/2}J_{\nu+1}(y)</td>
</tr>
<tr>
<td>2</td>
<td>\begin{cases} x^{\nu+1/2}(a^2 - x^2)\mu, &amp; 0 &lt; x &lt; a \ 0, &amp; a &lt; x \end{cases}, \quad \text{Re } \nu, \text{ Re } \mu &gt; -1,</td>
<td>2^\mu \Gamma(\mu + 1)a^{\nu+\mu+1}y^{\nu-\mu-1/2} \times J_{\nu+\mu+1}(ay)</td>
</tr>
<tr>
<td>3</td>
<td>\begin{cases} x^{\nu+1/2}(x^2 + a^2)^{-\nu-1/2}, &amp; \text{Re } a &gt; 0, \text{ Re } \nu &gt; -1/2 \end{cases}, \quad</td>
<td>\sqrt{\pi}y^{-\nu-1/2}e^{-\nu y} \times [\Gamma(\nu + 1/2)]^{-1}</td>
</tr>
<tr>
<td>4</td>
<td>\begin{cases} x^{\nu+1/2}e^{-ax}, &amp; \text{Re } a &gt; 0, \text{ Re } \nu &gt; -1 \end{cases},</td>
<td>a(\pi)^{-1/2}2^{-\nu+1/2}\Gamma(\nu + 3/2) \times (a^2 + y^2)^{-\nu-3/2}</td>
</tr>
<tr>
<td>5</td>
<td>\begin{cases} x^{\nu+1/2}e^{-ax^2}, &amp; \text{Re } a &gt; 0, \text{ Re } \nu &gt; -1 \end{cases},</td>
<td>y^{\nu+1/2}(2a)^{-\nu-1} \exp \left( -\frac{y^2}{4a} \right)</td>
</tr>
<tr>
<td>6</td>
<td>e^{-ax}/\sqrt{x}, \quad \text{Re } a &gt; 0, \text{ Re } \nu &gt; -1</td>
<td>y^{\nu+1/2} \left[ \sqrt{(a^2 + y^2) - a} \right]^{\nu} \times (a^2 + y^2)^{-1/2}</td>
</tr>
<tr>
<td>7</td>
<td>\begin{cases} x^{-\nu-1/2}\cos(ax), &amp; a &gt; 0, \text{ Re } \nu &gt; -1/2 \end{cases},</td>
<td>\sqrt{\pi}2^{-\nu}y^{-\nu-1/2}[\Gamma(\nu + 1/2)]^{-1} \times (y^2 - a^2)^{\nu-1/2}H(y - a)</td>
</tr>
<tr>
<td>8</td>
<td>\begin{cases} x^{1/2-\nu}\sin(ax), &amp; a &gt; 0, \text{ Re } \nu &gt; 1/2 \end{cases},</td>
<td>a2^{1-\nu}\sqrt{\pi}y^{\nu+1/2}[\Gamma(\nu - 1/2)]^{-1} \times (y^2 - a^2)^{-\nu-3/2}H(y - a)</td>
</tr>
<tr>
<td>9</td>
<td>\begin{cases} x^{-1/2}J_{\nu-1}(ax), &amp; a &gt; 0, \text{ Re } \nu &gt; -1 \end{cases},</td>
<td>a^{\nu-1}y^{-\nu+1/2}H(y - a)</td>
</tr>
<tr>
<td>10</td>
<td>\begin{cases} x^{-1/2}J_{\nu+1}(ax), &amp; a &gt; 0, \text{ Re } \nu &gt; -3/2 \end{cases},</td>
<td>\begin{cases} a^{\nu-1}y^{\nu+1/2}, &amp; 0 &lt; y &lt; a \ 0, &amp; a &lt; y \end{cases}</td>
</tr>
</tbody>
</table>
Hilbert transforms

\[ \mathcal{H}(f)(y) = F(y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{x-y} \, dx. \]

<table>
<thead>
<tr>
<th>No.</th>
<th>( f(x) )</th>
<th>( F(y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>( \begin{cases} 0, &amp; -\infty &lt; x &lt; a \ 1, &amp; a &lt; x &lt; b \ 0, &amp; b &lt; x &lt; \infty \end{cases} )</td>
<td>( \frac{1}{\pi} \log</td>
</tr>
<tr>
<td>3</td>
<td>( \begin{cases} 0, &amp; -\infty &lt; x &lt; a \ \frac{1}{x}, &amp; a &lt; x &lt; \infty \end{cases} )</td>
<td>( (\pi y)^{-1} \log</td>
</tr>
<tr>
<td>4</td>
<td>((x + a)^{-1})</td>
<td>(i(y + a)^{-1})</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{1}{1 + x^2} )</td>
<td>(-y/1 + y^2)</td>
</tr>
<tr>
<td>6</td>
<td>( \frac{1}{1 + x^4} )</td>
<td>(-\frac{y(1 + y^2)}{\sqrt{2}(1 + y^4)})</td>
</tr>
<tr>
<td>7</td>
<td>(\sin(ax), \ a &gt; 0)</td>
<td>(\cos(ay))</td>
</tr>
<tr>
<td>8</td>
<td>(\sin(ax)/x, \ a &gt; 0)</td>
<td>([\cos(ay) - 1]/y)</td>
</tr>
<tr>
<td>9</td>
<td>(\cos(ax), \ a &gt; 0)</td>
<td>(-\sin(ay))</td>
</tr>
<tr>
<td>10</td>
<td>([1 - \cos(ax)]/x, \ a &gt; 0)</td>
<td>(\sin(ay)/y)</td>
</tr>
<tr>
<td>11</td>
<td>(\text{sgn}(x) \sin(a</td>
<td>x</td>
</tr>
<tr>
<td>12</td>
<td>(e^{iax}, \ a &gt; 0)</td>
<td>(ie^{iay})</td>
</tr>
</tbody>
</table>
Mellin transforms

\[ f^*(s) = \mathcal{M}[f(x); s] = \int_0^\infty f(x)x^{s-1} \, dx. \]

<table>
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<tr>
<th>No.</th>
<th>( f(x) )</th>
<th>( f^*(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( ag(x) + bh(x) )</td>
<td>( ag^<em>(s) + bh^</em>(s) )</td>
</tr>
<tr>
<td>2</td>
<td>( f^{(n)}(x) \dagger )</td>
<td>( (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)} f^*(s-n) )</td>
</tr>
<tr>
<td>3</td>
<td>( x^n f^{(n)}(x) \dagger )</td>
<td>( (-1)^n \frac{\Gamma(s+n)}{\Gamma(s)} f^*(s) )</td>
</tr>
<tr>
<td>4</td>
<td>( I_n f(x) \ddagger )</td>
<td>( (-1)^n \frac{\Gamma(s)}{\Gamma(s+n)} f^*(s+n) )</td>
</tr>
<tr>
<td>5</td>
<td>( e^{-x} )</td>
<td>( \Gamma(s) ) Re ( s &gt; 0 )</td>
</tr>
<tr>
<td>6</td>
<td>( e^{-x^2} )</td>
<td>( \frac{1}{2}\Gamma\left(\frac{1}{2}s\right) ) Re ( s &gt; 0 )</td>
</tr>
<tr>
<td>7</td>
<td>( \cos x )</td>
<td>( \Gamma(s) \cos\left(\frac{1}{2}\pi s\right) ) 0 &lt; Re ( s &lt; 1 )</td>
</tr>
<tr>
<td>8</td>
<td>( \sin x )</td>
<td>( \Gamma(s) \sin\left(\frac{1}{2}\pi s\right) ) 0 &lt; Re ( s &lt; 1 )</td>
</tr>
<tr>
<td>9</td>
<td>( (1 - x)^{-1} )</td>
<td>( \pi \cot(\pi s) ) 0 &lt; Re ( s &lt; 1 )</td>
</tr>
<tr>
<td>10</td>
<td>( (1 + x)^{-1} )</td>
<td>( \pi \cosec(\pi s) ) 0 &lt; Re ( s &lt; 1 )</td>
</tr>
<tr>
<td>11</td>
<td>( (1 + x^a)^{-b} )</td>
<td>( \frac{\Gamma(s/a)\Gamma(b-s/a)}{a\Gamma(b)} ) 0 &lt; Re ( s &lt; ab )</td>
</tr>
<tr>
<td>12</td>
<td>( \log(1 + ax) )</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>( \tan^{-1} x )</td>
<td>( -\frac{1}{2}\pi s^{-1} \sec\left(\frac{1}{2}\pi s\right) ) (-1 &lt; \text{Re} , s &lt; 0 )</td>
</tr>
<tr>
<td>14</td>
<td>( \cot^{-1} x )</td>
<td>( \frac{1}{2}\pi s^{-1} \sec\left(\frac{1}{2}\pi s\right) ) 0 &lt; Re ( s &lt; 1 )</td>
</tr>
<tr>
<td>15</td>
<td>( \csc ax )</td>
<td>Re ( a &gt; 0 ) ( 2(1 - 2^{-s})a^{-s} \Gamma(s)\zeta(s) ) Re ( s &gt; 1 )</td>
</tr>
<tr>
<td>16</td>
<td>( \sec^2 ax )</td>
<td>Re ( a &gt; 0 ) ( 4(2a)^{-s} \Gamma(s)\zeta(s-1) ) Re ( s &gt; 2 )</td>
</tr>
<tr>
<td>17</td>
<td>( \csc^2 ax )</td>
<td>Re ( a &gt; 0 ) ( 4(2a)^{-s} \Gamma(s)\zeta(s-1) ) Re ( s &gt; 2 )</td>
</tr>
<tr>
<td>18</td>
<td>( K_v(ax) )</td>
<td>Re ( s &gt;</td>
</tr>
</tbody>
</table>

\dagger Assuming that \( \lim_{x \to 0} x^{s-r-1} f^{(r)}(x) = 0 \) for \( r = 0, 1, \ldots, n - 1 \).

\ddagger Where \( I_n \) denotes the \( n^{\text{th}} \) repeated integral of \( f(x) \): \( I_0 f(x) = f(x) \), \( I_n f(x) = \int_0^x I_{n-1}(t) \, dt \).
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7.1 PROBABILITY THEORY

7.1.1 INTRODUCTION

A sample space $S$ associated with an experiment is a set $S$ of elements such that any outcome of the experiment corresponds to a unique element of the set. An event $E$ is a subset of a sample space $S$. An element in a sample space is called a sample point or a simple event.

Definition of probability

If an experiment can occur in $n$ mutually exclusive and equally likely ways, and if exactly $m$ of these ways correspond to an event $E$, then the probability of $E$ is given by

$$P(E) = \frac{m}{n}.$$ 

If $E$ is a subset of $S$, and if to each unit subset of $S$, a nonnegative number, called the probability, is assigned, and if $E$ is the union of two or more different simple events, then the probability of $E$, denoted $P(E)$, is the sum of the probabilities of those simple events whose union is $E$.

Marginal and conditional probability

Suppose a sample space $S$ is partitioned into $rs$ disjoint subsets where the general subset is denoted $E_i \cap F_j$ (with $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, s$). Then the marginal probability of $E_i$ is defined as

$$P(E_i) = \sum_{j=1}^{s} P(E_i \cap F_j),$$

and the marginal probability of $F_j$ is defined as

$$P(F_j) = \sum_{i=1}^{r} P(E_i \cap F_j).$$

The conditional probability of $E_i$, given that $F_j$ has occurred, is defined as

$$P(E_i \mid F_j) = \frac{P(E_i \cap F_j)}{P(F_j)}, \quad \text{when } P(F_j) \neq 0$$

and that of $F_j$, given that $E_i$ has occurred, is defined as

$$P(F_j \mid E_i) = \frac{P(E_i \cap F_j)}{P(E_i)}, \quad \text{when } P(E_i) \neq 0.$$
Probability theorems

1. If \( \emptyset \) is the null set, then \( P(\emptyset) = 0 \).

2. If \( S \) is the sample space, then \( P(S) = 1 \).

3. If \( E \) and \( F \) are two events, then
\[
P(E \cup F) = P(E) + P(F) - P(E \cap F). \tag{7.1.5}
\]

4. If \( E \) and \( F \) are mutually exclusive events, then
\[
P(E \cup F) = P(E) + P(F). \tag{7.1.6}
\]

5. If \( E \) and \( E' \) are complementary events, then
\[
P(E) = 1 - P(E'). \tag{7.1.7}
\]

6. Two events are said to be independent if and only if
\[
P(E \cap F) = P(E) \cdot P(F). \tag{7.1.8}
\]

The event \( E \) is said to be statistically independent of the event \( F \) if \( P(E \mid F) = P(E) \) and \( P(F \mid E) = P(F) \).

7. The events \( \{E_1, \ldots, E_n\} \) are called mutually independent for all combinations if and only if every combination of these events taken any number of times is independent.

8. Bayes' rule: If \( \{E_1, \ldots, E_n\} \) are \( n \) mutually exclusive events whose union is the sample space \( S \), and if \( E \) is any arbitrary event of \( S \) such that \( P(E) \neq 0 \), then
\[
P(E_k \mid E) = \frac{P(E_k) \cdot P(E \mid E_k)}{P(E)} = \frac{P(E_k) \cdot P(E \mid E_k)}{\sum_{j=1}^{n} P(E_j) \cdot P(E \mid E_j)}. \tag{7.1.9}
\]

9. For a uniform probability distribution,
\[
P(A) = \frac{\text{Number of outcomes in event } A}{\text{Total number of outcomes}}
\]

Terminology

1. A function whose domain is a sample space \( S \) and whose range is some set of real numbers is called a random variable. This random variable is called discrete if it assumes only a finite or denumerable number of values. It is called continuous if it assumes a continuum of values.

2. “iid” or “i.i.d.” is often used for the phrase “independent and identically distributed”.

3. Random variables are usually represented by capital letters.
4. Many probability distribution have special representations:
   (a) \( \chi^2_n \): chi-square random variable with \( n \) degrees of freedom
   (b) \( E(\lambda) \): exponential distribution with parameter \( \lambda \)
   (c) \( N(\mu, \sigma) \): normal random variable with mean \( \mu \) and standard deviation \( \sigma \)
   (d) \( P(\lambda) \): Poisson distribution with parameter \( \lambda \)
   (e) \( U[a, b) \): uniform random variable on the interval \( [a, b) \)

Characterizing random variables

When \( X \) is a discrete random variable, let \( p_k \) for \( k = 0, 1, \ldots \) be the probability that \( X = x_k \) (with \( p_k \geq 0 \) and \( \sum_k p_k = 1 \)). For any event \( E \),

\[
P(E) = P(X \text{ is in } E) = \sum_{x_k \in E} p_k.
\]  
(7.1.10)

In the continuous case, \( f(x) \, dx \) is used to denote the probability that \( X \) lies in the region \( [x, x + dx] \); it is called the probability density function (with \( f(x) \geq 0 \) and \( \int f(x) \, dx = 1 \)). For any event \( E \),

\[
P(E) = P(X \text{ is in } E) = \int_E f(x) \, dx.
\]  
(7.1.11)

The cumulative distribution function, or simply the distribution function, is defined by

\[
F(x) = \text{Probability}(X \leq x) = \begin{cases} 
\sum_{x_k \leq x} p_k, & \text{in the discrete case,} \\
\int_{-\infty}^{x} f(t) \, dt, & \text{in the continuous case.}
\end{cases}
\]  
(7.1.12)

Note that \( F(-\infty) = 0 \) and \( F(\infty) = 1 \). The probability that \( X \) is between \( a \) and \( b \) is

\[
P(a \leq X \leq b) = P(X \leq b) - P(X \leq a) = F(b) - F(a).
\]  
(7.1.13)

Let \( g(X) \) be a function of \( X \). The expected value of \( g(X) \), denoted by \( E [g(X)] \), is defined by

\[
E [g(X)] = \begin{cases} 
\sum_k p_k g(x_k), & \text{in the discrete case,} \\
\int g(t) f(t) \, dt, & \text{in the continuous case.}
\end{cases}
\]  
(7.1.14)

1. \( E[aX + bY] = aE[X] + bE[Y] \).
2. \( E[XY] = E[X]E[Y] \) if \( X \) and \( Y \) are statistically independent.

The moments of \( X \) are defined by \( \mu_k = E[X^k] \). The first moment, \( \mu_1 \), is called the mean of \( X \); it is usually denoted by \( \mu = \mu_1 = E[X] \). The centered moments of \( X \) are defined by \( \mu_k = E[(X - \mu)^k] \). The second centered moment is called the variance and is denoted by \( \sigma^2 = \mu_2 = E[(X - \mu)^2] \). Here, \( \sigma \) is called the standard deviation. The skewness is \( \gamma_1 = \mu_3/\sigma^3 \), and the excess or kurtosis is \( \gamma_2 = (\mu_4/\sigma^4) - 3 \).
Using $\sigma_Z^2$ to denote the variance for the random variable $Z$.

1. $\sigma_{cX}^2 = c^2\sigma_X^2$.
2. $\sigma_{c+X}^2 = \sigma_X^2$.
3. $\sigma_{aX+b}^2 = a^2\sigma_X^2$.

Generating and characteristic functions

In the case of a discrete distribution, the generating function corresponding to $X$ (when it exists) is given by $G(s) = G_X(s) = \mathbb{E}[s^X] = \sum_{k=0}^{\infty} p_k s^k$. From this function, the moments may be found from

$$\mu'_n = \left. \left( s \frac{d}{ds} \right)^n G(s) \right|_{s=1}.$$ (7.1.15)

1. If $c$ is a constant, then the generating function of $c + X$ is $s^c G(s)$.
2. If $c$ is a constant, then the generating function of $cX$ is $G(cs)$.
3. If $Z = X + Y$ where $X$ and $Y$ are independent discrete random variables, then $G_Z(s) = G_X(s)G_Y(s)$.
4. If $Y = \sum_{i=1}^{n} X_i$, the $\{X_i\}$ are independent, and each $X_i$ has the common generating function $G_X(s)$, then the generating function of $Y$ is $[G_X(s)]^n$.

In the case of a continuous distribution, the characteristic function corresponding to $X$ is given by $\phi(t) = \mathbb{E}[e^{itX}] = \int_{-\infty}^{\infty} e^{itx} f(x) \, dx$. From this function, the moments may be found:

$$\mu'_n = i^{-n} \phi^{(n)}(0).$$

If $Z = X + Y$ where $X$ and $Y$ are independent continuous random variables, then $\phi_Z(t) = \phi_X(t)\phi_Y(t)$. The cumulant function is defined as the logarithm of the characteristic function. The $n^{th}$ cumulant, $\kappa_n$, is defined as a certain term in the Taylor series of the cumulant function,

$$\log \phi(t) = \sum_{n=0}^{\infty} \kappa_n \frac{(it)^n}{n!}.$$  

Note that $\kappa_1 = \mu$, $\kappa_2 = \sigma^2$, $\kappa_3 = \mu_3$, and $\kappa_4 = \mu_4 - 3\mu_2^2$. For a normal probability distribution, $\kappa_n = 0$ for $n \geq 3$. The centered moments in terms of cumulants are

$$\begin{align*}
\mu_2 &= \kappa_2, \\
\mu_3 &= \kappa_3, \\
\mu_4 &= \kappa_4 + 3\kappa_2^2, \\
\mu_5 &= \kappa_5 + 10\kappa_3\kappa_2, \\
\mu_6 &= \kappa_6 + 15\kappa_4\kappa_2 + 10\kappa_3^2 + 15\kappa_2^3. 
\end{align*}$$ (7.1.16)
7.1.2 MULTIVARIATE DISTRIBUTIONS

Discrete case

The $k$-dimensional random variable $(X_1, \ldots, X_k)$ is a $k$-dimensional discrete random variable if it assumes values only at a finite or denumerable number of points $(x_1, \ldots, x_k)$. Define

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k) = f(x_1, x_2, \ldots, x_k)$$

(7.1.17)

for every value that the random variable can assume. The function $f(x_1, \ldots, x_k)$ is called the joint density of the $k$-dimensional random variable. If $E$ is any subset of the set of values that the random variable can assume, then

$$P(E) = P[(X_1, \ldots, X_k) \text{ is in } E] = \sum_E f(x_1, \ldots, x_k)$$

(7.1.18)

where the sum is over all those points in $E$. The cumulative distribution function is defined as

$$F(x_1, x_2, \ldots, x_k) = \sum_{z_1 \leq x_1} \sum_{z_2 \leq x_2} \cdots \sum_{z_k \leq x_k} f(z_1, z_2, \ldots, z_k).$$

(7.1.19)

Continuous case

The $k$ random variables $(X_1, \ldots, X_k)$ are said to be jointly distributed if a function $f$ exists so that $f(x_1, \ldots, x_k) \geq 0$ for all $-\infty < x_i < \infty$ ($i = 1, \ldots, k$) and so that, for any given event $E$,

$$P(E) = P[(X_1, X_2, \ldots, X_k) \text{ is in } E] = \int_E \cdots \int f(x_1, x_2, \ldots, x_k) \, dx_1 \, dx_2 \cdots \, dx_k.$$  

(7.1.20)

The function $f(x_1, \ldots, x_k)$ is called the joint density of the random variables $X_1, X_2, \ldots, X_k$. The cumulative distribution function is defined as

$$F(x_1, x_2, \ldots, x_k) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_k} f(z_1, z_2, \ldots, z_k) \, dz_1 \, dz_2 \cdots \, dz_k.$$  

(7.1.21)

Given the cumulative distribution function, the probability density may be found from

$$f(x_1, x_2, \ldots, x_k) = \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} \cdots \frac{\partial}{\partial x_k} F(x_1, x_2, \ldots, x_k).$$

(7.1.22)

Moments

The $r$th moment of $X_i$ is defined as

$$E[X_i^r] = \begin{cases} \sum_{x_1} \cdots \sum_{x_k} x_i^r f(x_1, \ldots, x_k), & \text{in the discrete case}, \\ \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_k} x_i^r f(x_1, \ldots, x_k) \, dx_1 \cdots \, dx_k, & \text{in the continuous case}. \end{cases}$$

(7.1.23)
Joint moments about the origin are defined as $E\left[X_1^{r_1}X_2^{r_2}\cdots X_k^{r_k}\right]$ where $r_1 + r_2 + \cdots + r_k$ is the order of the moment. Joint moments about the mean are defined as $E\left[(X_1 - \mu_1)^{r_1}(X_2 - \mu_2)^{r_2}\cdots (X_k - \mu_k)^{r_k}\right]$, where $\mu_k = E[X_k]$.

**Marginal and conditional distributions**

If the random variables $X_1, X_2, \ldots, X_k$ have the joint density function $f(x_1, x_2, \ldots, x_k)$, then the marginal distribution of the subset of the random variables, say, $X_1, X_2, \ldots, X_p$ (with $p < k$), is given by

$$g(x_1, x_2, \ldots, x_p) = \begin{cases} \sum_{x_{p+1}} \sum_{x_{p+2}} \cdots \sum_{x_k} f(x_1, x_2, \ldots, x_k), & \text{in the discrete case,} \\ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_k) \, dx_{p+1} \cdots dx_k, & \text{in the continuous case.} \end{cases} \tag{7.1.24}$$

The conditional distribution of a certain subset of the random variables is the joint distribution of this subset under the condition that the remaining variables are given certain values. The conditional distribution of $X_1, X_2, \ldots, X_p$, given $X_{p+1}, X_{p+2}, \ldots, X_k$, is

$$h(x_1, \ldots, x_p \mid x_{p+1}, \ldots, x_k) = \frac{f(x_1, x_2, \ldots, x_k)}{g(x_{p+1}, x_{p+2}, \ldots, x_k)} \tag{7.1.25}$$

if $g(x_{p+1}, x_{p+2}, \ldots, x_k) \neq 0$.

The variance $\sigma_{ii}$ of $X_i$ and the covariance $\sigma_{ij}$ of $X_i$ and $X_j$ are given by

$$\sigma_{ii}^2 = \sigma_i^2 = E[(X_i - \mu_i)^2],$$
$$\sigma_{ij}^2 = \rho_{ij}\sigma_i\sigma_j = E[(X_i - \mu_i)(X_j - \mu_j)]. \tag{7.1.26}$$

where $\rho_{ij}$ is the correlation coefficient, and $\sigma_i$ and $\sigma_j$ are the standard deviations of $X_i$ and $X_j$.

**7.1.3 RANDOM SUMS OF RANDOM VARIABLES**

If $T = \sum_{i=1}^{N} X_i$, $N$ is an integer-valued random variable with generating function $G_N(s)$, and if the $\{X_i\}$ are discrete independent and identically distributed random variables with generating function $G_X(s)$, and the $\{X_i\}$ are independent of $N$, then the generating function for $T$ is $G_T(s) = G_N(G_X(s))$. (If the $\{X_i\}$ are continuous random variables, then $\phi_T(\xi) = G_N(\phi_X(\xi))$.) Hence,

- $\mu_T = \mu_N \mu_X$.
- $\sigma_T^2 = \mu_N \sigma_X^2 + \mu_X \sigma_N^2$.
7.1.4 TRANSFORMING VARIABLES

1. Suppose that the random variable \(X\) has the probability density function \(f_X(x)\) and the random variable \(Y\) is defined by \(Y = g(X)\). If \(g\) is measurable and one-to-one, then

\[
f_Y(y) = f_X(h(y)) \left| \frac{dh}{dy} \right| (7.1.27)
\]

where \(h(y) = g^{-1}(y)\).

2. If the random variables \(X\) and \(Y\) are independent and if their densities \(f_X\) and \(f_Y\), respectively, exist almost everywhere, then the probability density of their sum, \(Z = X + Y\), is given by the formula,

\[
f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) \, dx. (7.1.28)
\]

3. If the random variables \(X\) and \(Y\) are independent and if their densities \(f_X\) and \(f_Y\), respectively, exist almost everywhere, then the probability density of their product, \(Z = XY\), is given by the formula,

\[
f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{|x|} f_X(x) f_Y \left( \frac{z}{x} \right) \, dx. (7.1.29)
\]

7.1.5 CENTRAL LIMIT THEOREM

If \(\{X_i\}\) are independent and identically distributed random variables with mean \(\mu\) and finite variance \(\sigma^2\), then the random variable

\[
Z = \frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sqrt{n}\sigma} (7.1.30)
\]
tends (as \(n \to \infty\)) to a normal random variable with mean zero and variance one.

7.1.6 AVERAGES OVER VECTORS

Let \(\bar{f(n)}\) denote the average of the function \(f\) as the unit vector \(\mathbf{n}\) varies uniformly in all directions in three dimensions. If \(\mathbf{a}, \mathbf{b}, \mathbf{c},\) and \(\mathbf{d}\) are constant vectors, then

\[
\begin{align*}
|\mathbf{a} \cdot \mathbf{n}|^2 &= |\mathbf{a}|^2 / 3, \\
(\mathbf{a} \cdot \mathbf{n})(\mathbf{b} \cdot \mathbf{n}) &= (\mathbf{a} \cdot \mathbf{b}) / 3, \\
(\mathbf{a} \cdot \mathbf{n})\mathbf{n} &= \mathbf{a} / 3, \\
|\mathbf{a} \times \mathbf{n}|^2 &= 2|\mathbf{a}|^2 / 3, \\
(\mathbf{a} \times \mathbf{n}) \cdot (\mathbf{b} \times \mathbf{n}) &= 2\mathbf{a} \cdot \mathbf{b} / 3, \\
(\mathbf{a} \cdot \mathbf{n})(\mathbf{b} \cdot \mathbf{n})(\mathbf{c} \cdot \mathbf{n})(\mathbf{d} \cdot \mathbf{n}) &= [(\mathbf{a} \cdot \mathbf{b})(\mathbf{c} \cdot \mathbf{d}) + (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) + (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})] / 15.
\end{align*}
\]
Now let \( \overline{f(n)} \) denote the average of the function \( f \) as the unit vector \( n \) varies uniformly in all directions in two dimensions. If \( a \) and \( b \) are constant vectors, then

\[
\begin{align*}
|a \cdot n|^2 &= |a|^2 / 2, \\
(a \cdot n)(b \cdot n) &= (a \cdot b)/2, \\
(a \cdot n)n &= a/2.
\end{align*}
\]

(7.1.32)

### 7.1.7 INEQUALITIES

1. **Markov's Inequality**: If \( X \) is a random variable which takes only nonnegative values, then for any \( a > 0 \),

\[
P(X \geq a) \leq \frac{E[X]}{a}.
\]

(7.1.33)

2. **Cauchy–Schwartz Inequality**: Let \( X \) and \( Y \) be random variables for which \( E[X^2] \) and \( E[Y^2] \) exist, then

\[
(E[XY])^2 \leq E[X^2]E[Y^2].
\]

(7.1.34)

3. **One-Sided Chebyshev Inequality**: Let \( X \) be a random variable with zero mean (i.e., \( E[X] = 0 \)) and variance \( \sigma^2 \). Then, for any positive \( a \),

\[
P(X > a) \leq \frac{\sigma^2}{\sigma^2 + a^2}.
\]

(7.1.35)

4. **Chebyshev's Inequality**: Let \( c \) be any real number and let \( X \) be a random variable for which \( E[(X - c)^2] \) is finite. Then, for every \( \epsilon > 0 \) the following holds:

\[
P(|X - c| \geq \epsilon) \leq \frac{1}{\epsilon^2}E[(X - c)^2].
\]

(7.1.36)

5. **Bienaymé–Chebyshev's Inequality**: If \( E[|X|^r] < \infty \) for all \( r > 0 \) (\( r \) not necessarily an integer) then, for every \( \epsilon > 0 \)

\[
P(|X| \geq \epsilon) \leq \frac{E[|X|^r]}{\epsilon^r}.
\]

(7.1.37)

6. **Generalized Bienaymé–Chebyshev's Inequality**: Let \( g(x) \) be a non-decreasing nonnegative function defined on \((0, \infty)\). Then, for \( a \geq 0 \),

\[
P(|X| \geq a) \leq \frac{E[g(|X|)]}{g(a)}.
\]

(7.1.38)

7. **Chernoff bound**: This bound is useful for sums of random variables. Let \( Y_n = \sum_{i=1}^{n} X_i \) where each of the \( X_i \) is iid. Let \( M(t) = E[e^{tX}] \) be the common moment generating function for the \( \{X_i\} \), and define \( g(t) = \log M(t) \). Then,

\[
P(Y_n \geq ng'(t)) \leq e^{-n[g'(t) - g(t)]}, \quad \text{if } t \geq 0,
\]

\[
P(Y_n \leq ng'(t)) \leq e^{-n[g'(t) - g(t)]}, \quad \text{if } t \leq 0.
\]
8. **Kolmogorov’s Inequality**: Let $X_1, X_2, \ldots, X_n$ be $n$ independent random variables such that $E[X_i] = 0$ and $Var(X_i) = \sigma^2$ is finite. Then, for all $a > 0$,

$$P \left( \max_{i=1,\ldots,n} |X_1 + X_2 + \cdots + X_i| > a \right) \leq \sum_{i=1}^n \frac{\sigma^2}{a^2}.$$  

9. **Jensen’s Inequality**: If $E[X]$ exists, and if $f(x)$ is a convex function, then

$$E[f(X)] \geq f(E[X]). \quad (7.1.39)$$

### 7.1.8 GEOMETRIC PROBABILITY

1. **Points on a finite line**
   If $A$ and $B$ are uniformly chosen from the interval $[0, 1]$, and $X$ is the distance between $A$ and $B$ (that is, $X = |A - B|$) then the probability density of $X$ is $f_X(x) = 2(1 - x)$.

2. **Points on a finite line**
   Uniformly and independently choose $n - 1$ random values in the interval $[0, 1)$. This creates $n$ intervals. 

   $$P_k(x) = \text{Probability (exactly } k \text{ intervals have length larger than } x \text{)}$$

   $$= \binom{n}{k} \left\{ [1 - kx]^{n-1} - \binom{n-1}{1} [1 - (k+1)x]^{n-1} + \cdots + (-1)^s \binom{n-k}{s} [1 - (k+s)x]^{n-1} \right\},$$

   where $s = \left\lfloor \frac{1}{x} - k \right\rfloor$. From this, the probability that the largest interval length exceeds $x$ is

   $$1 - P_0(x) = \binom{n}{1} (1 - x)^{n-1} - \binom{n}{2} (1 - 2x)^{n-1} + \ldots.$$

3. **Points in the plane**
   Assume that the number of points in any region $A$ of the plane is a Poisson variate with mean $\lambda A$ ($\lambda$ is the “density” of the points). Given a fixed point $P$ define $R_1, R_2, \ldots,$ to be the distance to the point nearest to $P$, second nearest to $P$, etc. Then

   $$f_{R_1}(r) = \frac{2(\lambda \pi)^r}{(s-1)!} r^{2s-1} e^{-\lambda \pi r^2}.$$  

4. **Buffon’s needle problem**
   A needle of length $L$ is placed at random on a plane on which are ruled parallel lines at unit distance apart. Assume that $L < 1$ so that only one intersection is possible. The probability $P$ that the needle intersects a line is

   $$P = \frac{2}{\pi} \left[ \frac{\pi}{2} - \sin^{-1} L^{-1} + L - \sqrt{L^2 - 1} \right].$$

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5. Points in three-dimensional space

Assume that the number of points in any volume $V$ is a Poisson variate with mean $\lambda V$ ($\lambda$ is the “density” of the points). Given a fixed point $P$ define $R_1$, $R_2$, \ldots, to be the distance to the point nearest to $P$, second nearest to $P$, etc. Then

$$f_{R_s}(r) = \frac{3}{\Gamma(s)} \left( \frac{4 \lambda \pi}{3} \right)^s r^{3s-1} e^{-\frac{4 \lambda \pi}{3} r^3}.$$

7.1.9 CLASSIC PROBABILITY PROBLEMS

1. Birthday problem: The probability that $n$ people all have different birthdays is $q_n = \frac{365}{365} \frac{364}{365} \ldots \frac{365-n}{365}$. Let $p_n = 1 - q_n$. For 23 people the probability of at least two people having the same birthday is $p_{23} = 1 - q_{23} > 1/2$; more than half.

<table>
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<th>20</th>
<th>23</th>
<th>30</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
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<td>0.411</td>
<td>0.507</td>
<td>0.706</td>
<td>0.891</td>
<td>0.970</td>
</tr>
</tbody>
</table>

2. Raisin cookie problem: A baker creates enough cookie dough for 1000 raisin cookies. The number of raisins to be added to the dough, $R$, is to be determined.

- If you want to be 99% certain that the first cookie will have at least one raisin, then $1 - \left( \frac{999}{1000} \right)^R \geq 0.99$, or $R \geq 4603$.

- If you want to be 99% certain that every cookie will have at least one raisin, then $P(C, R) \geq 0.99$, where $C$ is the number of cookies and $P(C, R) = C^{-R} \sum_{i=0}^{C} \binom{C}{i} (-1)^i (C - i)^R$. Hence $R \geq 11508$.

7.2 PROBABILITY DISTRIBUTIONS

7.2.1 DISCRETE DISTRIBUTIONS

1. Discrete uniform distribution: If the random variable $X$ has a probability density function given by

$$P(X = x) = f(x) = \frac{1}{n}, \quad \text{for } x = x_1, x_2, \ldots, x_n,$$

then the variable $X$ is said to possess a discrete uniform probability distribution.
Properties: When \( x_i = i \) for \( i = 1, 2, \ldots, n \) then

\[
\text{Mean} = \mu = \frac{n + 1}{2}, \\
\text{Variance} = \sigma^2 = \frac{n^2 - 1}{12}, \\
\text{Standard deviation} = \sigma = \sqrt{\frac{n^2 - 1}{12}}, \\
\text{Moment generating function} = G(t) = e^t \left( 1 - e^{\theta t} \right).
\] (7.2.2)

2. **Binomial distribution:** If the random variable \( X \) has a probability density function given by

\[
P(X = x) = f(x) = \binom{n}{x} \theta^x (1 - \theta)^{n-x}, \quad \text{for } x = 0, 1, \ldots, n,
\] (7.2.3)

then the variable \( X \) is said to possess a binomial distribution. Note that \( f(x) \) is the general term in the expansion of \([\theta + (1 - \theta)]^n\).

**Properties:**

\[
\text{Mean} = \mu = n\theta, \\
\text{Variance} = \sigma^2 = n\theta(1 - \theta), \\
\text{Standard deviation} = \sigma = \sqrt{n\theta(1 - \theta)}, \\
\text{Moment generating function} = G(t) = [\theta e^t + (1 - \theta)]^n.
\] (7.2.4)

3. **Geometric distribution:** If the random variable \( X \) has a probability density function given by

\[
P(X = x) = f(x) = \theta (1 - \theta)^{x-1} \quad \text{for } x = 1, 2, 3, \ldots
\] (7.2.5)

then the variable \( X \) is said to possess a geometric distribution.

**Properties:**

\[
\text{Mean} = \mu = \frac{1}{\theta}, \\
\text{Variance} = \sigma^2 = \frac{1 - \theta}{\theta^2}, \\
\text{Standard deviation} = \sigma = \sqrt{\frac{1 - \theta}{\theta^2}}, \\
\text{Moment generating function} = G(t) = \frac{\theta e^t}{1 - e^t(1 - \theta)}.
\] (7.2.6)

4. **Hypergeometric distribution:** If the random variable \( X \) has a probability density function given by

\[
P(X = x) = f(x) = \binom{k}{x} \binom{N-k}{n-x} \binom{n}{x} \quad \text{for } x = 1, 2, 3, \ldots, \min(n, k)
\] (7.2.7)
then the variable $X$ is said to possess a hypergeometric distribution.

**Properties:**

\[
\text{Mean} = \mu = \frac{kn}{N}, \\
\text{Variance} = \sigma^2 = \frac{k(N - k)n(N - n)}{N^2(N - 1)}, \\
\text{Standard deviation} = \sigma = \sqrt{\frac{k(N - k)n(N - n)}{N^2(N - 1)}}.
\] (7.2.8)

5. **Negative binomial distribution:** If the random variable $X$ has a probability density function given by

\[
P(X = x) = f(x) = \binom{x + r - 1}{r - 1} \theta^r (1 - \theta)^x 
\text{ for } x = 0, 1, 2, \ldots
\] (7.2.9)

then the variable $X$ is said to possess a negative binomial distribution (also known as a Pascal or Polya distribution).

**Properties:**

\[
\text{Mean} = \mu = \frac{r}{\theta}, \\
\text{Variance} = \sigma^2 = \frac{r}{\theta} \left( \frac{1}{\theta} - 1 \right) = \frac{r(1 - \theta)}{\theta^2}, \\
\text{Standard deviation} = \sigma = \sqrt{\frac{r(1 - \theta)}{\theta^2}}, \\
\text{Moment generating function} = G(t) = e^{r(\theta e^t - 1)}.
\] (7.2.10)

6. **Poisson distribution:** If the random variable $X$ has a probability density function given by

\[
P(X = x) = f(x) = e^{-\lambda} \frac{\lambda^x}{x!} 
\text{ for } x = 0, 1, 2, \ldots
\] (7.2.11)

with $\lambda > 0$, then the variable $X$ is said to possess a Poisson distribution.

**Properties:**

\[
\text{Mean} = \mu = \lambda, \\
\text{Variance} = \sigma^2 = \lambda, \\
\text{Standard deviation} = \sigma = \sqrt{\lambda}, \\
\text{Moment generating function} = G(t) = e^{\lambda(e^t - 1)}.
\] (7.2.12)

7. **Multinomial distribution:** If a set of random variables $X_1, X_2, \ldots, X_n$ has a probability function given by

\[
P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = f(x_1, x_2, \ldots, x_n) \\
= \frac{n!}{x_1! \cdots x_n!} \prod_{i=1}^{n} \frac{q_i^{x_i}}{x_i!}
\] (7.2.13)
where the \( \{x_i\} \) are positive integers, each \( \theta_i > 0 \), and

\[
\sum_{i=1}^{n} \theta_i = 1 \quad \text{and} \quad \sum_{i=1}^{n} x_i = N,
\]

then the joint distribution of \( X_1, X_2, \ldots, X_n \) is called the multinomial distribution. Note that \( f(x_1, x_2, \ldots, x_n) \) is the general term in the expansion of \((\theta_1 + \theta_2 + \cdots + \theta_n)^N\).

**Properties:**

\[
\begin{align*}
\text{Mean of } X_i &= \mu_i = N\theta_i, \\
\text{Variance of } X_i &= \sigma^2_i = N\theta_i(1 - \theta_i), \\
\text{Covariance of } X_i \text{ and } X_j &= \sigma^2_{ij} = -N\theta_i\theta_j, \\
\text{Joint moment generating function} &= (\theta_1 e^t + \cdots + \theta_n e^t)^N.
\end{align*}
\]

### 7.2.2 CONTINUOUS DISTRIBUTIONS

1. **Uniform distribution:** If the random variable \( X \) has a density function of the form

\[
f(x) = \frac{1}{\beta - \alpha}, \quad \text{for } \alpha < x < \beta,
\]

then the variable \( X \) is said to possess a uniform distribution.

**Properties:**

\[
\begin{align*}
\text{Mean} &= \mu = \frac{\alpha + \beta}{2}, \\
\text{Variance} &= \sigma^2 = \frac{(\beta - \alpha)^2}{12}, \\
\text{Standard deviation} &= \sigma = \sqrt{\frac{(\beta - \alpha)^2}{12}}, \\
\text{Moment generating function} &= G(t) = \frac{e^{\beta t} - e^{\alpha t}}{(\beta - \alpha)t}, \\
&= \frac{2}{(\beta - \alpha)t} \sinh \left[ \frac{(\beta - \alpha)t}{2} \right] e^{(\alpha + \beta)t/2}.
\end{align*}
\]

2. **Normal distribution:** If the random variable \( X \) has a density function of the form

\[
f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left( -\frac{(x - \mu)^2}{2\sigma^2} \right), \quad \text{for } -\infty < x < \infty,
\]

then the variable \( X \) is said to possess a uniform distribution.
Properties:

- Mean = \( \mu \),
- Variance = \( \sigma^2 \),
- Standard deviation = \( \sigma \),
- Moment generating function = \( G(t) = \exp \left( \mu t + \frac{\sigma^2 t^2}{2} \right) \).

Set \( y = \frac{x - \mu}{\sigma} \) to obtain a standard normal distribution.

- The cumulative distribution function is
  \[
  F(x) = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp \left( -\frac{(t - \mu)^2}{2\sigma^2} \right) dt.
  \]

3. Multidimensional normal distribution:

The random vector \( \mathbf{X} \) is said to be multivariate normal if and only if the linear combination \( \mathbf{a}^T \mathbf{X} \) is normal for all vectors \( \mathbf{a} \). If the mean of \( \mathbf{X} \) is \( \mu \), and if the second moment matrix \( R = \text{E} \left[ (\mathbf{X} - \mu)(\mathbf{X} - \mu)^T \right] \) is nonsingular, the density function of \( \mathbf{X} \) is

\[
p(\mathbf{x}) = \frac{1}{(2\pi)^{n/2} \sqrt{\det R}} \exp \left[ -\frac{1}{2} (\mathbf{x} - \mu)^T R^{-1} (\mathbf{x} - \mu) \right].
\]

Sometimes integrals of the form \( I_n = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (\mathbf{x}^T \mathbf{M} \mathbf{x})^n p(\mathbf{x}) d\mathbf{x} \) are desired. Defining \( a_k = \text{tr}(M^k) \), we find:

- \( I_0 = 1 \),
- \( I_1 = a_1 \),
- \( I_2 = a_1^2 + 2a_2 \),
- \( I_3 = a_1^3 + 6a_1a_2 + 8a_3 \),
- \( I_4 = a_1^4 + 12a_1^2a_2 + 32a_1a_3 + 12a_2^2 + 48a_4 \).

4. Gamma distribution: If the random variable \( X \) has a density function of the form

\[
f(x) = \frac{1}{\Gamma(1 + \alpha)\beta^{1+\alpha}} x^\alpha e^{-x/\beta}, \quad \text{for } 0 < x < \infty,
\]

with \( \alpha > -1 \) and \( \beta > 0 \), then the variable \( X \) is said to possess a gamma distribution.

Properties:

- Mean = \( \mu = \beta(1 + \alpha) \),
- Variance = \( \sigma^2 = \beta^2(1 + \alpha) \),
- Standard deviation = \( \sigma = \beta \sqrt{1 + \alpha} \),
- Moment generating function = \( G(t) = (1 - \beta t)^{-1-\alpha} \), \quad \text{for } t < \beta^{-1} \).

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5. **Exponential distribution**: If the random variable $X$ has a density function of the form
\[
f(x) = \frac{e^{-x/\theta}}{\theta}, \quad \text{for } 0 < x < \infty,
\] (7.2.24)
where $\theta > 0$, then the variable $X$ is said to possess an exponential distribution.

**Properties:**

- Mean $\mu = \theta$,
- Variance $\sigma^2 = \theta^2$,
- Standard deviation $\sigma = \theta$,
- Moment generating function $G(t) = (1 - \theta t)^{-1}$. (7.2.25)

6. **Beta distribution**: If the random variable $X$ has a density function of the form
\[
f(x) = \frac{\Gamma(\alpha + \beta + 2)}{\Gamma(1 + \alpha)\Gamma(1 + \beta)} x^\alpha(1-x)\beta, \quad \text{for } 0 < x < 1,
\] (7.2.26)
where $\alpha > -1$ and $\beta > -1$, then the variable $X$ is said to possess a beta distribution.

**Properties:**

- Mean $\mu = \frac{1 + \alpha}{2 + \alpha + \beta}$,
- Variance $\sigma^2 = \frac{(1 + \alpha)(1 + \beta)}{(2 + \alpha + \beta)(3 + \alpha + \beta)}$,
- $r^{th}$ moment about the origin $\nu_r = \frac{\Gamma(2 + \alpha + \beta)\Gamma(1 + \alpha + r)}{\Gamma(2 + \alpha + \beta + r)\Gamma(1 + \alpha)}$. (7.2.27)

7. **Chi-square distribution**: If the random variable $X$ has a density function of the form
\[
f(x) = \frac{x^{(n-2)/2}e^{-x/2}}{2^{n/2}\Gamma(n/2)} \quad \text{for } 0 < x < \infty
\] (7.2.28)
then the variable $X$ is said to possess a chi-square ($\chi^2$) distribution with $n$ degrees of freedom.

**Properties:**

- Mean $\mu = n$,
- Variance $\sigma^2 = 2n$. (7.2.29)

(a) If $Y_1, Y_2, \ldots, Y_n$ are independent and identically distributed normal random variables with a mean of 0 and a variance of 1, then $\chi^2 = \sum_{i=1}^{n} Y_i^2$ is distributed as chi-square with $n$ degrees of freedom.
(b) If \( \chi_1^2, \chi_2^2, \ldots, \chi_k^2 \), are independent random variables and have chi-square distributions with \( n_1, n_2, \ldots, n_k \) degrees of freedom, then \( \sum_{i=1}^{k} \chi_i^2 \) has a chi-squared distribution with \( n = \sum_{i=1}^{k} n_i \) degrees of freedom.

8. Snedecor’s F-distribution: If the random variable \( X \) has a density function of the form

\[
    f(x) = \frac{\Gamma \left( \frac{n+m}{2} \right) \left( \frac{m}{n} \right)^{m/2} x^{(m-2)/2}}{\Gamma \left( \frac{m}{2} \right) \Gamma \left( \frac{n}{2} \right) \left( 1 + \frac{m}{n} x \right)^{(n+m)/2}}, \quad \text{for } 0 < x < \infty,
\]

then the variable \( X \) is said to possess a F-distribution with \( m \) and \( n \) degrees of freedom.  

Properties:

- Mean = \( \mu = \frac{n}{n-2} \), for \( n > 2 \),
- Variance = \( \sigma^2 = \frac{2n^2(m + n - 2)}{m(n(n-2)^2(n-4))} \), for \( n > 4 \).

(a) The transformation \( w = \frac{mx/n}{1 + \frac{mx}{n}} \) transforms the F-density to the beta density.

(b) If the random variable \( X \) has a \( \chi^2 \)-distribution with \( m \) degrees of freedom, the random variable \( Y \) has a \( \chi^2 \)-distribution with \( n \) degrees of freedom, and \( X \) and \( Y \) are independent, then \( F = \frac{X/m}{Y/n} \) is distributed as an \( F \)-distribution with \( m \) and \( n \) degrees of freedom.

9. Student’s t-distribution: If the random variable \( X \) has a density function of the form

\[
    f(x) = \frac{\Gamma \left( \frac{n+1}{2} \right)}{\sqrt{n\pi} \Gamma \left( \frac{n}{2} \right) \left( 1 + \frac{x^2}{n} \right)^{(n+1)/2}}, \quad \text{for } -\infty < x < \infty.
\]

then the variable \( X \) is said to possess a t-distribution with \( n \) degrees of freedom.

Properties:

- Mean = \( \mu = 0 \),
- Variance = \( \sigma^2 = \frac{n}{n-2} \), for \( n > 2 \).

- If the random variable \( X \) is normally distributed with mean 0 and variance \( \sigma^2 \), and if \( Y^2/\sigma^2 \) has a \( \chi^2 \) distribution with \( n \) degrees of freedom, and if \( X \) and \( Y \) are independent, then \( t = \frac{X\sqrt{n}}{Y} \) is distributed as a t-distribution with \( n \) degrees of freedom.
7.3 QUEUING THEORY

A queue is represented as $A/B/c/K/m/Z$ where

- $A$ and $B$ represent the interarrival times and service times:
  - $GI$: general independent interarrival time,
  - $G$: general service time distribution,
  - $H_k$: $k$-stage hyperexponential interarrival or service time distribution,
  - $E_k$: Erlang-$k$ interarrival or service time distribution,
  - $M$: exponential interarrival or service time distribution,
  - $D$: deterministic (constant) interarrival or service time distribution.

- $c$ is the number of identical servers.
- $K$ is the system capacity.
- $m$ is the number in the source.
- $Z$ is the queue discipline:
  - FCFS: first come, first served (also known as FIFO),
  - LIFO: last in, first out,
  - RSS: random,
  - PRI: priority service.

When not all variables are present, the trailing ones have the default values, $K = \infty$, $m = \infty$, $Z$ is RSS.

The variable of concern are

1. $a_n$: proportion of customers that find $n$ customers already in the system when they arrive
2. $d_n$: proportion of customers leaving behind $n$ customers in the system
3. $p_n$: proportion of time the system contains $n$ customers
4. $L$: average number of customers in the system
5. $L_Q$: average number of customers in the queue
6. $W$: average time for customer in system
7. \( W_Q \): average time for customer in the queue
8. \( \lambda \): average arrival rate of customers to the system (number per unit time)
9. \( \mu \): average service rate per server (number per unit time)
10. \( u \): traffic intensity, \( u = \lambda / \mu \)
11. \( \rho \): server utilization, the probability that any particular server is busy, \( \rho = u / c = (\lambda / \mu) / c \)

**Theorems:**

1. Little’s formula: \( L = \lambda W \) and \( L_Q = \lambda W_Q \).
2. For Poisson arrivals: \( p_n = a_n \).
3. If customers arrive one at a time and are served one at a time: \( a_n = d_n \).
4. For an \( M/M/1 \) queue with \( \lambda < \mu \),
   - \( p_n = (1 - u)u^n \)
   - \( L = u / (1 - u) \)
   - \( L_Q = L - (1 - p_0) \)
   - \( L_Q = u^2 / (1 - u) \)
   - \( W = 1 / (\mu - \lambda) \)
   - \( W_Q = \lambda W / (\mu - \lambda) \)
5. For an \( M/M/c \) queue (so that \( \mu_n = n\mu \) for \( n = 1, 2, \ldots, c \) and \( \mu_n = c\mu \) for \( n \geq c \),
   - \( p_0 = \left[ \frac{u^c}{c!(1 - \rho)} + \sum_{n=0}^{c-1} \frac{u^n}{n!} \right]^{-1} \)
   - \( p_n = p_0 u^n / n! \) for \( n = 0, 1, \ldots, c \)
   - \( p_n = p_0 u^n / c! e^{n-c} \) for \( n \geq c \)
   - \( L_Q = p_0 u^c \rho / c! (1 - \rho)^2 \)
   - \( W_Q = L_Q / \lambda \)
   - \( W = W_Q + 1 / \mu \)
   - \( L = \lambda W \)
7.4 MARKOV CHAINS

A discrete parameter stochastic process is a collection of random variables \( \{ X(t), t = 0, 1, 2, \ldots \} \). The values of \( X(t) \) are called the states of the process. The collection of states is called the state space. The values of \( t \) usually represent points in time. The number of states is either finite or countably infinite. A discrete parameter stochastic process is called a Markov chain if, for any set of \( n \) time points \( t_1 < t_2 < \cdots < t_n \), the conditional distribution of \( X(t_n) \) given values for \( X(t_1), X(t_2), \ldots, X(t_{n-1}) \) depends only on \( X(t_{n-1}) \). It is expressed by

\[
P \left[ X(t_n) \leq x_n \mid X(t_1) = x_1, \ldots, X(t_{n-1}) = x_{n-1} \right] = P \left[ X(t_n) \leq x_n \mid X(t_{n-1}) = x_{n-1} \right]. \tag{7.4.1}
\]

A Markov chain is said to be stationary if the value of the conditional probability \( P \left[ X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n \right] \) is independent of \( n \). This discussion will be restricted to stationary Markov chains.

7.4.1 TRANSITION FUNCTION

Let \( x \) and \( y \) be states and let \( \{ t_n \} \) be time points in \( T = \{ 0, 1, 2, \ldots \} \). The transition function, \( P(x, y) \), is defined by

\[
P(x, y) = P_{n,n+1}(x, y) = P \left[ X(t_{n+1}) = y \mid X(t_n) = x \right], \quad t_n, t_{n+1} \in T. \tag{7.4.2}
\]

\( P(x, y) \) is the probability that a Markov chain in state \( x \) at time \( n \) will be in state \( y \) at time \( n + 1 \). Some properties of the transition function are that \( P(x, y) \geq 0 \) and \( \sum_y P(x, y) = 1 \). The values of \( P(x, y) \) are commonly called the one-step transition probabilities.

The function \( \pi_0(x) = P(X(0) = x) \), with \( \pi_0(x) \geq 0 \) and \( \sum_x \pi_0(x) = 1 \) is called the initial distribution of the Markov chain. It is the probability distribution when the chain is started. Thus,

\[
P \left[ X(0) = x_0, X(1) = x_1, \ldots, X(n) = x_n \right] = \pi_0(x_0) P_{0,1}(x_0, x_1) P_{1,2}(x_1, x_2) \cdots P_{n-1,n}(x_{n-1}, x_n). \tag{7.4.3}
\]

7.4.2 TRANSITION MATRIX

A convenient way to summarize the transition function of a Markov chain is by using the one-step transition matrix. It is defined as

\[
P = \begin{bmatrix}
P(0, 0) & P(0, 1) & \cdots & P(0, n) \\
P(1, 0) & P(1, 1) & \cdots & P(1, n) \\
\vdots & \vdots & \ddots & \vdots \\
P(n, 0) & P(n, 1) & \cdots & P(n, n)
\end{bmatrix}.
\]

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Define the \textit{n–step transition matrix} by $P^{(n)}$ as the matrix with entries
\[
P^{(n)}(x, y) = P\left[ X(t_m+n) = y \mid X(t_m) = x \right].
\] (7.4.4)

This can be written in terms of the one-step transition matrix as $P^{(n)} = P^n$.

Suppose the state space is finite. The one-step transition matrix is said to be \textit{regular} if, for some positive power $m$, all of the elements of $P^m$ are strictly positive.

\textbf{THEOREM 7.4.1} \textit{(Chapman–Kolmogorov equation)}

\begin{quote}
Let $P(x, y)$ be the one-step transition function of a Markov chain and define $P^0(x, y) = 1$, if $x = y$, and 0, otherwise. Then, for any pair of nonnegative integers, $s$ and $t$, such that $s + t = n$,
\[
P^n(x, y) = \sum_z P^s(x, z) P^t(z, y).
\]
\end{quote}

\textbf{7.4.3 RECURRENCE}

Define the probability that a Markov chain starting in state $x$ returns to state $x$ for the first time after $n$ steps by
\[
f^n(x, x) = P\left[ X(t_n) = x, X(t_{n-1}) \neq x, \ldots, X(t_1) \neq x \mid X(t_0) = x \right].
\]

It follows that $P^n(x, x) = \sum_{k=0}^n f^k(x, x) P^{n-k}(x, x)$. A state $x$ is said to be \textit{recurrent} if $\sum_{n=0}^{\infty} f^n(x, x) = 1$. This means that a state $x$ is recurrent if, after starting in $x$, the probability of returning to it after some finite length of time is one. A state which is not recurrent is said to be \textit{transient}.

\textbf{THEOREM 7.4.2}

A state $x$ of a Markov chain is recurrent if and only if $\sum_{n=1}^{\infty} P^n(x, x) = \infty$.

Two states, $x$ and $y$, are said to \textit{communicate} if, for some $n \geq 0$, $P^n(x, y) > 0$. This theorem implies that, if $x$ is a recurrent state and $x$ communicates with $y$, $y$ is also a recurrent state. A Markov chain is said to be \textit{irreducible} if every state communicates with every other state and with itself.

Let $x$ be a recurrent state and define $T_x$ the \textit{(return time)} as the number of stages for a Markov chain to return to state $x$, having begun there. A recurrent state $x$ is said to be \textit{null recurrent} if $E[T_x] = \infty$. A recurrent state that is not null recurrent is said to be \textit{positive recurrent}.

\textbf{7.4.4 STATIONARY DISTRIBUTIONS}

Let $\{X(t), t = 0, 1, 2, \ldots\}$ be a Markov chain having a one-step transition function of $P(x, y)$. A function $\pi(x)$ where each $\pi(x)$ is nonnegative, $\sum_x \pi(x) P(x, y) = \pi(y)$, and $\sum_y \pi(y) = 1$, is called a \textit{stationary distribution}. If a Markov chain has a
stationary distribution and \( \lim_{n \to \infty} P^n(x, y) = \pi(y) \), then, regardless of the initial distribution, \( \pi_0(x) \), the distribution of \( X(t_n) \) approaches \( \pi(x) \) as \( n \) becomes infinite. When this happens, \( \pi(x) \) is often referred to as the steady state distribution. The following categorizes those Markov chains with stationary distributions.

**THEOREM 7.4.3**

Let \( X_P \) denote the set of positive recurrent states of a Markov chain.

1. If \( X_P \) is empty, the chain has no stationary distribution.
2. If \( X_P \) is a nonempty irreducible set, the chain has a unique stationary distribution.
3. If \( X_P \) is nonempty but not irreducible, the chain has an infinite number of distinct stationary distributions.

The period of a state \( x \) is denoted by \( d(x) \) and is defined as the greatest common divisor of all integers, \( n \geq 1 \), for which \( P^n(x, x) > 0 \). If \( P^n(x, x) = 0 \) for all \( n \geq 1 \), then define \( d(x) = 0 \). If each state of a Markov chain has \( d(x) = 1 \), the chain is said to be aperiodic. If each state has period \( d > 1 \), the chain is said to be periodic with period \( d \). The vast majority of Markov chains encountered in practice are aperiodic. An irreducible, positive recurrent, aperiodic Markov chain always possesses a steady-state distribution. An important special case occurs when the state space is finite. Suppose that \( X = \{1, 2, \ldots, K\} \). Let \( \pi_0 = \{\pi_0(1), \pi_0(2), \ldots, \pi_0(K)\} \).

**THEOREM 7.4.4**

Let \( P \) be a regular one-step transition matrix and \( \pi_0 \) be an arbitrary vector of initial probabilities. Then \( \lim_{n \to \infty} \pi_0(x) \ P^n = y \), where \( yP = y \), and \( \sum_{i=1}^{K} \pi_0(t_i) = 1 \).

**A simple three-state Markov chain**

A Markov chain having three states \{0, 1, 2\} with a one-step transition matrix of

\[
\begin{bmatrix}
\frac{1}{4} & 0 & \frac{3}{4} \\
\frac{1}{2} & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\]

is diagrammed below.

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The one-step transition matrix gives a two-step transition matrix of

\[ P^{(2)} = P^2 = \begin{bmatrix}
\frac{1}{4} & \frac{3}{8} & \frac{3}{8} \\
\frac{5}{16} & \frac{9}{16} & \frac{1}{8} \\
\frac{3}{16} & \frac{9}{16} & \frac{1}{8}
\end{bmatrix} \]

The one-step transition matrix is regular. This Markov chain is irreducible, and all three states are recurrent. In addition, all three states are positive recurrent. Since all states have period 1, the chain is aperiodic. The steady state distribution is \( \pi(0) = \frac{3}{11}, \pi(1) = \frac{6}{11}, \text{ and } \pi(2) = \frac{2}{11} \).

### 7.4.5 Random Walks

Let \( \eta(t_1), \eta(t_2), \ldots \) be independent random variables having a common density \( f(x) \), and let \( t_1, t_2, \ldots \) be integers. Let \( X(t_0) \) be an integer-valued random variable that is independent of \( \eta(t_1), \eta(t_2), \ldots, \) and \( X(t_n) = X_0 + \sum_{i=1}^{n} \eta(t_i) \). The sequence \( \{X(t_i), i = 0, 1, \ldots\} \) is called a random walk. An important special case is a simple random walk. It is defined by

\[ P(x, y) = \begin{cases}
p, & \text{if } y = x - 1, \\
r, & \text{if } y = x, \quad \text{where } p + q + r = 1, \quad \text{and } P(0, 0) = p + r. \\
q, & \text{if } y = x + 1,
\end{cases} \]

Here, an object begins at a certain point in a lattice and at each step either stays at that point or moves to a neighboring lattice point. In the case of a one- or two-dimensional lattice, it turns out that, if a random walk begins at a lattice point \( x \), it will return to that point with probability 1. In the case of a three-dimensional lattice, the probability that it will return to its starting point is only about 0.3405.

### 7.4.6 Ehrenfest Chain

A simple model of gas exchange between two isolated bodies is as follows. Suppose that there are two boxes, Box I and Box II, where Box I contains \( K \) molecules numbered \( 1, 2, \ldots, K \) and Box II contains \( N - K \) molecules numbered \( K + 1, K + 2, \ldots, N \). A number is chosen at random from \( \{1, 2, \ldots, N\} \), and the molecule with that number is transferred from its box to the other one. Let \( X(t_n) \) be the number of molecules in Box I after \( n \) trials. Then the sequence \( \{X(t_n), n = 0, 1, \ldots\} \) is a Markov chain with one-stage transition function of

\[ P(x, y) = \begin{cases}
\frac{x}{K}, & \text{if } y = x - 1, \\
1 - \frac{x}{K}, & \text{if } y = x + 1, \\
0, & \text{otherwise.}
\end{cases} \]
7.5 RANDOM NUMBER GENERATION

7.5.1 METHODS OF PSEUDORANDOM NUMBER GENERATION

Depending on the application, either integers in some range or floating point numbers in [0, 1) are the desired output from a pseudorandom number generator (PRNG). Since most PRNGs use integer recursions, a conversion into integers in a desired range or into a floating point number in [0, 1) is required. If \( x_n \) is an integer produced by some PRNG in the range \( 0 \leq x_n \leq M - 1 \), then an integer in the range \( 0 \leq x_n \leq N - 1 \), with \( N \leq M \), is given by \( y_n = \left\lfloor \frac{N x_n}{M} \right\rfloor \). If \( N \ll M \), then \( y_n = x_n \mod N \) may be used. Alternately, if a floating point value in [0, 1) is desired, let \( y_n = x_n / M \).

Linear congruential generators

Perhaps the oldest generator still in use is the linear congruential generator (LCG). The underlying integer recursion for LCGs is

\[
x_n = a x_{n-1} + b \pmod{M}.
\]

Equation (7.5.1) defines a periodic sequence of integers modulo \( M \) starting with \( x_0 \), the initial seed. The constants of the recursion are referred to as the modulus \( M \), multiplier \( a \), and additive constant \( b \). If \( M = 2^m \), a very efficient implementation is possible. Alternately, there are theoretical reasons why choosing \( M \) prime is optimal. Hence, the only moduli that are used in practical implementations are \( M = 2^m \) or the prime \( M = 2^p - 1 \) (i.e., \( M \) is a Mersenne prime). With a Mersenne prime, modular multiplication can be implemented at about twice the computational cost of multiplication modulo \( 2^p \).

Equation (7.5.1) yields a sequence \( \{x_n\} \) whose period, denoted \( \text{Per}(x_n) \), depends on \( M, a, \) and \( b \). The values of the maximal period for the three most common cases used and the conditions required to obtain them are

<table>
<thead>
<tr>
<th>( a )</th>
<th>( b )</th>
<th>( M )</th>
<th>( \text{Per}(x_n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Primitive root of ( M )</td>
<td>Anything</td>
<td>Prime</td>
<td>( M - 1 )</td>
</tr>
<tr>
<td>3 or 5 ( \pmod{8} )</td>
<td>0</td>
<td>( 2^m )</td>
<td>( 2^{m-2} )</td>
</tr>
<tr>
<td>1 ( \pmod{4} )</td>
<td>1 ( \pmod{2} )</td>
<td>( 2^m )</td>
<td>( 2^m )</td>
</tr>
</tbody>
</table>

A major shortcoming of LCGs modulo a power-of-two compared with prime modulus LCGs derives from the following theorem for LCGs:

**THEOREM 7.5.1**

Define the following LCG sequence: \( x_n = a x_{n-1} + b \pmod{M_1} \). If \( M_2 \) divides \( M_1 \) then \( y_n = x_n \pmod{M_2} \) satisfies \( y_n = a y_{n-1} + b \pmod{M_2} \).

Theorem 7.5.1 implies that the \( k \) least-significant bits of any power-of-two modulus LCG with \( \text{Per}(x_n) = 2^m = M \) has \( \text{Per}(y_n) = 2^k \), \( 0 \leq k \leq m \). Since a long period is
crucial in PRNGs, when these types of LCGs are employed in a manner that makes use of only a few least-significant-bits, their quality may be compromised. When $M$ is prime, no such problem arises.

Since LCGs are in such common usage, here is a list of parameter values mentioned in the literature. The Park–Miller LCG is widely considered a minimally acceptable PRNG.

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$M$</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>0</td>
<td>$2^{31} - 1$</td>
<td>Park–Miller</td>
</tr>
<tr>
<td>131</td>
<td>0</td>
<td>$2^{35}$</td>
<td>Neave</td>
</tr>
<tr>
<td>16333</td>
<td>25887</td>
<td>$2^{15}$</td>
<td>Oakenfull</td>
</tr>
<tr>
<td>3432</td>
<td>6789</td>
<td>9973</td>
<td>Oakenfull</td>
</tr>
<tr>
<td>171</td>
<td>0</td>
<td>30269</td>
<td>Wichman–Hill</td>
</tr>
</tbody>
</table>

**Shift register generators**

Another popular method of generating pseudorandom numbers is using binary shift register sequences to produce pseudorandom bits. A binary shift register sequence (SRS) is defined by a binary recursion of the type,

$$x_n = x_{n-j_1} \oplus x_{n-j_2} \oplus \cdots \oplus x_{n-j_k}, \quad j_1 < j_2 < \cdots < j_k = \ell,$$  \hspace{1cm} (7.5.2)

where $\oplus$ is the exclusive “or” operation. Note that $x \oplus y \equiv x + y \pmod{2}$. Thus the new bit, $x_n$, is produced by adding $k$ previously computed bits together modulo 2. The implementation of this recurrence requires keeping the last $\ell$ bits from the sequence in a shift register, hence the name. The longest possible period is equal to the number of nonzero $\ell$-dimensional binary vectors, namely $2^\ell - 1$.

A sufficient condition for achieving $\text{Per}(x_n) = 2^\ell - 1$ is that the characteristic polynomial, corresponding to Equation (7.5.2), be primitive modulo 2. Since primitive trinomials of nearly all degrees of interest have been found, SRSs are usually implemented using two-term recursions of the form,

$$x_n = x_{n-k} \oplus x_{n-\ell}, \quad 0 < k < \ell.$$ \hspace{1cm} (7.5.3)

In these two-term recursions, $k$ is the lag and $\ell$ is the register length. Proper choice of the pair $(\ell, k)$ leads to SRSs with $\text{Per}(x_n) = 2^\ell - 1$. Here is a list with suitable $(\ell, k)$ pairs:

<table>
<thead>
<tr>
<th>Primitive trinomial exponents</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5,2)</td>
</tr>
<tr>
<td>(7,1)</td>
</tr>
<tr>
<td>(7,3)</td>
</tr>
<tr>
<td>(17,3)</td>
</tr>
<tr>
<td>(17,5)</td>
</tr>
<tr>
<td>(17,6)</td>
</tr>
<tr>
<td>(31,3)</td>
</tr>
<tr>
<td>(31,6)</td>
</tr>
<tr>
<td>(31,7)</td>
</tr>
<tr>
<td>(31,13)</td>
</tr>
<tr>
<td>(127,1)</td>
</tr>
<tr>
<td>(521,32)</td>
</tr>
</tbody>
</table>

**Lagged-Fibonacci generators**

Another way of producing pseudorandom numbers uses lagged-Fibonacci generators. The term “lagged-Fibonacci” refers to two-term recurrences of the form,

$$x_n = x_{n-k} \phi x_{n-\ell}, \quad 0 < k < \ell,$$ \hspace{1cm} (7.5.4)
where $\diamond$ refers to three common methods of combination: (1) addition modulo $2^m$, (2) multiplication modulo $2^m$, or (3) bitwise exclusive ‘OR’ing of $m$-long bit vectors. Combination method (3) can be thought of as a special implementation of a two-term SRS.

Using combination method (1) leads to additive lagged-Fibonacci sequences (ALFS). If $x_n$ satisfies

$$x_n = x_{n-k} + x_{n-\ell} \pmod{2^m}, \quad 0 < k < \ell,$$

then the maximal period is $\text{Per}(x_n) = (2^\ell - 1)2^{m-1}$.

ALFS are especially suitable for producing floating point deviates using the real-valued recursion $y_n = y_{n-k} + y_{n-\ell} \pmod{1}$. This circumvents the need to convert from integers to floating point values and allows floating point hardware to be used. One caution with ALFS is that Theorem 7.5.1 holds, and so the low-order bits have periods that are shorter than the maximal period. However, this is not nearly the problem as in the LCG case. With ALFSs, the $j$ least-significant bits will have period $(2^\ell - 1)2^{m-1}$, so, if $\ell$ is large, there really is no problem. Note that one can use the table of primitive trinomial exponents to find $(\ell, k)$ pairs that give maximal period ALF sequences.

### 7.5.2 Generating Nonuniform Random Variables

Suppose we want deviates from a distribution with probability density function $f(x)$ and distribution function $F(x) = \int_{-\infty}^{x} f(u) \, du$. In the following $y$ is $U(0, 1)$ means $y$ is uniformly distributed on $[0, 1)$.

Two general techniques for converting uniform random variables into those from other distributions are as follows:

1. The inverse transform method:
   
   If $y$ is $U(0, 1)$, then the random variable $F^{-1}(y)$ will have its density equal to $f(x)$ (since $0 \leq F(x) \leq 1$).

2. The acceptance-rejection method:
   
   Suppose the density can be written as $f(x) = Ch(x)g(x)$ where $h(x)$ is the density of a computable random variable, the function $0 < g(x) \leq 1$, and $C^{-1} = \int_{-\infty}^{\infty} h(u)g(u) \, du$ is a normalization constant. If $x$ is $U(0, 1)$, $y$ has density $h(x)$, and if $x < g(y)$, then $x$ has density $f(x)$. Thus one generates $\{x, y\}$ pairs, rejecting both if $x \geq g(y)$ and returning $x$ if $x < g(y)$.

Examples of the inverse transform method:

1. (Exponential distribution) The exponential distribution with rate $\lambda$ is $f(x) = \lambda e^{-\lambda x}$ (for $x \geq 0$) and $F(x) = 1 - e^{-\lambda x}$. Thus $u = F(x)$ can be solved to give $x = F^{-1}(u) = -\lambda^{-1} \ln(1 - u)$. If $u$ is $U(0, 1)$, then so is $1 - u$. Hence $x = -\lambda^{-1} \ln u$ is exponentially distributed with rate $\lambda$.

2. (Normal distribution) Let $z_1$ be normally distributed with $f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$.

   If each of the pair $(z_1, z_2)$ is normally distributed, then the polar transformation gives random variables $r = \sqrt{z_1^2 + z_2^2}$ (exponentially distributed with
\( \lambda = 2 \) and \( \theta = \tan^{-1}(z_2/z_1) \) (uniformly distributed on \([0, 2\pi)\)). Inverting, \( z_1 = \sqrt{-2\ln x_1 \cos 2\pi x_2} \) and \( z_2 = \sqrt{-2\ln x_1 \sin 2\pi x_2} \) are normally distributed when \( x_1 \) and \( x_2 \) are \( U[0, 1) \).

Examples of the rejection method:

1. (Exponential distribution with \( \lambda = 1 \))
   (a) Generate random numbers \( \{U_i\}_{i=1}^N \) uniform on \([0, 1]\), stopping at \( N = \min\{n \mid U_1 \geq U_2 \geq U_{n-1} < U_n\} \).
   (b) If \( N \) is even, accept that run, and go to step (c). If \( N \) is odd reject the run, and return to step (a).
   (c) Set \( X \) equal to the number of failed runs plus the first random number in the successful run.

2. (Normal distribution)
   (a) Select two random variables \((V_1, V_2)\) from \( U[0, 1) \). Form \( R = V_1^2 + V_2^2 \).
   (b) If \( R > 1 \), then reject the \((V_1, V_2)\) pair, and select another pair.
   (c) If \( R < 1 \), then \( x = V_1\sqrt{-2\ln R} \) has a \( N(0, 1) \) distribution.

3. (Normal distribution)
   (a) Select two exponentially distributed random variables with rate 1: \((V_1, V_2)\).
   (b) If \( V_2 \geq (V_1 - 1)^2/2 \), then reject the \((V_1, V_2)\) pair, and select another pair.
   (c) Otherwise, \( V_1 \) has a \( N(0, 1) \) distribution.

4. (Cauchy distribution) To generate values of \( X \) from \( f(x) = \frac{1}{\pi(1+x^2)} \) on \(-\infty < x < \infty \),
   (a) Generate random numbers \( U_1, U_2 \) (uniform on \([0, 1]\)), and set \( Y_1 = U_1 - \frac{1}{2} \), \( Y_2 = U_2 - \frac{1}{2} \).
   (b) If \( Y_1^2 + Y_2^2 \leq \frac{1}{4} \), then return \( X = Y_1/Y_2 \). Otherwise return to step (a).

To generate values of \( X \) from a Cauchy distribution with parameters \( \beta \) and \( \theta \),
\[ f(x) = \frac{\beta}{\pi[\beta^2 + (x - \theta)^2]^{3/2}} \] for \(-\infty < x < \infty \), construct \( X \) as above, and then use \( \beta X + \theta \).

**Discrete random variables**

In general, the density function of a discrete random variable can be represented as a vector \( p = (p_0, p_1, \ldots, p_{n-1}, p_n) \) by defining the probabilities \( P(x = j) = p_j \) (for \( j = 0, \ldots, n \)). The distribution function can be defined by the vector \( c = (c_0, c_1, \ldots, c_{n-1}, 1) \), where \( c_j = \sum_{i=0}^j p_i \). Given this representation of \( F(x) \), we can apply the inverse transform by computing \( x \) to be \( U[0, 1) \), and then finding the index \( j \) so that \( c_j \leq x < c_{j+1} \). In this case event \( j \) will have occurred. Examples:
1. (Binomial distribution) The binomial distribution with \( n \) trials of mean \( p \) has
\[
p_j = \binom{n}{j} p^j (1 - p)^{n-j}, \quad \text{for } j = 0, \ldots, n.
\]
- As an example, consider the result of flipping a fair coin. In 2 flips, the probability of obtaining (0, 1, 2) heads is \( \mathbf{p} = (\frac{1}{4}, \frac{1}{2}, \frac{3}{4}) \). Hence \( \mathbf{e} = (\frac{1}{4}, \frac{3}{4}, 1) \). If \( x \) (chosen from \( U[0, 1] \)) turns out to be say, 0.4, then “1 head” is returned (since \( \frac{1}{4} < 0.4 < \frac{3}{4} \)).
- Note that, when \( n \) is large, it is costly to compute the density and distribution vectors. When \( n \) is large and relatively few binomially distributed pseudorandom numbers are desired, an alternative is to use the normal approximation to the binomial.
- Alternately, one can form the sum \( \sum_{i=1}^n [u_i + p] \), where each \( u_i \) is \( U[0, 1] \).

2. (Geometric distribution) To simulate a value from \( P(X = i) = p(1 - p)^{i-1} \) for \( i \geq 1 \), use
\[
X = 1 + \left\lceil \log U / \log(1 - p) \right\rceil.
\]

3. (Poisson distribution) The Poisson distribution with mean \( \lambda \) has
\[
p_j = \frac{\lambda^j}{j!} e^{-\lambda} \quad \text{for } j \geq 0.
\]
The Poisson distribution counts the number of events in a unit time interval if the times are exponentially distributed with rate \( \lambda \). Thus if the times \( t_i \) are exponentially distributed with rate \( \lambda \), then \( j \) will be Poisson distributed with mean \( \lambda \) when \( \sum_{i=0}^{j} t_i \leq 1 \leq \sum_{i=0}^{j+1} t_i \). Since \( t_i = -\lambda^{-1} \ln u_i \), where \( u_i \) is \( U[0, 1] \), the previous equation may be written as \( \prod_{i=0}^{j} u_i \geq e^{-\lambda} \geq \prod_{i=0}^{j+1} u_i \). This allows us to compute Poisson random variables by iteratively computing \( P_j = \prod_{i=0}^{j} u_i \) until \( P_j < e^{-\lambda} \). The first such \( j \) that makes this inequality true will have the desired distribution.

Random variables can be simulated using the following table (each \( U \) and \( U_i \) is uniform on the interval \([0, 1]\)):

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Density</th>
<th>Formula for deviate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Binomial</td>
<td>( p_j = \binom{n}{j} p^j (1 - p)^{n-j} )</td>
<td>( \sum_{i=1}^n [U_i + p] )</td>
</tr>
<tr>
<td>Cauchy</td>
<td>( f(x) = \frac{\sigma}{\pi(x^2 + \sigma^2)} )</td>
<td>( \sigma \tan(\pi U) )</td>
</tr>
<tr>
<td>Exponential</td>
<td>( f(x) = \lambda e^{-\lambda x} )</td>
<td>( -\lambda^{-1} \ln U )</td>
</tr>
<tr>
<td>Pareto</td>
<td>( f(x) = ab^x / x^{a+1} )</td>
<td>( b / U^{1/a} )</td>
</tr>
<tr>
<td>Rayleigh</td>
<td>( f(x) = x / \sigma e^{-x^2/2\sigma^2} )</td>
<td>( \sqrt{-\ln U} )</td>
</tr>
</tbody>
</table>
Testing pseudorandom numbers

The prudent way to check a complicated computation that makes use of pseudorandom numbers is to run it several times with different types of pseudorandom number generators and see if the results appear consistent across the generators. The fact that this is not always possible or practical has led researchers to develop statistical tests of randomness that should be passed by general purpose pseudorandom number generators. Some common tests are the spectral test, the equidistribution test, the serial test, the runs test, the coupon collector test, and the birthday spacing test.

7.6 CONTROL CHARTS

Control charts are graphical tools used to assess and maintain the stability of a process. They are used to separate random variation from specific causes. Data measurements are plotted versus time along with upper and lower control limits and a center line. If the process is in control and the underlying distribution is normal, then the control limits represent three standard deviations from the center line (mean).

If all of the data points are contained within the control limits, the process is considered stable and the mean and standard deviations can be reliably calculated. The variations between data points occur from random causes. Data outside the control limits or forming abnormal patterns point to unstable, out-of-control processes.

In the tables, $k$ denotes the number of samples taken, $i$ is an index for the samples ($i = 1 \ldots k$), $n$ is the sample size (number of elements in each sample), and $R$ is the range of the values in a sample (maximum element value minus minimum element value). The mean is $\mu$ and the standard deviation is $\sigma$. Control chart upper and lower control limits are denoted UCL and LCL.
## Types of Control Charts, Their Statistics, and Uses

<table>
<thead>
<tr>
<th>Chart</th>
<th>Statistics</th>
<th>Statistical quantity</th>
<th>Applications</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{x} - R$</td>
<td>Gaussian</td>
<td>Average value and range</td>
<td>Charts continuous measurable quantities. Measurements taken on small sample sets.</td>
</tr>
<tr>
<td>$\tilde{x} - R$</td>
<td>Gaussian</td>
<td>Median value and range</td>
<td>Similar to $\bar{x} - R$ chart but fewer calculations needed for plotting.</td>
</tr>
<tr>
<td>$x - Rs$</td>
<td>Gaussian</td>
<td>Individual measured values</td>
<td>Similar to $\bar{x} - R$ chart but single measurements are made. Used when measurements are expensive or dispersion of measured values is small. $Rs =</td>
</tr>
<tr>
<td>$pn$</td>
<td>Binomial</td>
<td>Number of defective units</td>
<td>Charts number of defective units in sets of fixed size.</td>
</tr>
<tr>
<td>$p$</td>
<td>Binomial</td>
<td>Percent defective</td>
<td>Charts number of defective units in sets of varying size.</td>
</tr>
<tr>
<td>$c$</td>
<td>Poisson</td>
<td>Number of defects</td>
<td>Charts number of flaws in a product of fixed size.</td>
</tr>
<tr>
<td>$u$</td>
<td>Poisson</td>
<td>Defect density (defects per quantity unit)</td>
<td>Charts the defect density on a product of varying size.</td>
</tr>
</tbody>
</table>

## Types of Control Charts and Limits ("P" stands for parameter)

<table>
<thead>
<tr>
<th>Chart</th>
<th>$(\mu, \sigma)$ known?</th>
<th>$P$</th>
<th>Centerline</th>
<th>UCL</th>
<th>LCL</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{x} - R$</td>
<td>No</td>
<td>$\bar{x}$</td>
<td>$\bar{x} = \frac{\sum x}{k}$</td>
<td>$\bar{x} + A_2 \bar{R}$</td>
<td>$\bar{x} - A_2 \bar{R}$</td>
</tr>
<tr>
<td>$\bar{x} - R$</td>
<td>No</td>
<td>$R$</td>
<td>$\bar{R} = \frac{\sum R}{k}$</td>
<td>$D_4 \bar{R}$</td>
<td>$D_3 \bar{R}$</td>
</tr>
<tr>
<td>$\tilde{x} - R$</td>
<td>Yes</td>
<td>$\tilde{x}$</td>
<td>$\tilde{x} = \mu$</td>
<td>$\mu + \frac{3\sigma}{\sqrt{n}}$</td>
<td>$\mu - \frac{3\sigma}{\sqrt{n}}$</td>
</tr>
<tr>
<td>$\bar{x} - R$</td>
<td>Yes</td>
<td>$R$</td>
<td>$\bar{R} = d_2 \sigma$</td>
<td>$D_2 \bar{R}$</td>
<td>$D_1 \sigma$</td>
</tr>
<tr>
<td>$\bar{x} - R$</td>
<td>No</td>
<td>$\bar{x}$</td>
<td>$\bar{x} = \frac{\sum x}{k}$</td>
<td>$\bar{x} + m_3 A_2$</td>
<td>$\bar{x} - m_3 A_2$</td>
</tr>
<tr>
<td>$\bar{x} - Rs$</td>
<td>No</td>
<td>Rs</td>
<td>$\bar{Rs} = \frac{\sum Rs}{k}$</td>
<td>3.27 $\bar{Rs}$</td>
<td>–</td>
</tr>
<tr>
<td>$\bar{x} - R$</td>
<td>No</td>
<td>$pn$</td>
<td>$\bar{pn} = \frac{\sum pn}{k}$</td>
<td>$\bar{pn} + \sqrt{\bar{pn}(1 - \bar{p})}$</td>
<td>$\bar{pn} + \sqrt{\bar{pn}(1 - \bar{p})}$</td>
</tr>
<tr>
<td>$\bar{x} - R$</td>
<td>No</td>
<td>$p$</td>
<td>$\bar{p} = \frac{\sum pn}{\sum n}$</td>
<td>$\bar{p}n + 3 \sqrt{\frac{\bar{p}(1 - \bar{p})}{n}}$</td>
<td>$\bar{p}n - 3 \sqrt{\frac{\bar{p}(1 - \bar{p})}{n}}$</td>
</tr>
<tr>
<td>$\bar{x} - R$</td>
<td>No</td>
<td>$c$</td>
<td>$\bar{c} = \frac{\sum c}{k}$</td>
<td>$\bar{c} + 3 \sqrt{\bar{c}}$</td>
<td>$\bar{c} - 3 \sqrt{\bar{c}}$</td>
</tr>
<tr>
<td>$\bar{x} - R$</td>
<td>No</td>
<td>$u$</td>
<td>$\bar{u} = \frac{\sum u}{\sum n}$</td>
<td>$\bar{u} + 3 \sqrt{\bar{u}/n}$</td>
<td>$\bar{u} - 3 \sqrt{\bar{u}/n}$</td>
</tr>
</tbody>
</table>

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### Abnormal Distributions of Points in Control Charts

<table>
<thead>
<tr>
<th>Abnormality</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Sequence</strong></td>
<td>Seven or more consecutive points on one side of the center line. Denotes the average value has shifted.</td>
</tr>
<tr>
<td><strong>Bias</strong></td>
<td>Fewer than seven consecutive points on one side of the center line, but most of the points are on that side.</td>
</tr>
<tr>
<td></td>
<td>• 10 of 11 consecutive points</td>
</tr>
<tr>
<td></td>
<td>• 12 or more of 14 consecutive points</td>
</tr>
<tr>
<td></td>
<td>• 14 or more of 17 consecutive points</td>
</tr>
<tr>
<td></td>
<td>• 16 or more of 20 consecutive points</td>
</tr>
<tr>
<td><strong>Trend</strong></td>
<td>Seven or more consecutive rising or falling points.</td>
</tr>
<tr>
<td><strong>Approaching the limit</strong></td>
<td>Two out of three or three or more out of seven consecutive points are more than two-thirds the distance between the center line and a control limit.</td>
</tr>
<tr>
<td><strong>Periodicity</strong></td>
<td>The data points vary in a regular periodic pattern.</td>
</tr>
</tbody>
</table>
7.7 STATISTICS

7.7.1 DESCRIPTIVE STATISTICS

1. Sample distribution and density functions
   - Sample distribution function:
     \[
     \hat{F}(x) = \frac{1}{n} \sum_{i=1}^{n} u(x - x_i)
     \]  
     where \( u(x) \) be the unit step function defined by \( u(x) = 0 \) for \( x \leq 0 \) and \( u(x) = 1 \) for \( x > 0 \).
   - Sample density function or histogram:
     \[
     \hat{f}(x) = \frac{\hat{F}(x_0 + (i + 1)w) - \hat{F}(x_0 + iw)}{w}
     \] 
     for \( x \in [x_0 + iw, x_0 + (i + 1)w) \). The interval \( [x_0 + iw, x_0 + (i + 1)w) \) is called the \( i^{th} \) bin, \( w \) is the bin width, and \( f_i = \hat{F}(x_0 + (i + 1)w) - \hat{F}(x_0 + iw) \) is the bin frequency.

2. Order statistics and quantiles
   - Order statistics are obtained by arranging the sample values \( \{x_1, x_2, \ldots, x_n\} \) in increasing order, denoted by
     \[
     x(1) \leq x(2) \leq \cdots \leq x(n).
     \] 
     (a) \( x(1) \) and \( x(n) \) are the minimum and maximum data values, respectively.
     (b) For \( i = 1, \ldots, n \), \( x(i) \) is called the \( i^{th} \) order statistic.
   - Quantiles: If \( 0 < p < 1 \), then the quantile of order \( p \), \( \xi_p \), is given by the \( p(n + 1) \) order statistic. It may be necessary to interpolate between successive values.
     (a) If \( p = j/4 \) for \( j = 1, 2, \) or \( 3 \), then \( \xi_{4j} \) is called the \( j \)th quartile.
     (b) If \( p = j/10 \) for \( j = 1, 2, \ldots, 9 \), then \( \xi_{10j} \) is called the \( j \)th decile.
     (c) If \( p = j/100 \) for \( j = 1, 2, \ldots, 99 \), then \( \xi_{100j} \) is called the \( j \)th percentile.

3. Measures of central tendency
   - Arithmetic mean:
     \[
     \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i = \frac{x_1 + x_2 + \cdots + x_n}{n}.
     \]
• \( \alpha \)-trimmed mean:
\[
\bar{x}_\alpha = \frac{1}{n(1-2\alpha)} \left( (1-r) \left( x_{(k+1)} + x_{(n-k)} \right) + \sum_{i=k+2}^{n-k-1} x_{(i)} \right),
\]
where \( k = \lfloor \alpha n \rfloor \) is the greatest integer less than or equal to \( \alpha n \), and \( r = \alpha n - k \). If \( \alpha = 0 \) then \( \bar{x}_\alpha = \bar{x} \).

• Weighted mean: If to each \( x_i \) is associated a weight \( w_i \geq 0 \) so that
\[
\sum_{i=1}^{n} w_i = 1, \quad \text{then} \quad \bar{x}_w = \sum_{i=1}^{n} w_i x_i.
\]

• Geometric mean:
\[
\text{G.M.} = \left( \prod_{i=1}^{n} x_i \right)^{\frac{1}{n}} = (x_1 x_2 \cdots x_n)^{\frac{1}{n}}.
\]

• Harmonic mean:
\[
\text{H.M.} = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}} = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}.
\]

• Relationship between arithmetic, geometric, and harmonic means:
\[
\text{H.M.} \leq \text{G.M.} \leq \bar{x}
\]
with equality holding only when all sample values are equal.

• The mode is the data value that occurs with the greatest frequency. Note that the mode may not be unique.

• Median:
(a) If \( n \) is odd and \( n = 2k + 1 \), then \( M = x_{(k+1)} \).
(b) If \( n \) is even and \( n = 2k \), then \( M = (x_{(k)} + x_{(k+1)})/2 \).

• Midrange:
\[
\text{mid} = \frac{x_{(1)} + x_{(n)}}{2}.
\]

4. Measures of dispersion

• Mean deviation or absolute deviation:
\[
\text{M.D.} = \frac{1}{n} \sum_{i=1}^{n} |x_i - M|, \quad \text{or} \quad \text{M.D.} = \frac{1}{n} \sum_{i=1}^{n} |x_i - \bar{x}|.
\]
• Sample standard deviation:

\[ s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2} = \sqrt{\frac{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2}{n-1}}. \] (7.7.11)

• The sample variance is the square of the sample standard deviation.

• Root mean square: \( \text{R.M.S.} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} x_i^2} \).

• Sample range: \( x_{(n)} - x_{(1)} \).

• Interquartile range: \( \xi_{\frac{3}{4}} - \xi_{\frac{1}{4}} \).

• The quartile deviation or semi-interquartile range is one half the interquartile range.

5. Higher-order statistics

• Sample moments: \( m_k = \frac{1}{n} \sum_{i=1}^{n} x_i^k \).

• Sample central moments, or sample moments about the mean:

\[ \mu_k = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^k. \] (7.7.12)

### 7.7.2 STATISTICAL ESTIMATORS

**Definitions**

1. A function of a set of random variables is a statistic. It is a function of observable random variables that does not contain any unknown parameters. A statistic is itself an observable random variable.

2. Let \( \theta \) be a parameter appearing in the density function for the random variable \( X \). Suppose that we know a formula for computing an approximate value \( \hat{\theta} \) of \( \theta \) from a given sample \( \{x_1, \ldots, x_n\} \) (call such a function \( g \)). Then \( \hat{\theta} = g(x_1, x_2, \ldots, x_n) \) can be considered as a single observation of the random variable \( \hat{\Theta} = g(X_1, X_2, \ldots, X_n) \). The random variable \( \hat{\Theta} \) is an estimator for the parameter \( \theta \).

3. A hypothesis is an assumption about the distribution of a random variable \( X \). This may usually be cast into the form \( \theta \in \Theta_0 \). We use \( H_0 \) to denote the null hypothesis and \( H_1 \) to denote an alternative hypothesis.

4. In significance testing, a test statistic \( T = T(X_1, \ldots, X_n) \) is used to reject \( H_0 \), or to not reject \( H_0 \). Generally, if \( T \in C \), where \( C \) is a critical region, then \( H_0 \) is rejected.

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5. A **Type I error**, denoted $\alpha$, is to reject $H_0$ when it should not be rejected. A **Type II error**, denoted $\beta$, is to not reject $H_0$ when it should be rejected.

6. The power of a test is $\eta = 1 - \beta$.

<table>
<thead>
<tr>
<th>Unknown truth</th>
<th>$H_0$</th>
<th>$H_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Do not reject $H_0$</td>
<td>True decision.</td>
<td>Type II error.</td>
</tr>
<tr>
<td></td>
<td>Probability is $1 - \alpha$</td>
<td>Probability is $\beta$</td>
</tr>
<tr>
<td>Reject $H_0$</td>
<td>Type I error.</td>
<td>True decision.</td>
</tr>
<tr>
<td></td>
<td>Probability is $\alpha$</td>
<td>Probability is $\eta = 1 - \beta$</td>
</tr>
</tbody>
</table>

**Consistent estimators**

Let $\hat{\Theta} = g(X_1, X_2, \ldots, X_n)$ be an estimator for the parameter $\theta$, and suppose that $g$ is defined for arbitrarily large values of $n$. If the estimator has the property, $E \left[ (\hat{\Theta} - \theta)^2 \right] \rightarrow 0$, as $n \rightarrow \infty$, then the estimator is called a **consistent estimator**.

1. A consistent estimator is not unique.
2. A consistent estimator may be meaningless.
3. A consistent estimator is not necessarily unbiased.

**Efficient estimators**

An unbiased estimator $\hat{\Theta} = g(X_1, X_2, \ldots, X_n)$ for a parameter $\theta$ is said to be **efficient** if it has finite variance ($E \left[ (\hat{\Theta} - \theta)^2 \right] < \infty$) and if there does not exist another estimator $\hat{\Theta}^* = g^*(X_1, X_2, \ldots, X_n)$ for $\theta$, whose variance is smaller than that of $\hat{\Theta}$. The **efficiency** of an unbiased estimator is the ratio,

$$\frac{\text{Cramér–Rao lower bound}}{\text{Actual variance}}.$$ 

The relative efficiency of two unbiased estimators is the ratio of their variances.

**Maximum likelihood estimators (MLE)**

Suppose $X$ is a random variable whose density function is $f(x; \theta)$, where $\theta = (\theta_1, \ldots, \theta_r)$. If the independent sample values $x_1, \ldots, x_n$ are obtained, then define the likelihood function as $L = \prod_{i=1}^{n} f(x_i; \theta)$. The MLE estimate for $\theta$ is the solution of the simultaneous equations, $\frac{\partial}{\partial \theta_i} \log L = 0$, for $i = 1, \ldots, r$.

1. A MLE need not be consistent.
2. A MLE may not be unbiased.
3. A MLE need not be unique.
4. If a single sufficient statistic $T$ exists for the parameter $\theta$, the MLE of $\theta$ must be a function of $T$. 

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5. Let $\hat{\Theta}$ be a MLE of $\theta$. If $\tau(\cdot)$ is a function with a single-valued inverse, then a MLE of $\tau(\theta)$ is $\tau(\hat{\Theta})$.

Define $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{x})^2$ (note that $S \neq s$). Then:

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Estimated parameter</th>
<th>MLE estimate of parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponential $E(\lambda)$</td>
<td>$1/\lambda$</td>
<td>$1/\bar{x}$</td>
</tr>
<tr>
<td>Exponential $E(\lambda)$</td>
<td>$\lambda^2 = \sigma^2$</td>
<td>$\bar{x}^2$</td>
</tr>
<tr>
<td>Normal $N(\mu, \sigma)$</td>
<td>$\mu$</td>
<td>$\bar{x}$</td>
</tr>
<tr>
<td>Normal $N(\mu, \sigma)$</td>
<td>$\sigma^2$</td>
<td>$S^2$</td>
</tr>
<tr>
<td>Poisson $P(\lambda)$</td>
<td>$\lambda$</td>
<td>$\bar{x}$</td>
</tr>
<tr>
<td>Uniform $U(0, \theta)$</td>
<td>$\theta$</td>
<td>$X_{\text{max}}$</td>
</tr>
</tbody>
</table>

**Method of moments (MOM)**

Let $\{X_i\}$ be independent and identically distributed random variables with density $f(x; \theta)$. Let $\mu_r(\theta) = E[X^r]$ be the $r^{\text{th}}$ population moment (if it exists). Let $m_r = \frac{1}{n} \sum_{i=1}^{n} x_i^r$ be the $r^{\text{th}}$ sample moment. Form the $k$ equations, $\mu_r = m_r$, and solve to obtain an estimate of $\theta$.

1. MOM estimators are not necessarily uniquely defined.
2. MOM estimators may not be functions of sufficient or complete statistics.

**Sufficient statistics**

A statistic $G = g(X_1, \ldots, X_n)$ is defined as a *sufficient statistic* if, and only if, the conditional distribution of $H$, given $G$, does not depend on $\theta$ for any statistic $H = h(X_1, \ldots, X_n)$.

Let $\{X_i\}$ be independent and identically distributed random variables, with density $f(x; \theta)$. The statistics $\{G_1, \ldots, G_r\}$ are defined as *jointly sufficient statistics* if, and only if, the conditional distribution of $X_1, X_2, \ldots, X_n$ given $G_1 = g_1, G_1 = g_2, \ldots, G_r = g_r$ does not depend on $\theta$.

1. A single sufficient statistic may not exist.

**Unbiased estimators**

An estimator $g(X_1, X_2, \ldots, X_n)$ for a parameter $\theta$ is said to be *unbiased* if

\[ E[g(X_1, X_2, \ldots, X_n)] = \theta. \]
1. An unbiased estimator may not exist.
2. An unbiased estimator is not unique.
3. An unbiased estimator may be meaningless.
4. An unbiased estimator is not necessarily consistent.

**UMVU estimators**

A *uniformly minimum variance unbiased* estimator, called a UMVU estimator, is unbiased and has the minimum variance among all unbiased estimators.

Define, as usual, \( \bar{x} = \frac{\sum_{i=1}^{n} X_i}{n} \) and \( s^2 = \frac{\sum_{i=1}^{n} (X_i - \bar{x})^2}{n - 1} \). Then:

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Estimated parameter</th>
<th>UMVU estimate of parameter</th>
<th>Variance of estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponential ( E(\lambda) )</td>
<td>( \lambda )</td>
<td>( \frac{n - 1}{s} )</td>
<td>( \frac{\lambda^2}{n - 2} )</td>
</tr>
<tr>
<td>Exponential ( E(\lambda) )</td>
<td>( \frac{1}{\lambda} )</td>
<td>( \bar{x} )</td>
<td>( \frac{1}{n\lambda^2} )</td>
</tr>
<tr>
<td>Normal ( N(\mu, \sigma) )</td>
<td>( \mu )</td>
<td>( \bar{x} )</td>
<td>( \frac{\sigma^2}{n} )</td>
</tr>
<tr>
<td>Normal ( N(\mu, \sigma) )</td>
<td>( \sigma^2 )</td>
<td>( s^2 )</td>
<td>( \frac{2\sigma^4}{n - 1} )</td>
</tr>
<tr>
<td>Poisson ( P(\lambda) )</td>
<td>( \lambda )</td>
<td>( \bar{x} )</td>
<td>( \frac{\lambda^2}{n} )</td>
</tr>
<tr>
<td>Uniform ( U(0, \theta) )</td>
<td>( \theta )</td>
<td>( \frac{n + 1}{n} \bar{x}_{\text{max}} )</td>
<td>( \frac{\theta^2}{n(n + 2)} )</td>
</tr>
</tbody>
</table>

### 7.7.3 Cramer–Rao Bound

The Cramer–Rao bound gives a lower bound on the variance of an unknown unbiased statistical parameter, when \( n \) samples are taken. When the single unknown parameter is \( \theta \),

\[
\sigma^2(\theta) \geq \frac{1}{-nE\left[ \frac{\partial^2}{\partial \theta^2} \log f(x; \theta) \right]} = \frac{1}{nE\left[ \left( \frac{\partial}{\partial \theta} \log f(x; \theta) \right)^2 \right]}. 
\]

(7.7.13)

**Examples**

1. For a normal random variable with unknown mean \( \theta \) and known variance \( \sigma^2 \), the density is \( f(x; \theta) = \frac{1}{\sqrt{2\pi\sigma}} \exp \left( -\frac{(x-\theta)^2}{2\sigma^2} \right) \). Hence, \( \frac{\partial}{\partial \theta} \log f(x; \theta) = (x-\theta)/\sigma^2 \).
   
   The computation
   
   \[
   E\left[ \frac{(x-\theta)^2}{\sigma^4} \right] = \int_{-\infty}^{\infty} (x-\theta)^2 \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\theta)^2}{2\sigma^2}} \, dx = \frac{1}{\sigma^2}
   \]
   
   results in \( \sigma^2(\theta) \geq \frac{\sigma^2}{\sigma} \).

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For a normal random variable with known mean $\mu$ and unknown variance $\theta = \sigma^2$, the density is $f(x; \theta) = \frac{1}{\sqrt{2\pi \theta}} \exp \left( -\frac{(x-\mu)^2}{2\theta} \right)$. Hence, $\frac{\partial}{\partial \theta} \log f(x; \theta) = \frac{(x-\mu)^2 - 2\theta}{2\theta^2}$. The computation $E \left[ \frac{(x-\mu)^2 - 2\theta}{2\theta^2} \right] = \frac{1}{2\theta^2} = \frac{1}{2\sigma^4}$ results in $\sigma^2(\theta) \geq \frac{2\sigma^4}{n}$.

3. For a Poisson random variable with unknown mean $\theta$, the density is $f(x; \theta) = \frac{\theta^x e^{-\theta}}{x!}$. Hence, $\frac{\partial}{\partial \theta} \log f(x; \theta) = x/\theta - 1$. The computation $E \left[ (\frac{x}{\theta} - 1)^2 \right] = \frac{1}{\theta} \sum_{x=0}^{\infty} \left( \frac{x}{\theta} - 1 \right)^2 e^{-\theta} x! = \frac{1}{\theta}$ results in $\sigma^2(\theta) \geq \theta/n$.

### 7.7.4 ORDER STATISTICS

When $\{X_i\}$ are $n$ independent and identically distributed random variables with the common distribution function $F_X(x)$, let $Z_m$ be the $m$th largest of the values ($m = 0, 1, \ldots, n$). Hence $Z_1$ is the maximum of the $n$ values and $Z_n$ is the minimum of the $n$ values. Then $F_{Z_m}(x) = \sum_{i=1}^{\min(n,m)} \binom{n}{i} \left[ F_X(x) \right]^i \left[ 1 - F_X(x) \right]^{n-i}$. Hence

\[
F_{Z_m}(z) = [F_X(z)]^m, \quad f_{Z_m}(z) = n [F_X(z)]^{m-1} f_X(z),
\]

\[
F_{Z_n}(z) = 1 - [1 - F_X(z)]^n, \quad f_{Z_n}(z) = n [1 - F_X(z)]^{n-1} f_X(z).
\]

The expected value of the $i$th order statistic is given by

\[
E[x(i)] = \frac{n!}{(i-1)!(n-i)!} \int_{-\infty}^{\infty} x f(x) F^{i-1}(x) [1 - F(x)]^{n-i} \, dx.
\]  

(7.7.14)

#### Uniform distribution:

If $X$ is uniformly distributed on the interval $[0, 1]$ then $E[x(i)] = \frac{n!}{(i-1)!(n-i)!} \int_0^1 x^i (1-x)^{n-i} \, dx$. The expected value of the largest of $n$ samples is $\frac{n}{n+1}$; the expected value of the least of $n$ samples is $\frac{1}{n+1}$.

#### Normal distribution:

The following table gives values of $E[x(i)]$ for a standard normal distribution. Missing values (indicated by a dash) may be obtained from $E[x(i)] = -E[x(n-i+1)]$.

For example, if an average person takes five intelligence tests (each test having a normal distribution with a mean of 100 and a standard deviation of 20), then the expected value of the largest score is $100 + (1.1630)(20) \approx 123$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$n=2$</th>
<th>$3$</th>
<th>$4$</th>
<th>$5$</th>
<th>$6$</th>
<th>$7$</th>
<th>$8$</th>
<th>$10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5642</td>
<td>0.8463</td>
<td>1.0294</td>
<td>1.1630</td>
<td>1.2672</td>
<td>1.3522</td>
<td>1.4236</td>
<td>1.5388</td>
</tr>
<tr>
<td>2</td>
<td>—</td>
<td>0.0000</td>
<td>0.2970</td>
<td>0.4950</td>
<td>0.6418</td>
<td>0.7574</td>
<td>0.8522</td>
<td>1.0014</td>
</tr>
<tr>
<td>3</td>
<td>—</td>
<td>—</td>
<td>0.0000</td>
<td>0.2016</td>
<td>0.3527</td>
<td>0.4728</td>
<td>0.6561</td>
<td>—</td>
</tr>
<tr>
<td>4</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>0.0000</td>
<td>0.1522</td>
<td>0.3756</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>5</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>0.1226</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>6</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>7</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

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7.7.5 CLASSIC STATISTICS PROBLEMS

Sample size problem

Suppose that a Bernoulli random variable is to be estimated from a population. What sample size \( n \) is required so that, with 99% certainty, the error is no more than \( e = 5 \) percentage points (i.e., \( \text{Prob}(|\hat{p} - p| < 0.05) > 0.99) \)?

If an \emph{a priori} estimate of \( p \) is available, then the minimum sample size is \( n_p = \frac{z_{\alpha/2}^2 p(1 - p)}{e^2} \). If no \emph{a priori} estimate is available, then \( n = \frac{z_{\alpha/2}^2}{4e^2} \geq n_p \). For the numbers above, \( n \geq n_p = 664 \).

Large scale testing with infrequent success

Suppose that a disease occurs in one person out of every 1000. Suppose that a test for this disease has a type I and a type II error of 1% (that is, \( \alpha = \beta = 0.01 \)). Imagine that 100,000 people are tested. Of the 100 people who have the disease, 99 will be diagnosed as having it. Of the 99,900 people who do not have the disease, 999 will be diagnosed as having it. Hence, only \( \frac{99}{999} \approx 9\% \) of the people who test positive for the disease actually have it.

7.8 CONFIDENCE INTERVALS

A probability distribution may have one or more unknown parameters. A confidence interval is an assertion that an unknown parameter lies in a computed range, with a specified probability. Before constructing a confidence interval, first select a confidence coefficient, denoted \( 1 - \alpha \). Typically, \( 1 - \alpha = 0.95, 0.99 \), or the like. For the definitions of \( z_{\alpha}, t_{\alpha}, \) and \( \chi_{\alpha}^2 \) see Section 7.12.1.

7.8.1 CONFIDENCE INTERVAL: SAMPLE FROM ONE POPULATION

The following confidence intervals assume a random sample of size \( n \), given by \( \{x_1, x_2, \ldots, x_n\} \).

1. Find mean \( \mu \) of the normal distribution with known variance \( \sigma^2 \).

   - Determine the critical value \( z_{\alpha/2} \) such that \( \Phi(z_{\alpha/2}) = 1 - \alpha/2 \), where \( \Phi(z) \) is the standard normal distribution function.
   - Compute the mean \( \bar{x} \) of the sample.
   - Compute \( k = z_{\alpha/2} \sigma / \sqrt{n} \).
   - The 100(1-\( \alpha \)) percent confidence interval for \( \mu \) is given by \( [\bar{x} - k, \bar{x} + k] \).
2. Find mean $\mu$ of the normal distribution with unknown variance $\sigma^2$.

- Determine the critical value $t_{\alpha/2}$ such that $F \left( t_{\alpha/2} \right) = 1 - \alpha/2$, where $F \left( t \right)$ is the $t$-distribution with $n - 1$ degrees of freedom.
- Compute the mean $\bar{x}$ and standard deviation $s$ of the sample.
- Compute $k = t_{\alpha/2}s/\sqrt{n}$.
- The 100$(1 - \alpha)$ percent confidence interval for $\mu$ is given by $[\bar{x} - k, \bar{x} + k]$.

3. Find the probability of success $p$ for Bernoulli trials with large sample size.

- Determine the critical value $z_{\alpha/2}$ such that $\Phi \left( z_{\alpha/2} \right) = 1 - \alpha/2$, where $\Phi \left( z \right)$ is the standard normal distribution function.
- Compute the proportion $\hat{p}$ of “successes” out of $n$ trials.
- Compute $k = z_{\alpha/2}\sqrt{\frac{\hat{p}(1 - \hat{p})}{n}}$.
- The 100$(1 - \alpha)$ percent confidence interval for $\hat{p}$ is given by $[\hat{p} - k, \hat{p} + k]$.

4. Find variance $\sigma^2$ of the normal distribution.

- Determine the critical values $\chi^2_{\alpha/2}$ and $\chi^2_{1 - \alpha/2}$ such that $F \left( \chi^2_{\alpha/2} \right) = 1 - \alpha/2$ and $F \left( \chi^2_{1 - \alpha/2} \right) = \alpha/2$, where $F \left( z \right)$ is the chi-square distribution function with $n - 1$ degrees of freedom.
- Compute the standard deviation $s$.
- Compute $k_1 = \frac{(n - 1)s^2}{\chi^2_{\alpha/2}}$ and $k_2 = \frac{(n - 1)s^2}{\chi^2_{1 - \alpha/2}}$.
- The 100$(1 - \alpha)$ percent confidence interval for $\sigma^2$ is given by $[k_1, k_2]$.
- The 100$(1 - \alpha)$ percent confidence interval for the standard deviation $\sigma$ is given by $[\sqrt{k_1}, \sqrt{k_2}]$.

5. Find quantile $\xi_p$ of order $p$ for large sample sizes.

- Determine the critical value $z_{\alpha/2}$ such that $\Phi \left( z_{\alpha/2} \right) = 1 - \alpha/2$, where $\Phi \left( z \right)$ is the standard normal distribution function.
- Compute the order statistics $x_{(1)}$, $x_{(2)}$, $\ldots$, $x_{(n)}$.
- Compute $k_1 = \left[ np - z_{\alpha/2}\sqrt{np(1 - p)} \right]$ and $k_2 = \left[ np + z_{\alpha/2}\sqrt{np(1 - p)} \right]$.
- The 100$(1 - \alpha)$ percent confidence interval for $\xi_p$ is given by $[x_{(k_1)}, x_{(k_2)}]$.

6. Find median $M$ based on the Wilcoxon one-sample statistic for a large sample.

- Determine the critical value $z_{\alpha/2}$ such that $\Phi \left( z_{\alpha/2} \right) = 1 - \alpha/2$, where $\Phi \left( z \right)$ is the standard normal distribution function.
• Compute the order statistics \( w(1), w(2), \ldots, w(N) \) of the \( N = \frac{n(n - 1)}{2} \) averages \( \frac{x_i + x_j}{2} \) for \( 1 \leq i < j \leq n \).

• Compute \( k_1 = \left\lfloor \frac{N}{2} - \frac{z_{\alpha/2}N}{\sqrt{3n}} \right\rfloor \) and \( k_2 = \left\lceil \frac{N}{2} + \frac{z_{\alpha/2}N}{\sqrt{3n}} \right\rceil \).

• The 100(1 - \( \alpha \)) percent confidence interval for \( M \) is given by \( [w(k_1), w(k_2)] \).

### 7.8.2 CONFIDENCE INTERVAL: SAMPLES FROM TWO POPULATIONS

The following confidence intervals assume random samples from two large populations: one sample of size \( n \), given by \( \{x_1, x_2, \ldots, x_n\} \), and one sample of size \( m \), given by \( \{y_1, y_2, \ldots, y_m\} \).

1. Find the difference in population means \( \mu_x \) and \( \mu_y \) from independent samples with known variances \( \sigma_x^2 \) and \( \sigma_y^2 \).

   • Determine the critical value \( z_{\alpha/2} \) such that \( \Phi \left( z_{\alpha/2} \right) = 1 - \alpha/2 \), where \( \Phi (z) \) is the standard normal distribution function.

   • Compute the means \( \bar{x} \) and \( \bar{y} \).

   • Compute \( k = z_{\alpha/2} \sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}} \).

   • The 100(1 - \( \alpha \)) percent confidence interval for \( \mu_x - \mu_y \) is given by \( [(\bar{x} - \bar{y}) - k, (\bar{x} - \bar{y}) + k] \).

2. Find the difference in population means \( \mu_x \) and \( \mu_y \) from independent samples with unknown variances \( \sigma_x^2 \) and \( \sigma_y^2 \).

   • Determine the critical value \( z_{\alpha/2} \) such that \( \Phi \left( z_{\alpha/2} \right) = 1 - \alpha/2 \), where \( \Phi (z) \) is the standard normal distribution function.

   • Compute the means \( \bar{x} \) and \( \bar{y} \), and the standard deviations \( s_x \) and \( s_y \).

   • Compute \( k = z_{\alpha/2} \sqrt{\frac{s_x^2}{n} + \frac{s_y^2}{m}} \).

   • The 100(1 - \( \alpha \)) percent confidence interval for \( \mu_x - \mu_y \) is given by \( [(\bar{x} - \bar{y}) - k, (\bar{x} - \bar{y}) + k] \).

3. Find the difference in population means \( \mu_x \) and \( \mu_y \) from independent samples with unknown but equal variances \( \sigma_x^2 = \sigma_y^2 \).

   • Determine the critical value \( t_{\alpha/2} \) such that \( F \left( t_{\alpha/2} \right) = 1 - \alpha/2 \), where \( F (t) \) is the \( t \)-distribution with \( n + m - 2 \) degrees of freedom.
Compute the means $\bar{x}$ and $\bar{y}$, the standard deviations $s_x$ and $s_y$, and the pooled standard deviation estimate,

$$s = \sqrt{\frac{(n-1)s_x^2 + (m-1)s_y^2}{n+m-2}}. \quad (7.8.1)$$

- Compute $k = t_{a/2} s \sqrt{\frac{1}{n} + \frac{1}{m}}$.
- The 100$(1 - \alpha)$ percent confidence interval for $\mu_x - \mu_y$ is given by $[(\bar{x} - \bar{y}) - k, (\bar{x} - \bar{y}) + k]$.

4. Find the difference in population means $\mu_x$ and $\mu_y$ for paired samples with unknown but equal variances $\sigma^2_x = \sigma^2_y$.
- Determine the critical value $t_{a/2}$ such that $F(t_{a/2}) = 1 - \alpha/2$, where $F(t)$ is the $t$-distribution with $n - 1$ degrees of freedom.
- Compute the mean $\bar{\mu}_d$ and standard deviation $s_d$ of the paired differences $x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n$.
- Compute $k = t_{a/2} s_d / \sqrt{n}$.
- The 100$(1 - \alpha)$ percent confidence interval for $\mu_d = \mu_x - \mu_y$ is given by $[\bar{\mu}_d - k, \bar{\mu}_d + k]$.

5. Find the difference in Bernoulli trial success rates, $p_x - p_y$, for large, independent samples.
- Determine the critical value $z_{a/2}$ such that $\Phi(z_{a/2}) = 1 - \alpha/2$, where $\Phi(z)$ is the standard normal distribution function.
- Compute the proportions $\hat{p}_x$ and $\hat{p}_y$ of “successes” for the samples.
- Compute $k = z_{a/2} \sqrt{\frac{\hat{p}_x (1 - \hat{p}_x)}{n} + \frac{\hat{p}_y (1 - \hat{p}_y)}{m}}$.
- The 100$(1 - \alpha)$ percent confidence interval for $p_x - p_y$ is given by $[\hat{p}_x - \hat{p}_y - k, (\hat{p}_x - \hat{p}_y) + k]$.

6. Find the difference in medians $M_x - M_y$ based on the Mann–Whitney–Wilcoxon procedure.
- Determine the critical value $z_{a/2}$ such that $\Phi(z_{a/2}) = 1 - \alpha/2$, where $\Phi(z)$ is the standard normal distribution function.
- Compute the order statistics $w_{(1)}, w_{(2)}, \ldots, w_{(N)}$ of the $N = nm$ differences $x_i - y_j$, for $1 \leq i \leq n$ and $1 \leq j \leq m$.
- Compute

$$k_1 = \frac{nm}{2} + \left[0.5 - z_{a/2} \sqrt{\frac{nm(n + m + 1)}{12}}\right]$$
and
\[ k_2 = \left\lceil \frac{nm}{2} - 0.5 + \frac{z_{a/2}}{\sqrt{nm(n + m + 1)}} \right\rceil. \]

- The 100(1 − α) percent confidence interval for \( \mu_x - \mu_y \) is given by \([w(k_1), w(k_2)]\).

7. Find the ratio of variances \( \sigma_x^2 / \sigma_y^2 \), for independent samples.

- Determine the critical values \( F_{a/2} \) and \( F_{1-\alpha/2} \) such that \( F(F_{1-\alpha/2}) = \alpha/2 \) and \( F(F_{a/2}) = 1 - \alpha/2 \), where \( F \) is the \( F \)-distribution with \( m - 1 \) and \( n - 1 \) degrees of freedom.
- Compute the standard deviations \( s_x \) and \( s_y \) of the samples.
- Compute \( k_1 = F_{1-\alpha/2} \) and \( k_2 = F_{a/2} \)
- The 100(1 − α) percent confidence interval for \( \sigma_x^2 / \sigma_y^2 \) is given by \([s_x^2/k_1, s_y^2/k_2]\).

### 7.9 TESTS OF HYPOTHESES

A statistical hypothesis is an assumption about the distribution of a random variable. A statistical test of a hypothesis is a procedure in which a sample is used to determine whether we should “reject” or “not reject” the hypothesis. Before employing a hypothesis test, first select a significance level \( \alpha \). Typically, \( \alpha = 0.05, 0.01 \), or the like.

#### 7.9.1 HYPOTHESIS TESTS: PARAMETER FROM ONE POPULATION

The following hypothesis tests assume a random sample of size \( n \), given by \( \{x_1, x_2, \ldots, x_n\} \).

1. Test of the hypothesis \( \mu = \mu_0 \) against the alternative \( \mu \neq \mu_0 \) of the mean of a normal distribution with known variance \( \sigma^2 \):
   - Determine the critical value \( z_{a/2} \) such that \( \Phi(z_{a/2}) = 1 - \alpha/2 \), where \( \Phi(z) \) is the standard normal distribution function.
   - Compute the mean \( \bar{x} \) of the sample.
   - Compute the test statistic \( z = \frac{(\bar{x} - \mu_0)}{\sigma} \sqrt{n} \).
If \(|z| > z_{\alpha/2}\), then reject the hypothesis. If \(|z| \leq z_{\alpha/2}\), then do not reject the hypothesis.

2. Test of the hypothesis \(\mu = \mu_0\) against the alternative \(\mu > \mu_0\) (or \(\mu < \mu_0\)) of the mean of a normal distribution with known variance \(\sigma^2\):
   - Determine the critical value \(z_\alpha\) such that \(\Phi (z_\alpha) = 1 - \alpha\), where \(\Phi (z)\) is the standard normal distribution function.
   - Compute the mean \(\bar{x}\) of the sample.
   - Compute the test statistic \(z = \frac{\bar{x} - \mu_0}{\sigma \sqrt{n}}\). (For the alternative \(\mu < \mu_0\), multiply \(z\) by \(-1\).)
   - If \(z > z_\alpha\), then reject the hypothesis. If \(z \leq z_\alpha\), then do not reject the hypothesis.

3. Test of the hypothesis \(\mu = \mu_0\) against the alternative \(\mu \neq \mu_0\) of the mean of a normal distribution with unknown variance \(\sigma^2\):
   - Determine the critical value \(t_{\alpha/2}\) such that \(F (t_{\alpha/2}) = 1 - \alpha/2\), where \(F (t)\) is the \(t\)-distribution with \(n - 1\) degrees of freedom.
   - Compute the mean \(\bar{x}\) and standard deviation \(s\) of the sample.
   - Compute the test statistic \(t = \frac{\bar{x} - \mu_0}{s \sqrt{n}}\).
   - If \(|t| > t_{\alpha/2}\), then reject the hypothesis. If \(|t| \leq t_{\alpha/2}\), then do not reject the hypothesis.

4. Test of the hypothesis \(\mu = \mu_0\) against the alternative \(\mu > \mu_0\) (or \(\mu < \mu_0\)) of the mean of a normal distribution with unknown variance \(\sigma^2\):
   - Determine the critical value \(t_a\) such that \(F (t_a) = 1 - \alpha\), where \(F (t)\) is the \(t\)-distribution with \(n - 1\) degrees of freedom.
   - Compute the mean \(\bar{x}\) and standard deviation \(s\) of the sample.
   - Compute the test statistic \(t = \frac{\bar{x} - \mu_0}{s \sqrt{n}}\). (For the alternative \(\mu < \mu_0\), multiply \(t\) by \(-1\).)
   - If \(t > t_a\), then reject the hypothesis. If \(t \leq t_a\), then do not reject the hypothesis.

5. Test of the hypothesis \(p = p_0\) against the alternative \(p \neq p_0\) of the probability of success for a binomial distribution, large sample:
   - Determine the critical value \(z_{\alpha/2}\) such that \(\Phi (z_{\alpha/2}) = 1 - \alpha/2\), where \(\Phi (z)\) is the standard normal distribution function.
   - Compute the proportion \(\hat{p}\) of "successes" for the sample.
   - Compute the test statistic \(z = \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0) / n}}\).
• If \( |z| > z_{\alpha/2} \), then reject the hypothesis. If \( |z| \leq z_{\alpha/2} \), then do not reject the hypothesis.

6. Test of the hypothesis \( p = p_0 \) against the alternative \( p > p_0 \) (or \( p < p_0 \)) of the probability of success for a binomial distribution, large sample:

- Determine the critical value \( z_\alpha \) such that \( \Phi(z_\alpha) = 1 - \alpha \), where \( \Phi(z) \) is the standard normal distribution function.
- Compute the proportion \( \hat{p} \) of “successes” for the sample.
- Compute the test statistic \( z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \). (For the alternative \( p < p_0 \), multiply \( z \) by \(-1\).)
- If \( z > z_\alpha \), then reject the hypothesis. If \( z \leq z_\alpha \), then do not reject the hypothesis.

7. Wilcoxon signed rank test of the hypothesis \( M = M_0 \) against the alternative \( M \neq M_0 \) of the median of a population, large sample:

- Determine the critical value \( z_{\alpha/2} \) such that \( \Phi(z_{\alpha/2}) = 1 - \alpha/2 \), where \( \Phi(z) \) is the standard normal distribution.
- Compute the quantities \( |x_i - M_0| \), and keep track of the sign of \( x_i - M_0 \). If \( |x_i - M_0| = 0 \), then remove it from the list and reduce \( n \) by one.
- Order the \( |x_i - M_0| \) from smallest to largest, assigning rank 1 to the smallest and rank \( n \) to the largest; \( |x_i - M_0| \) has rank \( r_i \) if it is the \( r_i \)th entry in the ordered list. If \( |x_i - M_0| = |x_j - M_0| \), then assign each the average of their ranks.
- Compute the sum of the signed ranks \( R = \sum_{i=1}^{n} \text{sign}(x_i - M_0) r_i \).
- Compute the test statistic \( z = \frac{R}{\sqrt{\frac{n(n+1)(2n+1)}{6}}} \).
- If \( |z| > z_{\alpha/2} \), then reject the hypothesis. If \( |z| \leq z_{\alpha/2} \), then do not reject the hypothesis.

8. Wilcoxon signed rank test of the hypothesis \( M = M_0 \) against the alternative \( M > M_0 \) (or \( M < M_0 \)) of the median of a population, large sample:

- Determine the critical value \( z_\alpha \) such that \( \Phi(z_\alpha) = 1 - \alpha \), where \( \Phi(z) \) is the standard normal distribution.
- Compute the quantities \( |x_i - M_0| \), and keep track of the sign of \( x_i - M_0 \). If \( |x_i - M_0| = 0 \), then remove it from the list and reduce \( n \) by one.
- Order the \( |x_i - M_0| \) from smallest to largest, assigning rank 1 to the smallest and rank \( n \) to the largest; \( |x_i - M_0| \) has rank \( r_i \) if it is the \( r_i \)th entry in the ordered list. If \( |x_i - M_0| = |x_j - M_0| \), then assign each the average of their ranks.
• Compute the sum of the signed ranks \( R = \sum_{i=1}^{n} \text{sign} (x_i - M_0) r_i \).

• Compute the test statistic \( z = \frac{R}{\sqrt{\frac{n(n+1)(2n+1)}{6}}} \). (For the alternative \( M < M_0 \), multiply the test statistic by \(-1\).)

• If \( z > z_{\alpha} \), then reject the hypothesis. If \( z \leq z_{\alpha} \), then do not reject the hypothesis.

9. Test of the hypothesis \( \sigma^2 = \sigma_0^2 \) against the alternative \( \sigma^2 \neq \sigma_0^2 \) of the variance of a normal distribution:

• Determine the critical values \( \chi^2_{\alpha/2} \) and \( \chi^2_{1-\alpha/2} \) such that \( F(\chi^2_{\alpha/2}) = 1 - \alpha/2 \) and \( F(\chi^2_{1-\alpha/2}) = \alpha/2 \), where \( F(x) \) is the chi-square distribution function with \( n - 1 \) degrees of freedom.

• Compute the standard deviation \( s \) of the sample.

• Compute the test statistic \( \chi^2 = \frac{(n-1)s^2}{\sigma_0^2} \).

• If \( \chi^2 < \chi^2_{1-\alpha/2} \) or \( \chi^2 > \chi^2_{\alpha/2} \), then reject the hypothesis.

• If \( \chi^2_{1-\alpha/2} \leq \chi^2 \leq \chi^2_{\alpha/2} \), then do not reject the hypothesis.

10. Test of the hypothesis \( \sigma^2 = \sigma_0^2 \) against the alternative \( \sigma^2 > \sigma_0^2 \) (or \( \sigma^2 < \sigma_0^2 \)) of the variance of a normal distribution:

• Determine the critical value \( \chi^2_\alpha \) (\( \chi^2_{\alpha} \) for the alternative \( \sigma^2 < \sigma_0^2 \)) such that \( F(\chi^2_\alpha) = 1 - \alpha \) \( F(\chi^2_{1-\alpha}) = \alpha \), where \( F(x) \) is the chi-square distribution function with \( n - 1 \) degrees of freedom.

• Compute the standard deviation \( s \) of the sample.

• Compute the test statistic \( \chi^2 = \frac{(n-1)s^2}{\sigma_0^2} \).

• If \( \chi^2 > \chi^2_\alpha \) \( (\chi^2 < \chi^2_{1-\alpha}) \), then reject the hypothesis.

• If \( \chi^2 \leq \chi^2_\alpha \) \( (\chi^2 \leq \chi^2_{1-\alpha}) \), then do not reject the hypothesis.

7.9.2 HYPOTHESIS TESTS: PARAMETERS FROM TWO POPULATIONS

The following hypothesis tests assume a random sample of size \( n \), given by \( \{x_1, x_2, \ldots, x_n\} \), and a random sample of size \( m \), given by \( \{y_1, y_2, \ldots, y_m\} \).

1. Test of the hypothesis \( \mu_x = \mu_y \) against the alternative \( \mu_x \neq \mu_y \) of the means of independent normal distributions with known variances \( \sigma_x^2 \) and \( \sigma_y^2 \):

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• Determine the critical value $z_{\alpha/2}$ such that $\Phi \left( z_{\alpha/2} \right) = 1 - \alpha/2$, where $\Phi \left( z \right)$ is the standard normal distribution function.

• Compute the means, $\bar{x}$ and $\bar{y}$, of the samples.

• Compute the test statistic $z = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{s_x^2}{n} + \frac{s_y^2}{m}}}$.

• If $|z| > z_{\alpha/2}$, then reject the hypothesis. If $|z| \leq z_{\alpha/2}$, then do not reject the hypothesis.

2. Test of the hypothesis $\mu_x = \mu_y$ against the alternative $\mu_x > \mu_y$ (or $\mu_x < \mu_y$) of the means of independent normal distributions with known variances $\sigma_x^2$ and $\sigma_y^2$:

• Determine the critical value $z_{\alpha}$ such that $\Phi \left( z_{\alpha} \right) = 1 - \alpha$, where $\Phi \left( z \right)$ is the standard normal distribution function.

• Compute the means $\bar{x}$ and $\bar{y}$ of the samples.

• Compute the test statistic $z = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{s_x^2}{n} + \frac{s_y^2}{m}}}$. (For the alternative $\mu_x < \mu_y$, multiply $z$ by $-1$.)

• If $z > z_{\alpha}$, then reject the hypothesis. If $z \leq z_{\alpha}$, then do not reject the hypothesis.

3. Test of the hypothesis $\mu_x = \mu_y$ against the alternative $\mu_x \neq \mu_y$ of the means of independent normal distributions with unknown variances $\sigma_x^2$ and $\sigma_y^2$, large sample:

• Determine the critical value $z_{\alpha/2}$ such that $\Phi \left( z_{\alpha/2} \right) = 1 - \alpha/2$, where $\Phi \left( z \right)$ is the standard normal distribution.

• Compute the means, $\bar{x}$ and $\bar{y}$, and standard deviations, $s_x^2$ and $s_y^2$, of the samples.

• Compute the test statistic $z = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{s_x^2}{n} + \frac{s_y^2}{m}}}$.

• If $|z| > z_{\alpha/2}$, then reject the hypothesis. If $|z| \leq z_{\alpha/2}$, then do not reject the hypothesis.

4. Test of the hypothesis $\mu_x = \mu_y$ against the alternative $\mu_x > \mu_y$ (or $\mu_x < \mu_y$) of the means of independent normal distributions with unknown variances $\sigma_x^2$ and $\sigma_y^2$, large sample:

• Determine the critical value $z_{\alpha}$ such that $\Phi \left( z_{\alpha} \right) = 1 - \alpha$, where $\Phi \left( z \right)$ is the standard normal distribution function.

• Compute the means, $\bar{x}$ and $\bar{y}$, and standard deviations, $s_x^2$ and $s_y^2$, of the samples.
• Compute the test statistic $z = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{s_x^2}{n} + \frac{s_y^2}{m}}}$. (For the alternative $\mu_x < \mu_y$, multiply $z$ by $-1$.)

• If $z > z_{a/2}$, then reject the hypothesis. If $z \leq z_{a/2}$, then do not reject the hypothesis.

5. Test of the hypothesis $\mu_x = \mu_y$ against the alternative $\mu_x \neq \mu_y$ of the means of independent normal distributions with unknown variances $\sigma_x^2 = \sigma_y^2$:

• Determine the critical value $t_{a/2}$ such that $F(t_{a/2}) = 1 - \alpha/2$, where $F(t)$ is the $t$-distribution with $n + m - 2$ degrees of freedom.

• Compute the means, $\bar{x}$ and $\bar{y}$, and standard deviations, $s_x^2$ and $s_y^2$, of the samples.

• Compute the test statistic $t = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{(n-1)s_x^2 + (m-1)s_y^2}{n+m-2} \left( \frac{1}{n} + \frac{1}{m} \right)}}$.

• If $|t| > t_{a/2}$, then reject the hypothesis. If $|t| \leq t_{a/2}$, then do not reject the hypothesis.

6. Test of the hypothesis $\mu_x = \mu_y$ against the alternative $\mu_x > \mu_y$ (or $\mu_x < \mu_y$) of the means of independent normal distributions with unknown variances $\sigma_x^2 = \sigma_y^2$:

• Determine the critical value $t_a$ such that $F(t_a) = 1 - \alpha$, where $F(t)$ is the $t$-distribution with $n + m - 2$ degrees of freedom.

• Compute the means, $\bar{x}$ and $\bar{y}$, and standard deviations, $s_x^2$ and $s_y^2$, of the samples.

• Compute the test statistic $t = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{(n-1)s_x^2 + (m-1)s_y^2}{n+m-2} \left( \frac{1}{n} + \frac{1}{m} \right)}}$. (For the alternative $\mu_x < \mu_y$, multiply $t$ by $-1$.)

• If $t > t_a$, then reject the hypothesis. If $t \leq t_a$, then do not reject the hypothesis.

7. Test of the hypothesis $\mu_x = \mu_y$ against the alternative $\mu_x \neq \mu_y$ of the means of paired normal samples:

• Determine the critical value $t_{a/2}$ so that $F(t_{a/2}) = 1 - \alpha/2$, where $F(t)$ is the $t$-distribution with $n - 1$ degrees of freedom.

• Compute the mean, $\hat{\mu}_d$, and standard deviation, $s_d$, of the differences $x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n$.

• Compute the test statistic $t = \frac{\hat{\mu}_d \sqrt{n}}{s_d}$.

• If $|t| > t_{a/2}$, then reject the hypothesis. If $|t| \leq t_{a/2}$, then do not reject the hypothesis.
8. Test of the hypothesis $\mu_x = \mu_y$ against the alternative $\mu_x > \mu_y$ (or $\mu_x < \mu_y$) of the means of paired normal samples:
   - Determine the critical value $t_\alpha$ so that $F(t_\alpha) = 1 - \alpha$, where $F(t)$ is the $t$-distribution with $n + m - 2$ degrees of freedom.
   - Compute the mean, $\hat{\mu}_d$, and standard deviation, $s_d$, of the differences $x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n$.
   - Compute the test statistic $t = \frac{\hat{\mu}_d \sqrt{n}}{s_d}$. (For the alternative $\mu_x < \mu_y$, multiply $t$ by $-1$.)
   - If $t > t_\alpha$, then reject the hypothesis. If $t \leq t_\alpha$, then do not reject the hypothesis.

9. Test of the hypothesis $p_x = p_y$ against the alternative $p_x \neq p_y$ of the probability of success for a binomial distribution, large sample:
   - Determine the critical value $z_{\alpha/2}$ such that $\Phi(z_{\alpha/2}) = 1 - \alpha/2$, where $\Phi(z)$ is the standard normal distribution function.
   - Compute the proportions, $\hat{p}_x$ and $\hat{p}_y$, of “successes” for the samples.
   - Compute the test statistic $z = \frac{\hat{p}_x - \hat{p}_y}{\sqrt{\frac{\hat{p}_x(1-\hat{p}_x)}{n} + \frac{\hat{p}_y(1-\hat{p}_y)}{m}}}$.
   - If $|z| > z_{\alpha/2}$, then reject the hypothesis. If $|z| \leq z_{\alpha/2}$, then do not reject the hypothesis.

10. Test of the hypothesis $p_x = p_y$ against the alternative $p_x > p_y$ (or $p_x < p_y$) of the probability of success for a binomial distribution, large sample:
    - Determine the critical value $z_\alpha$ such that $\Phi(z_\alpha) = 1 - \alpha$, where $\Phi(z)$ is the standard normal distribution function.
    - Compute the proportions, $\hat{p}_x$ and $\hat{p}_y$, of “successes” for the samples.
    - Compute the test statistic $z = \frac{\hat{p}_x - \hat{p}_y}{\sqrt{\frac{\hat{p}_x(1-\hat{p}_x)}{n} + \frac{\hat{p}_y(1-\hat{p}_y)}{m}}}$.
    - If $z > z_\alpha$, then reject the hypothesis. If $z \leq z_\alpha$, then do not reject the hypothesis.

11. Mann-Whitney-Wilcoxon test of the hypothesis $M_x = M_y$ against the alternative $M_x \neq M_y$ of the medians of independent samples, large sample:
    - Determine the critical value $z_{\alpha/2}$ such that $\Phi(z_{\alpha/2}) = 1 - \alpha/2$, where $\Phi(z)$ is the standard normal distribution.
    - Pool the $N = m + n$ observations, but keep track of which sample the observation was drawn from.

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• Order the pooled observations from smallest to largest, assigning rank 1 to the smallest and rank \( N \) to the largest; an observation has rank \( r_i \) if it is the \( r_i \)th entry in the ordered list. If two observations are equal, then assign each the average of their ranks.

• Compute the sum of the ranks from the first sample \( T_x \).

• Compute the test statistic \( z = \frac{T_x - \frac{m(N+1)}{2}}{\sqrt{\frac{mn(N+1)}{12}}} \).

• If \(|z| > z_{\alpha/2} \), then reject the hypothesis. If \(|z| \leq z_{\alpha/2} \), then do not reject the hypothesis.

12. Mann–Whitney–Wilcoxon test of the hypothesis \( M_x = M_y \) against the alternative \( M_x > M_y \) (or \( M_x < M_y \)) of the medians of independent samples, large sample:

- Determine the critical value \( z_{\alpha} \) such that \( \Phi(z_{\alpha}) = 1 - \alpha \), where \( \Phi(z) \) is the standard normal distribution.

- Pool the \( N = m + n \) observations, but keep track of which sample the observation was drawn from.

- Order the pooled observations from smallest to largest, assigning rank 1 to the smallest and rank \( N \) to the largest; an observation has rank \( r_i \) if it is the \( r_i \)th entry in the ordered list. If two observations are equal, then assign each the average of their ranks.

- Compute the sum of the ranks from the first sample \( T_x \).

- Compute the test statistic \( z = \frac{T_x - \frac{m(N+1)}{2}}{\sqrt{\frac{mn(N+1)}{12}}} \). (For the alternative \( M_x < M_y \), multiply the test statistic by \(-1\).)

- If \(|z| > z_{\alpha} \), then reject the hypothesis. If \(|z| \leq z_{\alpha} \), then do not reject the hypothesis.

13. Wilcoxon signed rank test of the hypothesis \( M_x = M_y \) against the alternative \( M_x \neq M_y \) of the medians of paired samples, large sample:

- Determine the critical value \( z_{\alpha/2} \) such that \( \Phi(z_{\alpha/2}) = 1 - \alpha/2 \), where \( \Phi(z) \) is the standard normal distribution.

- Compute the paired differences \( d_i = x_i - y_i \), for \( i = 1, 2, \ldots, n \).

- Compute the quantities \(|d_i|\) and keep track of the sign of \( d_i \). If \( d_i = 0 \), then remove it from the list and reduce \( n \) by one.

- Order the \(|d_i|\) from smallest to largest, assigning rank 1 to the smallest and rank \( n \) to the largest; \( |d_i| \) has rank \( r_i \) if it is the \( r_i \)th entry in the ordered list. If \(|d_i| = |d_j| \), then assign each the average of their ranks.

- Compute the sum of the signed ranks \( R = \sum_{i=1}^{n} \text{sign}(d_i) r_i \).
• Compute the test statistic $z = \frac{R}{\sqrt{\frac{n(n+1)(2n+1)}{6}}}$.

• If $|z| > z_{\alpha/2}$, then reject the hypothesis. If $|z| \leq z_{\alpha/2}$, then do not reject the hypothesis.

14. Wilcoxon signed rank test of the hypothesis $M_x = M_y$ against the alternative $M_x > M_y$ (or $M_x < M_y$) of the medians of paired samples, large sample:

- Determine the critical value $z_{\alpha}$ such that $\Phi(z_{\alpha}) = 1 - \alpha$, where $\Phi(z)$ is the standard normal distribution.
- Compute the paired differences $d_i = x_i - y_i$, for $i = 1, 2, \ldots, n$.
- Compute the quantities $|d_i|$ and keep track of the sign of $d_i$. If $d_i = 0$, then remove it from the list and reduce $n$ by one.
- Order the $|d_i|$ from smallest to largest, assigning rank 1 to the smallest and rank $n$ to the largest; $|d_i|$ has rank $r_i$ if it is the $r_{th}$ entry in the ordered list. If $|d_i| = |d_j|$, then assign each the average of their ranks.
- Compute the sum of the signed ranks $R = \sum_{i=1}^{n} \text{sign}(d_i) r_i$.
- Compute the test statistic $z = \frac{R}{\sqrt{\frac{n(n+1)(2n+1)}{6}}}$. (For the alternative $M_x < M_y$, multiply the test statistic by $-1$.)
- If $z > z_{\alpha}$, then reject the hypothesis. If $z \leq z_{\alpha}$, then do not reject the hypothesis.

15. Test of the hypothesis $\sigma^2_x = \sigma^2_y$ against the alternative $\sigma^2_x \neq \sigma^2_y$ (or $\sigma^2_x > \sigma^2_y$) of the variances of independent normal samples:

- Determine the critical value $F_{\alpha/2}$ ($F_{\alpha}$ for the alternative $\sigma^2_x > \sigma^2_y$) such that $F(F_{\alpha/2}) = 1 - \alpha/2$ ($F(F_{\alpha}) = 1 - \alpha$), where $F(F)$ is the $F$-distribution function with $n-1$ and $m-1$ degrees of freedom.
- Compute the standard deviations $s_x$ and $s_y$ of the samples.
- Compute the test statistic $F = \frac{s_x^2}{s_y^2}$. (For the two-sided test, put the larger value in the numerator.)
- If $F > F_{\alpha/2}$ ($F > F_{\alpha}$), then reject the hypothesis. If $F \leq F_{\alpha/2}$ ($F \leq F_{\alpha}$), then do not reject the hypothesis.
7.9.3 HYPOTHESIS TESTS: DISTRIBUTION OF A POPULATION

The following hypothesis tests assume a random sample of size $n$, given by $\{x_1, x_2, \ldots, x_n\}$.

1. Run test for randomness of a sample of binary values, large sample:
   - Determine the critical value $z_{\alpha/2}$ such that $\Phi \left( z_{\alpha/2} \right) = 1 - \alpha/2$, where $\Phi (z)$ is the standard normal distribution function.
   - Since the data are binary, denote the possible values of $x_i$ by 0 and 1. Count the total number of zeros, and call this $n_1$; count the total number of ones, and call this $n_2$. Group the data into maximal sub-sequences of zeros and ones, and call each such sub-sequence a run. Let $R$ be the number of runs in the sample.
   - Compute $\mu_R = \frac{2n_1n_2}{n_1 + n_2} + 1$, and $\sigma_R^2 = \frac{\mu_R - 1)(\mu_R - 2)}{n_1 + n_2 - 1}$.
   - Compute the test statistic $z = \frac{R - \mu_R}{\sigma_R}$.
   - If $|z| > z_{\alpha/2}$, then reject the hypothesis. If $|z| \leq z_{\alpha/2}$, then do not reject the hypothesis.

2. Run test for randomness against an alternative that a trend is present in a sample of binary values, large sample:
   - Determine the critical value $z_\alpha$ such that $\Phi (z_\alpha) = 1 - \alpha$, where $\Phi (z)$ is the standard normal distribution function.
   - Since the data are binary, denote the possible values of $x_i$ by 0 and 1. Count the total number of zeros, and call this $n_1$; count the total number of ones, and call this $n_2$. Group the data into maximal sub-sequences of zeros and ones, and call each such sub-sequence a run. Let $R$ be the number of runs in the sample.
   - Compute $\mu_R = \frac{2n_1n_2}{n_1 + n_2} + 1$, and $\sigma_R^2 = \frac{\mu_R - 1)(\mu_R - 2)}{n_1 + n_2 - 1}$.
   - Compute the test statistic $z = \frac{R - \mu_R}{\sigma_R}$.
   - If $z < -z_\alpha$, then reject the hypothesis (this suggests the presence of a trend in the data). If $z \geq -z_\alpha$, then do not reject the hypothesis.

3. Run test for randomness against an alternative that the data are periodic for a sample of binary values, large sample:
   - Determine the critical value $z_\alpha$ such that $\Phi (z_\alpha) = 1 - \alpha$, where $\Phi (z)$ is the standard normal distribution function.
   - Since the data are binary, denote the possible values of $x_i$ by 0 and 1. Count the total number of zeros, and call this $n_1$; count the total number of ones,
and call this $n_2$. Group the data into maximal sub-sequences of zeros and ones, and call each such sub-sequence a run. Let $R$ be the number of runs in the sample.

- Compute $\mu_R = \frac{2n_1n_2}{n_1+n_2} + 1$, and $\sigma^2_R = \frac{(\mu_R-1)(\mu_R-2)}{n_1+n_2-1}$.
- Compute the test statistic $z = \frac{R - \mu_R}{\sigma_R}$.
- If $z > z_{\alpha}$, then reject the hypothesis (this suggests the data are periodic). If $z \leq z_{\alpha}$, then do not reject the hypothesis.

4. Chi-square test that the data are drawn from a specific $k$-parameter multinomial distribution, large sample:

- Determine the critical value $\chi^2_\alpha$ such that $F(\chi^2_\alpha) = 1 - \alpha$, where $F(x)$ is the chi-square distribution with $k-1$ degrees of freedom.
- The $k$-parameter multinomial has $k$ possible outcomes $A_1, A_2, \ldots, A_k$ with probabilities $p_1, p_2, \ldots, p_k$. For $i = 1, 2, \ldots, k$, compute $n_i$, the number of $x_j$s corresponding to $A_i$.
- For $i = 1, 2, \ldots, k$, compute the sample multinomial parameters $\hat{p}_i = \frac{n_i}{n}$.
- Compute the test statistic $\chi^2 = \sum_{i=1}^{k} \frac{(n_i - np_i)^2}{np_i}$.
- If $\chi^2 > \chi^2_\alpha$, then reject the hypothesis. If $\chi^2 \leq \chi^2_\alpha$, then do not reject the hypothesis.

5. Chi-square test for independence of attributes $A$ and $B$ having possible outcomes $A_1, A_2, \ldots, A_k$ and $B_1, B_2, \ldots, B_m$:

- Determine the critical value $\chi^2_\alpha$ such that $F(\chi^2_\alpha) = 1 - \alpha$, where $F(x)$ is the chi-square distribution with $(k-1)(m-1)$ degrees of freedom.
- For $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, m$, define $o_{ij}$ to be the number of observations having attributes $A_i$ and $B_j$, and define $o_i = \sum_{j=1}^{m} o_{ij}$ and $o_j = \sum_{i=1}^{k} o_{ij}$.
- The variables defined above are often collected into a table, called a contingency table:

<table>
<thead>
<tr>
<th>Attribute</th>
<th>$B_1$</th>
<th>$B_2$</th>
<th>$\cdots$</th>
<th>$B_m$</th>
<th>Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>$o_{11}$</td>
<td>$o_{12}$</td>
<td>$\cdots$</td>
<td>$o_{1m}$</td>
<td>$o_1$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$o_{21}$</td>
<td>$o_{22}$</td>
<td>$\cdots$</td>
<td>$o_{2m}$</td>
<td>$o_2$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\cdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$A_k$</td>
<td>$o_{k1}$</td>
<td>$o_{k2}$</td>
<td>$\cdots$</td>
<td>$o_{km}$</td>
<td>$o_k$</td>
</tr>
<tr>
<td>Totals</td>
<td>$o_{1}$</td>
<td>$o_{2}$</td>
<td>$\cdots$</td>
<td>$o_{m}$</td>
<td>$n$</td>
</tr>
</tbody>
</table>
• For $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, m$, compute the sample mean number of observations in the $ij$th cell of the contingency table $e_{ij} = \frac{o_i \cdot o_j}{n}$.

• Compute the test statistic, $\chi^2 = \sum_{i=1}^{k} \sum_{j=1}^{m} \frac{(o_{ij} - e_{ij})^2}{e_{ij}}$.

• If $\chi^2 > \chi^2_\alpha$, then reject the hypothesis (that is, conclude that the attributes are not independent). If $\chi^2 \leq \chi^2_\alpha$, then do not reject the hypothesis.

6. Kolmogorov–Smirnov test that $F_0(x)$ is the distribution of the population from which the sample was drawn:

• Determine the critical value $D_\alpha$ such that $Q(D_\alpha) = 1 - \alpha$, where $Q(D)$ is the distribution function for the Kolmogorov–Smirnov test statistic $D$.

• Compute the sample distribution function $\hat{F}(x)$.

• Compute the test statistic, given the maximum deviation of the sample and target distribution functions $D = \max |\hat{F}(x) - F_0(x)|$.

• If $D > D_\alpha$, then reject the hypothesis (this suggests the data are periodic). If $D \leq D_\alpha$, then do not reject the hypothesis.

7.9.4 HYPOTHESIS TESTS: DISTRIBUTIONS OF TWO POPULATIONS

The following hypothesis tests assume a random sample of size $n$, given by $\{x_1, x_2, \ldots, x_n\}$, and a random sample of size $m$, given by $\{y_1, y_2, \ldots, y_m\}$.

1. Chi-square test that two $k$-parameter multinomial distributions are equal, large sample:

• Determine the critical value $\chi^2_\alpha$ such that $F(\chi^2_\alpha) = 1 - \alpha$, where $F(x)$ is the chi-square distribution with $k - 1$ degrees of freedom.

• The $k$-parameter multinomials have $k$ possible outcomes $A_1, A_2, \ldots, A_k$. For $i = 1, 2, \ldots, k$, compute $n_i$, the number of $x_j$s corresponding to $A_i$, and compute $m_i$, the number of $y_j$s corresponding to $A_i$.

• For $i = 1, 2, \ldots, k$, compute the sample multinomial parameters $\hat{p}_i = \frac{n_i + m_i}{n + m}$.

• Compute the test statistic,

$$
\chi^2 = \sum_{i=1}^{k} \frac{(n_i - n \hat{p}_i)^2}{n \hat{p}_i} + \sum_{i=1}^{k} \frac{(m_i - m \hat{p}_i)^2}{m \hat{p}_i}.
$$

• If $\chi^2 > \chi^2_\alpha$, then reject the hypothesis. If $\chi^2 \leq \chi^2_\alpha$, then do not reject the hypothesis.
2. Mann–Whitney–Wilcoxon test for equality of independent continuous distributions, large sample:
   • Determine the critical value $z_{a/2}$ such that $\Phi \left( \frac{z_{a/2}}{2} \right) = 1 - \alpha$, where $\Phi (z)$ is the normal distribution function.
   • For $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, define $S_{ij} = 1$ if $x_i < y_j$ and $S_{ij} = 0$ if $x_i > y_j$.
   • Compute $U = \sum_{i=1}^{n} \sum_{j=1}^{m} S_{ij}$.
   • Compute the test statistic $z = \frac{U - \frac{mn}{2}}{\sqrt{\frac{mn(m+n+1)}{12}}}$.
   • If $|z| > z_{a/2}$, then reject the hypothesis. If $|z| \leq z_{a/2}$, then do not reject the hypothesis.

3. Spearman rank correlation coefficient for independence of paired samples, large sample:
   • Determine the critical value $R_{a/2}$ such that $F \left( R_{a/2} \right) = 1 - \alpha$, where $F (R)$ is the distribution function for the Spearman rank correlation coefficient.
   • The samples are ordered, with the smallest $x_i$ assigned the rank $r_i$ and the largest assigned the rank $r_n$; for $i = 1, 2, \ldots, n$, $x_i$ is assigned rank $r_i$ if it occupies the $i$th position in the ordered list. Similarly the $y_i$s are assigned ranks $s_i$. In case of a tie within a sample, the ranks are averaged.
   • Compute the test statistic
     $$ R = \frac{n \sum_{i=1}^{n} r_is_i - \left( \sum_{i=1}^{n} r_i \right) \left( \sum_{i=1}^{n} s_i \right)}{\sqrt{\left(n \sum_{i=1}^{n} r_i^2 - \left( \sum_{i=1}^{n} r_i \right)^2 \right) \left(n \sum_{i=1}^{n} s_i^2 - \left( \sum_{i=1}^{n} s_i \right)^2 \right)}}. $$
   • If $|R| > R_{a/2}$, then reject the hypothesis. If $|R| \leq R_{a/2}$, then do not reject the hypothesis.

### 7.9.5 SEQUENTIAL PROBABILITY RATIO TESTS

Given two simple hypotheses and $m$ observations, compute:
   • $P_{0m} = \text{Prob (observations | } H_0 \text{)}$.
   • $P_{1m} = \text{Prob (observations | } H_1 \text{)}$.
   • $v_m = P_{1m} / P_{0m}$.

and then make one of the following decisions:
• If $v_m \geq \frac{1 - \beta}{\alpha}$ then reject $H_0$.
• If $v_m \leq \frac{\beta}{1 - \alpha}$ then reject $H_1$.
• If $\frac{\beta}{1 - \alpha} < v_m < \frac{1 - \beta}{\alpha}$ then make another observation.

Hence, the number of samples taken is not a priori fixed, but determined as sampling occurs. For example:

• Let $\theta$ denote the fraction of defective items. Two simple hypotheses are $H_0$: $\theta = \theta_0 = 0.05$ and $H_1$: $\theta = \theta_1 = 0.15$. Choose $\alpha = 5\%$ and $\beta = 10\%$ (i.e., reject lot with $\theta = \theta_0$ about 5\% of the time; accept lot with $\theta = \theta_1$ about 10\% of the time). If, after $m$ observations, there are $d$ defective items, then

$$P_{im} = \binom{m}{d} \theta_1^d (1 - \theta_1)^{m-d}$$
and

$$v_m = \left(\frac{\theta_1}{\theta_0}\right)^d \left(\frac{1 - \theta_1}{1 - \theta_0}\right)^{m-d}$$

or $v_m = 3^d (0.895)^{m-d}$, using the above numbers. The critical values are $\frac{\beta}{1 - \alpha} = 0.105$ and $\frac{1 - \beta}{\alpha} = 18$. The decision to perform another observation depends on whether or not

$$0.105 \leq 3^d (0.895)^{m-d} \leq 18.$$  

Taking logarithms, a $(m - d, d)$ control chart can be drawn with the following lines: $d = 0.101(m - d) - 2.049$ and $d = 0.101(m - d) + 2.63$. On the figure below, a sample path leading to rejection of $H_0$ has been indicated:

Let $X$ be normally distributed with unknown mean $\mu$ and known standard deviation $\sigma$. Consider the two simple hypotheses, $H_0 : \mu = \mu_0$ and $H_1 : \mu = \mu_1$. If $Y$ is the sum of the first $m$ observations of $X$, then a $(Y, m)$ control chart is constructed with the two lines:

$$Y = \frac{\mu_0 + \mu_1}{2} m + \frac{\sigma^2}{\mu_1 - \mu_0} \log \frac{\beta}{1 - \alpha}$$
$$Y = \frac{\mu_0 + \mu_1}{2} m + \frac{\sigma^2}{\mu_1 - \mu_0} \log \frac{1 - \beta}{\alpha}.$$  

(7.9.2)
7.10 LINEAR REGRESSION

1. The general linear statistical model assumes that the observed data values \( \{y_1, y_2, \ldots, y_m\} \) are of the form
\[
y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_n x_{in} + \epsilon_i,
\]
for \( i = 1, 2, \ldots, m \).

2. For \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n \), the independent variables \( x_{ij} \) are known (nonrandom).

3. \( \{\beta_0, \beta_1, \beta_2, \ldots, \beta_n\} \) are unknown parameters.

4. For each \( i \), \( \epsilon_i \) is a zero-mean normal random variable with unknown variance \( \sigma^2 \).

7.10.1 LINEAR MODEL \( y = \beta_0 + \beta_1 x + \epsilon \)

1. Point estimate of \( \beta_1 \):
\[
\hat{\beta}_1 = \frac{\sum_{i=1}^{m} x_{i1} y_i - \left( \sum_{i=1}^{m} x_{i1} \right) \left( \sum_{i=1}^{m} y_i \right)}{m \left( \sum_{i=1}^{m} x_{i1}^2 - \left( \sum_{i=1}^{m} x_{i1} \right)^2 \right)}.
\]

(7.10.2)

2. Point estimate of \( \beta_0 \):
\[
\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}.
\]

(7.10.3)

3. Point estimate of the correlation coefficient:
\[
r = \hat{r} = \frac{m \sum_{i=1}^{m} x_{i1} y_i - \left( \sum_{i=1}^{m} x_{i1} \right) \left( \sum_{i=1}^{m} y_i \right)}{\sqrt{m \left( \sum_{i=1}^{m} x_{i1}^2 - \left( \sum_{i=1}^{m} x_{i1} \right)^2 \right)} \sqrt{m \left( \sum_{i=1}^{m} y_i^2 - \left( \sum_{i=1}^{m} y_i \right)^2 \right)}}.
\]

(7.10.4)

4. Point estimate of error variance \( \sigma^2 \):
\[
\hat{\sigma}^2 = \frac{\sum_{i=1}^{m} \left( y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i \right)^2}{m - 2}.
\]

(7.10.5)

5. The standard error of the estimate is defined as \( s_e = \sqrt{\hat{\sigma}^2} \).

6. Least-squares regression line: \( \hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x \).

7. Confidence interval for \( \beta_0 \):
   - Determine the critical value \( t_{a/2} \) such that \( F \left( t_{a/2} \right) = 1 - \alpha/2 \), where \( F (t) \) is the \( t \)-distribution with \( m - 2 \) degrees of freedom.
   - Compute the point estimate \( \hat{\beta}_0 \).  

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• Compute \( k = t_{a/2}se \sqrt{ \frac{1}{m} + \frac{\bar{x}^2}{\sum_{i=1}^{m} (x_i - \bar{x})^2} } \).

• The 100(1 - \( \alpha \)) percent confidence interval for \( \beta_0 \) is given by \( [\hat{\beta}_0 - k, \hat{\beta}_0 + k] \).

8. Confidence interval for \( \beta_1 \):
   • Determine the critical value \( t_{a/2} \) such that \( F \left( t_{a/2} \right) = 1 - \alpha/2 \), where \( F \) is the \( t \)-distribution with \( m - 2 \) degrees of freedom.
   • Compute the point estimate \( \hat{\beta}_1 \).
   • Compute \( k = t_{a/2} \frac{se}{\sqrt{\sum_{i=1}^{m} (x_i - \bar{x})^2}} \).
   • The 100(1 - \( \alpha \)) percent confidence interval for \( \beta_1 \) is given by \( [\hat{\beta}_1 - k, \hat{\beta}_1 + k] \).

9. Confidence interval for \( \sigma^2 \):
   • Determine the critical values \( \chi^2_{a/2} \) and \( \chi^2_{1-a/2} \) such that \( F \left( \chi^2_{a/2} \right) = 1 - \alpha/2 \) and \( F \left( \chi^2_{1-a/2} \right) = \alpha/2 \), where \( F \) is the chi-square distribution function with \( m - 2 \) degrees of freedom.
   • Compute the point estimate \( \hat{\sigma}^2 \).
   • Compute \( k_1 = \frac{(n-2)\hat{\sigma}^2}{\chi^2_{a/2}} \) and \( k_2 = \frac{(n-2)\hat{\sigma}^2}{\chi^2_{1-a/2}} \).
   • The 100(1 - \( \alpha \)) percent confidence interval for \( \sigma^2 \) is given by \( [k_1, k_2] \).

10. Confidence interval (predictive interval) for \( \gamma \), given \( x_0 \):
    • Determine the critical value \( t_{a/2} \) such that \( F \left( t_{a/2} \right) = 1 - \alpha/2 \), where \( F \) is the \( t \)-distribution with \( m - 2 \) degrees of freedom.
    • Compute the point estimates \( \hat{\beta}_0, \hat{\beta}_1, \) and \( s_e \).
    • Compute \( k = t_{a/2}se \sqrt{ \frac{1}{m} + \frac{(x_0 - \bar{x})^2}{\sum_{i=1}^{m} (x_i - \bar{x})^2} } \) and \( \hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_0 \).
    • The 100(1 - \( \alpha \)) percent confidence interval for \( \beta_1 \) is given by \( [\hat{y} - k, \hat{y} + k] \).

11. Test of the hypothesis \( \beta_1 = 0 \) against the alternative \( \beta_1 \neq 0 \):
    • Determine the critical value \( t_{a/2} \) such that \( F \left( t_{a/2} \right) = 1 - \alpha/2 \), where \( F \) is the \( t \)-distribution with \( m - 2 \) degrees of freedom.
    • Compute the point estimates \( \hat{\beta}_1 \) and \( s_e \).
    • Compute the test statistic \( t = \frac{\hat{\beta}_1}{se} \sqrt{\sum_{i=1}^{m} (x_i - \bar{x})^2} \).
    • If \( |t| > t_{a/2} \), then reject the hypothesis. If \( |t| \leq t_{a/2} \), then do not reject the hypothesis.
7.10.2 GENERAL MODEL  

\[ y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n + \epsilon \]

1. The \( m \) equations \((i = 1, 2, \ldots, m)\)

\[ y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_n x_{in} + \epsilon_i \quad (7.10.6) \]

can be written in matrix notation as \( y = X\beta + \epsilon \) where

\[
    \begin{bmatrix}
    y_1 \\
    y_2 \\
    \vdots \\
    y_m \\
    \end{bmatrix} =
    \begin{bmatrix}
    \beta_0 \\
    \beta_1 \\
    \vdots \\
    \beta_n \\
    \end{bmatrix} +
    \begin{bmatrix}
    \epsilon_1 \\
    \epsilon_2 \\
    \vdots \\
    \epsilon_m \\
    \end{bmatrix},
\]

(7.10.7)

\[
    X =
    \begin{bmatrix}
    1 & x_{11} & x_{12} & \cdots & x_{1n} \\
    1 & x_{21} & x_{22} & \cdots & x_{2n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    1 & x_{m1} & x_{m2} & \cdots & x_{mn} \\
    \end{bmatrix},
\]

(7.10.8)

2. Throughout the remainder of the section, we assume \( X \) has full column rank.

3. The least-squares estimate \( \hat{\beta} \) satisfies the normal equations \( X^T X \hat{\beta} = X^T y \). That is, \( \hat{\beta} = (X^T X)^{-1} X^T y \).

4. Point estimator of \( \sigma^2 \):

\[ \hat{\sigma}^2 = \frac{1}{m-n-1} (y^T y - \hat{\beta}^T (X^T y)) . \]

(7.10.9)

5. The standard error of the estimate is defined as \( s_e = \sqrt{\hat{\sigma}^2} \).

6. Least-squares regression line:  \( \hat{y} = X^T \hat{\beta} \).

7. In the following, let \( c_{ij} \) denote the \((i, j)^{th}\) entry in the matrix \((X^T X)^{-1}\).

8. Confidence interval for \( \beta_i \):

- Determine the critical value \( t_{\alpha/2} \) such that \( F \left( t_{\alpha/2} \right) = 1 - \alpha/2 \), where \( F \) is the \( t \)-distribution with \( m - n - 1 \) degrees of freedom.
- Compute the point estimate \( \hat{\beta}_i \) by solving the normal equations, and compute \( s_e \).
- Compute \( k_i = t_{\alpha/2} s_e \sqrt{c_{ii}} \).
- The 100(1 - \( \alpha \)) percent confidence interval for \( \beta_i \) is given by \([ \hat{\beta}_i - k_i, \hat{\beta}_i + k_i ]\).

9. Confidence interval for \( \sigma^2 \):

- Determine the critical values \( \chi^2_{\alpha/2} \) and \( \chi^2_{1-\alpha/2} \) such that \( F \left( \chi^2_{\alpha/2} \right) = 1 - \alpha/2 \)
  and \( F \left( \chi^2_{1-\alpha/2} \right) = \alpha/2 \), where \( F \) is the chi-square distribution function with \( m - n - 1 \) degrees of freedom.
• Compute the point estimate $\hat{\sigma}^2$.

• Compute $k_1 = \frac{(m - n - 1)\hat{\sigma}^2}{\chi^2_{\alpha/2}}$ and $k_2 = \frac{(m - n - 1)\hat{\sigma}^2}{\chi^2_{1-\alpha/2}}$.

• The 100$(1 - \alpha)$ percent confidence interval for $\sigma^2$ is given by $[k_1, k_2]$.

10. Confidence interval (predictive interval) for $y$, given $x_0$:

• Determine the critical value $t_{a/2}$ such that $F(t_{a/2}) = 1 - \alpha/2$, where $F(t)$ is the $t$-distribution with $m - 2$ degrees of freedom.

• Compute the point estimate $\hat{\beta}_i$ by solving the normal equations, and compute $s_e$.

• Compute $k = t_{a/2} s_e \sqrt{1 + \mathbf{x}_0^T (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{x}_0}$ and $\hat{y} = \mathbf{x}_0^T \hat{\beta}$.

• The 100$(1 - \alpha)$ percent confidence interval for $\beta_1$ is given by $[\hat{y} - k, \hat{y} + k]$.

11. Test of the hypothesis $\beta_i = 0$ against the alternative $\beta_i \neq 0$:

• Determine the critical value $t_{a/2}$ such that $F(t_{a/2}) = 1 - \alpha/2$, where $F(t)$ is the $t$-distribution with $m - n - 1$ degrees of freedom.

• Compute the point estimates $\hat{\beta}_i$ and $s_e$ by solving the normal equations.

• Compute the test statistic $t = \frac{\hat{\beta}_i}{s_e \sqrt{c_{ii}}}$.

• If $|t| > t_{a/2}$, then reject the hypothesis. If $|t| \leq t_{a/2}$, then do not reject the hypothesis.

### 7.11 ANALYSIS OF VARIANCE (ANOVA)

#### 7.11.1 ONE-FACTOR ANOVA

1. Suppose we have $k$ samples from $k$ populations, with the $j^{th}$ population consisting of $n_j$ observations,

\[
\begin{align*}
  y_{11}, y_{21}, \ldots, y_{n_1} & \\
  y_{12}, y_{22}, \ldots, y_{n_2} & \\
  \vdots & \\
  y_{1k}, y_{2k}, \ldots, y_{nk} & 
\end{align*}
\]
2. One-factor model:

- The one-factor ANOVA assumes that the \(i^{th}\) observation from the \(j^{th}\) sample is of the form \(y_{ij} = \mu + \tau_j + e_{ij}\).
- For \(j = 1, 2, \ldots, k\), the parameter \(\mu_j = \mu + \tau_j\) is the unknown mean of the \(j^{th}\) population, and \(\sum_{j=1}^{k} \tau_j = 0\).
- For \(j = 1, 2, \ldots, k\) and \(i = 1, 2, \ldots, n_j\), the random variable \(e_{ij}\) is normally distributed with mean zero and variance \(\sigma^2\).
- For \(j = 1, 2, \ldots, k\) and \(i = 1, 2, \ldots, n_j\), the random variables \(e_{ij}\) are independent.
- The total number of observations is \(n = n_1 + n_2 + \cdots + n_k\).

3. Point estimates of means:

- Total sample mean \(\hat{\gamma} = \frac{1}{n} \sum_{j=1}^{k} \sum_{i=1}^{n_j} y_{ij}\).
- Sample mean of \(j^{th}\) sample \(\hat{\gamma}_j = \frac{1}{n_j} \sum_{i=1}^{n_j} y_{ij}\).

4. Sums of squares:

- Sum of squares between samples \(SS_b = \sum_{j=1}^{k} n_j (\hat{\gamma}_j - \hat{\gamma})^2\).
- Sum of squares within samples \(SS_w = \sum_{j=1}^{k} \sum_{i=1}^{n_j} (y_{ij} - \hat{\gamma}_j)^2\).
- Total sum of squares \(Total SS = \sum_{j=1}^{k} \sum_{i=1}^{n_j} (y_{ij} - \hat{\gamma})^2\).
- Partition of total sum of squares \(Total SS = SS_b + SS_w\).

5. Degrees of freedom:

- Between samples, \(k - 1\).
- Within samples, \(n - k\).
- Total, \(n - 1\).
6. Mean squares:

- Obtained by dividing sums of squares by their respective degrees of freedom.
- Between samples, \( MS_b = \frac{SS_b}{k - 1} \).
- Within samples (also called the residual mean square),
  \( MS_w = \frac{SS_w}{n - k} \).

7. Test of the hypothesis \( \mu_1 = \mu_2 = \cdots = \mu_k \) against the alternative \( \mu_i \neq \mu_j \) for some \( i \) and \( j \); equivalently, test the null hypothesis \( \tau_1 = \tau_2 = \cdots = \tau_k = 0 \) against the hypothesis \( \tau_j \neq 0 \) for some \( j \):

- Determine the critical value \( F_\alpha \) such that \( F (F_\alpha) = 1 - \alpha \), where \( F (F) \) is the \( F \)-distribution with \( k - 1 \) and \( n - k \) degrees of freedom.
- Compute the point estimates \( \hat{\mu}_i \) and \( \hat{\mu}_j \).
- Compute the sums of squares \( SS_b \) and \( SS_w \).
- Compute the mean squares \( MS_b \) and \( MS_w \).
- Compute the test statistic \( F = \frac{MS_b}{MS_w} \).
- If \( F > F_\alpha \), then reject the hypothesis. If \( F \leq F_\alpha \), then do not reject the hypothesis.
- The above computations are often organized into an ANOVA table:

<table>
<thead>
<tr>
<th>Source</th>
<th>SS</th>
<th>D.O.F.</th>
<th>MS</th>
<th>F Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Between samples</td>
<td>( SS_b )</td>
<td>( k - 1 )</td>
<td>( MS_b )</td>
<td>( F = \frac{MS_b}{MS_w} )</td>
</tr>
<tr>
<td>Within samples</td>
<td>( SS_w )</td>
<td>( n - k )</td>
<td>( MS_w )</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>Total SS</td>
<td>( n - 1 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

8. Confidence interval for \( \mu_i - \mu_j \), for \( i \neq j \):

- Determine the critical value \( t_{\alpha/2} \) such that \( F (t_{\alpha/2}) = 1 - \alpha/2 \), where \( F (t) \) is the \( t \)-distribution with \( n - k \) degrees of freedom.
- Compute the point estimates \( \hat{\mu}_i \) and \( \hat{\mu}_j \).
- Compute the residual mean square \( MS_w \).
- Compute \( k = t_{\alpha/2} \sqrt{MS_w \left( \frac{1}{n_i} + \frac{1}{n_j} \right)} \).
- The \( 100(1 - \alpha) \) percent confidence interval for \( \mu_i - \mu_j \) is given by \( [\hat{\mu}_i - \hat{\mu}_j - k, \hat{\mu}_i - \hat{\mu}_j + k] \).

9. Confidence interval for contrast in the means, defined by \( C = c_1 \mu_1 + c_2 \mu_2 + \cdots + c_k \mu_k \), where \( c_1 + c_2 + \cdots + c_k = 0 \):

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• Determine the critical value $F_\alpha$ such that $F(F_\alpha) = 1 - \alpha$, where $F(F)$ is the $F$-distribution with $k - 1$ and $n - k$ degrees of freedom.

• Compute the point estimates $\hat{y}_j$ for $j = 1, 2, \ldots, k$.

• Compute the residual mean square $MS_w$.

• Compute $k = \sqrt{F_\alpha MS_w \left( \frac{k - 1}{n} \sum_{j=1}^{k} c_j^2 \right)}$.

• The 100$(1 - \alpha)$ percent confidence interval for the contrast $C$ is given by

$$\left[ \sum_{j=1}^{k} c_j \hat{y}_j - k, \sum_{j=1}^{k} c_j \hat{y}_j + k \right].$$

### 7.11.2 UNREPLICATED TWO-FACTOR ANOVA

1. Suppose we have a sample of observations $y_{ij}$ indexed by two factors $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

2. Unreplicated two-factor model:

   • The unreplicated two-factor ANOVA assumes that the $ij^{th}$ observation is of the form $y_{ij} = \mu + \beta_i + \tau_j + e_{ij}$.

   • $\mu$ is the overall mean, $\beta_i$ is the $i^{th}$ differential effect of factor one, $\tau_j$ is the $j^{th}$ differential effect of factor two, and

   $$\sum_{i=1}^{m} \beta_i = \sum_{j=1}^{n} \tau_j = 0.$$

   • For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, the random variable $e_{ij}$ is normally distributed with mean zero and variance $\sigma^2$.

   • For $j = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, the random variables $e_{ij}$ are independent.

   • Total number of observations is $mn$.

3. Point estimates of means:

   • Total sample mean $\hat{y} = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}$.

   • $i^{th}$ factor-one sample mean $\hat{y}_i = \frac{1}{n} \sum_{j=1}^{n} y_{ij}$.

   • $j^{th}$ factor-two sample mean $\hat{y}_j = \frac{1}{m} \sum_{i=1}^{m} y_{ij}$. 

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4. Sums of squares:

- Factor-one sum of squares $SS_1 = n \sum_{i=1}^{m} (\hat{y}_i - \bar{y})^2$.
- Factor-two sum of squares $SS_2 = m \sum_{j=1}^{n} (\hat{y}_j - \bar{y})^2$.
- Residual sum of squares $SS_r = \sum_{i=1}^{m} \sum_{j=1}^{n} (y_{ij} - \hat{y}_i - \hat{y}_j + \bar{y})^2$.
- Total sum of squares $Total SS = \sum_{j=1}^{k} \sum_{i=1}^{n_j} (y_{ij} - \bar{y})^2$.
- Partition of total sum of squares $Total SS = SS_1 + SS_2 + SS_r$.

5. Degrees of freedom:

- Factor one, $m - 1$.
- Factor two, $n - 1$.
- Residual, $(m - 1)(n - 1)$.
- Total, $mn - 1$.

6. Mean squares:

- Obtained by dividing sums of squares by their respective degrees of freedom.
- Factor-one mean square $MS_1 = \frac{SS_1}{m - 1}$.
- Factor-two mean square $MS_2 = \frac{SS_2}{n - 1}$.
- Residual mean square $MS_r = \frac{SS_r}{(m - 1)(n - 1)}$.

7. Test of the null hypothesis $\beta_1 = \beta_2 = \cdots = \beta_m = 0$ (no factor-one effects) against the alternative hypothesis $\beta_i \neq 0$ for some $i$:

- Determine the critical value $F_\alpha$ such that $F (F_\alpha) = 1 - \alpha$, where $F (F)$ is the $F$-distribution with $m - 1$ and $(m - 1)(n - 1)$ degrees of freedom.
- Compute the point estimates $\hat{y}$ and $\hat{y}_i$ for $i = 1, 2, \ldots, m$.
- Compute the sums of squares $SS_1$ and $SS_r$.
- Compute the mean squares $MS_1$ and $MS_r$.
- Compute the test statistic $F = \frac{MS_1}{MS_r}$.
- If $F > F_\alpha$, then reject the hypothesis. If $F \leq F_\alpha$, then do not reject the hypothesis.
8. Test of the null hypothesis \( \tau_1 = \tau_2 = \cdots = \tau_n = 0 \) (no factor-two effects) against the alternative hypothesis \( \tau_j \neq 0 \) for some \( j \):

- Determine the critical value \( F_\alpha \) such that \( F(F_\alpha) = 1 - \alpha \), where \( F(F) \) is the \( F \)-distribution with \( n - 1 \) and \((m - 1)(n - 1)\) degrees of freedom.
- Compute the point estimates \( \hat{y} \) and \( \hat{y}_j \) for \( j = 1, 2, \ldots, n \).
- Compute the sums of squares \( SS_2 \) and \( SS_r \).
- Compute the mean squares \( MS_2 \) and \( MS_r \).
- Compute the test statistic \( F = \frac{MS_2}{MS_r} \).
- If \( F > F_\alpha \), then reject the hypothesis. If \( F \leq F_\alpha \), then do not reject the hypothesis.

The above computations are often organized into an ANOVA table:

<table>
<thead>
<tr>
<th>Source</th>
<th>SS</th>
<th>D.O.F.</th>
<th>MS</th>
<th>F Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factor one</td>
<td>( SS_1 )</td>
<td>( m - 1 )</td>
<td>( MS_1 )</td>
<td>( F = \frac{MS_1}{MS_r} )</td>
</tr>
<tr>
<td>Factor two</td>
<td>( SS_2 )</td>
<td>( n - 1 )</td>
<td>( MS_2 )</td>
<td>( F = \frac{MS_2}{MS_r} )</td>
</tr>
<tr>
<td>Residual</td>
<td>( SS_r )</td>
<td>((m - 1)(n - 1))</td>
<td>( MS_r )</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>Total SS</td>
<td>( mn - 1 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

9. Confidence interval for contrast in the factor-one means, defined by \( C = c_1\beta_1 + c_2\beta_2 + \cdots + c_m\beta_m \), where \( c_1 + c_2 + \cdots + c_m = 0 \):

- Determine the critical value \( F_\alpha \) such that \( F(F_\alpha) = 1 - \alpha \), where \( F(F) \) is the \( F \)-distribution with \( m - 1 \) and \((m - 1)(n - 1)\) degrees of freedom.
- Compute the point estimates \( \hat{y}_i \) for \( i = 1, 2, \ldots, m \).
- Compute the residual mean square \( MS_r \).
- Compute \( k = \sqrt{\frac{F_\alpha MS_r}{\left(\frac{m - 1}{n} \sum_{i=1}^{m} c_i^2\right)}} \).
- The 100(1 - \( \alpha \)) percent confidence interval for the contrast \( C \) is given by
  \[
  \left[ \sum_{i=1}^{m} c_i \hat{y}_i - k, \sum_{i=1}^{m} c_i \hat{y}_i + k \right].
  \]

10. Confidence interval for contrast in the factor-two means, defined by \( C = c_1\tau_1 + c_2\tau_2 + \cdots + c_n\tau_n \), where \( c_1 + c_2 + \cdots + c_n = 0 \):

- Determine the critical value \( F_\alpha \) such that \( F(F_\alpha) = 1 - \alpha \), where \( F(F) \) is the \( F \)-distribution with \( n - 1 \) and \((m - 1)(n - 1)\) degrees of freedom.
- Compute the point estimates \( \hat{y}_j \) for \( j = 1, 2, \ldots, n \).
• Compute the residual mean square $MS_r$.

• Compute $k = \sqrt{F_{\alpha} MS_r \left( \frac{n - 1}{m} \sum_{j=1}^{n} c_j^2 \right)}$.

• The $100(1 - \alpha)$ percent confidence interval for the contrast $C$ is given by

$$\left[ \sum_{j=1}^{n} c_j \hat{y}_j - k, \sum_{j=1}^{n} c_j \hat{y}_j + k \right].$$

### 7.11.3 REPLICATED TWO-FACTOR ANOVA

1. Suppose we have a sample of observations $y_{ijk}$ indexed by two factors $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Moreover, there are $p$ observations per factor pair $(i, j)$, indexed by $k = 1, 2, \ldots, p$.

2. Replicated two-factor model:

   - The replicated two-factor ANOVA assumes that the $ij^k$th observation is of the form $y_{ijk} = \mu + \beta_i + \tau_j + \gamma_{ij} + e_{ijk}$.
   - $\mu$ is the overall mean, $\beta_i$ is the $i$th differential effect of factor one, $\tau_j$ is the $j$th differential effect of factor two, and

$$\sum_{i=1}^{m} \beta_i = \sum_{j=1}^{n} \tau_j = 0.$$  

   - For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, $\gamma_{ij}$ is the $ij$th interaction effect of factors one and two.
   - For $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, and $k = 1, 2, \ldots, p$, the random variable $e_{ijk}$ is normally distributed with mean zero and variance $\sigma^2$.
   - For $j = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, and $k = 1, 2, \ldots, p$, the random variables $e_{ijk}$ are independent.

   - Total number of observations is $mnp$.

3. Point estimates of means:

   - Total sample mean $\hat{y} = \frac{1}{mnp} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} y_{ijk}$.

   - $i$th factor-one sample mean $\hat{y}_i = \frac{1}{np} \sum_{j=1}^{n} \sum_{k=1}^{p} y_{ijk}$.

   - $j$th factor-two sample mean $\hat{y}_j = \frac{1}{mp} \sum_{i=1}^{m} \sum_{k=1}^{p} y_{ijk}$.
\[ i,j \text{th interaction mean } \hat{\gamma}_{ij} = \frac{1}{p} \sum_{k=1}^{p} y_{ijk}. \]

4. Sums of squares:
   - Factor-one sum of squares \( SS_1 = np \sum_{i=1}^{m} (\hat{y}_i - \hat{y})^2. \)
   - Factor-two sum of squares \( SS_2 = mp \sum_{j=1}^{n} (\hat{y}_j - \hat{y})^2. \)
   - Interaction sum of squares \( SS_{12} = p \sum_{i=1}^{m} \sum_{j=1}^{n} (\hat{y}_{ij} - \hat{y})^2. \)
   - Residual sum of squares \( SS_r = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} (y_{ijk} - \hat{y}_{i.} - \hat{y}_{.j} + \hat{y})^2. \)
   - Total sum of squares \( \text{Total SS} = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} (y_{ijk} - \hat{y})^2. \)
   - Partition of total sum of squares \( \text{Total SS} = SS_1 + SS_2 + SS_{12} + SS_r. \)

5. Degrees of freedom:
   - Factor one, \( m - 1. \)
   - Factor two, \( n - 1. \)
   - Interaction, \( (m - 1)(n - 1). \)
   - Residual, \( mn(p - 1). \)
   - Total, \( mnp - 1. \)

6. Mean squares:
   - Obtained by dividing sums of squares by their respective degrees of freedom.
   - Factor-one mean square \( MS_1 = \frac{SS_1}{m - 1}. \)
   - Factor-two mean square \( MS_2 = \frac{SS_2}{n - 1}. \)
   - Interaction mean square \( MS_{12} = \frac{SS_{12}}{(m - 1)(n - 1)}. \)
   - Residual mean square \( MS_r = \frac{SS_r}{mn(p - 1)}. \)
7. Test of the null hypothesis \( \beta_1 = \beta_2 = \cdots = \beta_m = 0 \) (no factor-one effects) against the alternative hypothesis \( \beta_i \neq 0 \) for some \( i \):

- Determine the critical value \( F_\alpha \) such that \( F (F_\alpha) = 1 - \alpha \), where \( F (F) \) is the \( F \)-distribution with \( m - 1 \) and \( mn(p - 1) \) degrees of freedom.
- Compute the point estimates \( \hat{y} \) and \( \hat{y}_i \) for \( i = 1, 2, \ldots, m \).
- Compute the sums of squares \( SS_1 \) and \( SS_r \).
- Compute the mean squares \( MS_1 \) and \( MS_r \).
- Compute the test statistic \( F = \frac{MS_1}{MS_r} \).
- If \( F > F_\alpha \), then reject the hypothesis. If \( F \leq F_\alpha \), then do not reject the hypothesis.

8. Test of the null hypothesis \( \tau_1 = \tau_2 = \cdots = \tau_n = 0 \) (no factor-two effects) against the alternative hypothesis \( \tau_j \neq 0 \) for some \( j \):

- Determine the critical value \( F_\alpha \) such that \( F (F_\alpha) = 1 - \alpha \), where \( F (F) \) is the \( F \)-distribution with \( n - 1 \) and \( mn(p - 1) \) degrees of freedom.
- Compute the point estimates \( \hat{y} \) and \( \hat{y}_j \) for \( j = 1, 2, \ldots, n \).
- Compute the sums of squares \( SS_2 \) and \( SS_r \).
- Compute the mean squares \( MS_2 \) and \( MS_r \).
- Compute the test statistic \( F = \frac{MS_2}{MS_r} \).
- If \( F > F_\alpha \), then reject the hypothesis. If \( F \leq F_\alpha \), then do not reject the hypothesis.

9. Test of the null hypothesis \( \gamma_{ij} = 0 \) for \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n \) (no factor-one effects) against the alternative hypothesis \( \gamma_{ij} \neq 0 \) for some \( i \) and \( j \):

- Determine the critical value \( F_\alpha \) such that \( F (F_\alpha) = 1 - \alpha \), where \( F (F) \) is the \( F \)-distribution with \( (m - 1)(n - 1) \) and \( mn(p - 1) \) degrees of freedom.
- Compute the point estimates \( \hat{y}, \hat{y}_i, \hat{y}_j, \) and \( \hat{y}_{ij} \) for \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n \).
- Compute the sums of squares \( SS_{12} \) and \( SS_r \).
- Compute the mean squares \( MS_{12} \) and \( MS_r \).
- Compute the test statistic \( F = \frac{MS_{12}}{MS_r} \).
- If \( F > F_\alpha \), then reject the hypothesis. If \( F \leq F_\alpha \), then do not reject the hypothesis.
• The above computations are often organized into an ANOVA table:

<table>
<thead>
<tr>
<th>Source</th>
<th>SS</th>
<th>D.O.F.</th>
<th>MS</th>
<th>F Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factor one</td>
<td>SS1</td>
<td>m−1</td>
<td>MS1</td>
<td>( F = \frac{MS_1}{MS_r} )</td>
</tr>
<tr>
<td>Factor two</td>
<td>SS2</td>
<td>n−1</td>
<td>MS2</td>
<td>( F = \frac{MS_2}{MS_r} )</td>
</tr>
<tr>
<td>Interaction</td>
<td>SS12</td>
<td>(m−1)(n−1)</td>
<td>MS12</td>
<td>( F = \frac{MS_{12}}{MS_r} )</td>
</tr>
<tr>
<td>Residual</td>
<td>SSr</td>
<td>mn(p−1)</td>
<td>MSr</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>Total SS</td>
<td>mnp−1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

10. Confidence interval for contrast in the factor-one means, defined by \( C = c_1\beta_1 + c_2\beta_2 + \cdots + c_m\beta_m \), where \( c_1 + c_2 + \cdots + c_m = 0 \):
   • Determine the critical value \( F_a \) such that \( F (F_a) = 1 - \alpha \), where \( F (F) \) is the \( F \)-distribution with \( m - 1 \) and \( mn(p−1) \) degrees of freedom.
   • Compute the point estimates \( \hat{y}_i \) for \( i = 1, 2, \ldots, m \).
   • Compute the residual mean square \( MS_r \).
   • Compute \( k = \sqrt{\frac{F_a MS_r}{mnp} \left( \sum_{i=1}^{m} c_i^2 \right)} \).
   • The 100(1 − \alpha) percent confidence interval for the contrast \( C \) is given by
   \[
   \left[ \sum_{i=1}^{m} c_i \hat{y}_i - k, \sum_{i=1}^{m} c_i \hat{y}_i + k \right].
   \]

11. Confidence interval for contrast in the factor-two means, defined by \( C = c_1\tau_1 + c_2\tau_2 + \cdots + c_n\tau_n \), where \( c_1 + c_2 + \cdots + c_n = 0 \):
   • Determine the critical value \( F_a \) such that \( F (F_a) = 1 - \alpha \), where \( F (F) \) is the \( F \)-distribution with \( n - 1 \) and \( mn(p−1) \) degrees of freedom.
   • Compute the point estimates \( \hat{y}_{.j} \) for \( j = 1, 2, \ldots, n \).
   • Compute the residual mean square \( MS_r \).
   • Compute \( k = \sqrt{\frac{F_a MS_r}{mp} \left( \sum_{j=1}^{n} c_j^2 \right)} \).
   • The 100(1 − \alpha) percent confidence interval for the contrast \( C \) is given by
   \[
   \left[ \sum_{j=1}^{n} c_j \hat{y}_{.j} - k, \sum_{j=1}^{n} c_j \hat{y}_{.j} + k \right].
   \]
7.12 PROBABILITY TABLES

7.12.1 CRITICAL VALUES

The critical value

• $z_\alpha$ satisfies $\Phi(z_\alpha) = 1 - \alpha$ (where, as usual, $\Phi(z)$ is the distribution function for the standard normal).

• $t_\alpha$ satisfies $F(t_\alpha) = 1 - \alpha$ where $F(t)$ is the distribution function for the $t$-distribution (with some specified number of degrees of freedom).

• $\chi^2_\alpha$ satisfies $F(\chi^2_\alpha) = 1 - \alpha$ where $F(t)$ is the distribution function for the $\chi^2$-distribution (with some specified number of degrees of freedom).

7.12.2 TABLE OF THE NORMAL DISTRIBUTION

For a standard normal random variable:

<table>
<thead>
<tr>
<th>Limits</th>
<th>Proportional of the total area (%)</th>
<th>Remaining area (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu - \lambda \sigma$</td>
<td>$\mu + \lambda \sigma$</td>
<td>68.26</td>
</tr>
<tr>
<td>$\mu - \sigma$</td>
<td>$\mu + \sigma$</td>
<td>90</td>
</tr>
<tr>
<td>$\mu - 1.65\sigma$</td>
<td>$\mu + 1.65\sigma$</td>
<td>95</td>
</tr>
<tr>
<td>$\mu - 1.96\sigma$</td>
<td>$\mu + 1.96\sigma$</td>
<td>95.44</td>
</tr>
<tr>
<td>$\mu - 2\sigma$</td>
<td>$\mu + 2\sigma$</td>
<td>99</td>
</tr>
<tr>
<td>$\mu - 2.58\sigma$</td>
<td>$\mu + 2.58\sigma$</td>
<td>99.73</td>
</tr>
<tr>
<td>$\mu - 3\sigma$</td>
<td>$\mu + 3\sigma$</td>
<td>99.8</td>
</tr>
<tr>
<td>$\mu - 3.09\sigma$</td>
<td>$\mu + 3.09\sigma$</td>
<td>99.9</td>
</tr>
<tr>
<td>$\mu - 3.29\sigma$</td>
<td>$\mu + 3.29\sigma$</td>
<td>10</td>
</tr>
</tbody>
</table>

For large values of $x$:

$$\left[\frac{e^{-x^2/2}}{\sqrt{2\pi}} \left(\frac{1}{x} - \frac{1}{x^3}\right)\right] < 1 - \Phi(x) < \left[\frac{e^{-x^2/2}}{\sqrt{2\pi}} \left(\frac{1}{x}\right)\right]$$
f (x)

x

F (x)

0.5000
0.4960
0.4920
0.4880
0.4840
0.4801
0.4761
0.4721
0.4681
0.4641

0.3989
0.3989
0.3989
0.3988
0.3986
0.3984
0.3982
0.3980
0.3977
0.3973

0.50
0.51
0.52
0.53
0.54
0.55
0.56
0.57
0.58
0.59

0.6915
0.6950
0.6985
0.7019
0.7054
0.7088
0.7123
0.7157
0.7190
0.7224

0.3085
0.3050
0.3015
0.2981
0.2946
0.2912
0.2877
0.2843
0.2810
0.2776

0.3521
0.3503
0.3485
0.3467
0.3448
0.3429
0.3411
0.3391
0.3372
0.3352

0.5398
0.5438
0.5478
0.5517
0.5557
0.5596
0.5636
0.5675
0.5714
0.5754

0.4602
0.4562
0.4522
0.4483
0.4443
0.4404
0.4364
0.4325
0.4286
0.4247

0.3970
0.3965
0.3961
0.3956
0.3951
0.3945
0.3939
0.3932
0.3925
0.3918

0.60
0.61
0.62
0.63
0.64
0.65
0.66
0.67
0.68
0.69

0.7258
0.7291
0.7324
0.7357
0.7389
0.7421
0.7454
0.7486
0.7518
0.7549

0.2742
0.2709
0.2676
0.2643
0.2611
0.2579
0.2546
0.2514
0.2482
0.2451

0.3332
0.3312
0.3292
0.3271
0.3251
0.3230
0.3209
0.3187
0.3166
0.3144

0.20
0.21
0.22
0.23
0.24
0.25
0.26
0.27
0.28
0.29

0.5793
0.5832
0.5871
0.5909
0.5948
0.5987
0.6026
0.6064
0.6103
0.6141

0.4207
0.4168
0.4129
0.4091
0.4052
0.4013
0.3974
0.3936
0.3897
0.3859

0.3910
0.3902
0.3894
0.3885
0.3876
0.3867
0.3857
0.3847
0.3836
0.3825

0.70
0.71
0.72
0.73
0.74
0.75
0.76
0.77
0.78
0.79

0.7580
0.7611
0.7642
0.7673
0.7703
0.7734
0.7764
0.7793
0.7823
0.7852

0.2420
0.2389
0.2358
0.2327
0.2296
0.2266
0.2236
0.2207
0.2177
0.2148

0.3123
0.3101
0.3079
0.3056
0.3034
0.3011
0.2989
0.2966
0.2943
0.2920

0.30
0.31
0.32
0.33
0.34
0.35
0.36
0.37
0.38
0.39

0.6179
0.6217
0.6255
0.6293
0.6331
0.6368
0.6406
0.6443
0.6480
0.6517

0.3821
0.3783
0.3745
0.3707
0.3669
0.3632
0.3594
0.3557
0.3520
0.3483

0.3814
0.3802
0.3790
0.3778
0.3765
0.3752
0.3739
0.3725
0.3711
0.3697

0.80
0.81
0.82
0.83
0.84
0.85
0.86
0.87
0.88
0.89

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0.7967
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0.8133

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0.2061
0.2033
0.2004
0.1977
0.1949
0.1921
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0.1867

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0.2874
0.2850
0.2827
0.2803
0.2780
0.2756
0.2732
0.2709
0.2685

0.40
0.41
0.42
0.43
0.44
0.45
0.46
0.47
0.48
0.49

0.6554
0.6591
0.6628
0.6664
0.6700
0.6736
0.6772
0.6808
0.6844
0.6879

0.3446
0.3409
0.3372
0.3336
0.3300
0.3264
0.3228
0.3192
0.3156
0.3121

0.3683
0.3668
0.3653
0.3637
0.3621
0.3605
0.3589
0.3572
0.3555
0.3538

0.90
0.91
0.92
0.93
0.94
0.95
0.96
0.97
0.98
0.99

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0.8186
0.8212
0.8238
0.8264
0.8289
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0.8340
0.8365
0.8389

0.1841
0.1814
0.1788
0.1762
0.1736
0.1711
0.1685
0.1660
0.1635
0.1611

0.2661
0.2637
0.2613
0.2589
0.2565
0.2541
0.2516
0.2492
0.2468
0.2444

0.50

0.6915

0.3085

0.3521

1.00

0.8413

0.1587

0.2420

x

F (x)

0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09

0.5000
0.5040
0.5080
0.5120
0.5160
0.5199
0.5239
0.5279
0.5319
0.5359

0.10
0.11
0.12
0.13
0.14
0.15
0.16
0.17
0.18
0.19

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1 − F (x)

1 − F (x)

f (x)


f (x)

x

F (x)

0.1587
0.1562
0.1539
0.1515
0.1492
0.1469
0.1446
0.1423
0.1401
0.1379

0.2420
0.2396
0.2371
0.2347
0.2323
0.2299
0.2275
0.2251
0.2226
0.2203

1.50
1.51
1.52
1.53
1.54
1.55
1.56
1.57
1.58
1.59

0.9332
0.9345
0.9357
0.9370
0.9382
0.9394
0.9406
0.9418
0.9429
0.9441

0.0668
0.0655
0.0643
0.0630
0.0618
0.0606
0.0594
0.0582
0.0570
0.0559

0.1295
0.1276
0.1257
0.1238
0.1219
0.1200
0.1182
0.1163
0.1145
0.1127

0.8643
0.8665
0.8686
0.8708
0.8729
0.8749
0.8770
0.8790
0.8810
0.8830

0.1357
0.1335
0.1314
0.1292
0.1271
0.1251
0.1230
0.1210
0.1190
0.1170

0.2178
0.2155
0.2131
0.2107
0.2083
0.2059
0.2036
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1.61
1.62
1.63
1.64
1.65
1.66
1.67
1.68
1.69

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0.9474
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0.9495
0.9505
0.9515
0.9525
0.9535
0.9545

0.0548
0.0537
0.0526
0.0515
0.0505
0.0495
0.0485
0.0475
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0.0455

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1.23
1.24
1.25
1.26
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1.28
1.29

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0.8869
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0.8962
0.8980
0.8997
0.9015

0.1151
0.1131
0.1112
0.1094
0.1075
0.1056
0.1038
0.1020
0.1003
0.0985

0.1942
0.1919
0.1895
0.1872
0.1849
0.1827
0.1804
0.1781
0.1759
0.1736

1.70
1.71
1.72
1.73
1.74
1.75
1.76
1.77
1.78
1.79

0.9554
0.9564
0.9573
0.9582
0.9591
0.9599
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0.9616
0.9625
0.9633

0.0446
0.0436
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0.0384
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0.0940
0.0925
0.0909
0.0893
0.0878
0.0863
0.0848
0.0833
0.0818
0.0804

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1.32
1.33
1.34
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1.37
1.38
1.39

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0.0968
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0.0823

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0.1540
0.1518

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1.81
1.82
1.83
1.84
1.85
1.86
1.87
1.88
1.89

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0.0694
0.0681
0.0669

1.40
1.41
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1.43
1.44
1.45
1.46
1.47
1.48
1.49

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0.9319

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0.0778
0.0764
0.0749
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0.0722
0.0708
0.0694
0.0681

0.1497
0.1476
0.1456
0.1435
0.1415
0.1394
0.1374
0.1354
0.1334
0.1315

1.90
1.91
1.92
1.93
1.94
1.95
1.96
1.97
1.98
1.99

0.9713
0.9719
0.9726
0.9732
0.9738
0.9744
0.9750
0.9756
0.9761
0.9767

0.0287
0.0281
0.0274
0.0268
0.0262
0.0256
0.0250
0.0244
0.0238
0.0233

0.0656
0.0644
0.0632
0.0619
0.0608
0.0596
0.0584
0.0573
0.0562
0.0551

1.50

0.9332

0.0668

0.1295

2.00

0.9772

0.0227

0.0540

x

F (x)

1.00
1.01
1.02
1.03
1.04
1.05
1.06
1.07
1.08
1.09

0.8413
0.8438
0.8461
0.8485
0.8508
0.8531
0.8554
0.8577
0.8599
0.8621

1.10
1.11
1.12
1.13
1.14
1.15
1.16
1.17
1.18
1.19

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1 − F (x)

1 − F (x)

f (x)


f (x)

x

F (x)

0.0227
0.0222
0.0217
0.0212
0.0207
0.0202
0.0197
0.0192
0.0188
0.0183

0.0540
0.0529
0.0519
0.0508
0.0498
0.0488
0.0478
0.0468
0.0459
0.0449

2.50
2.51
2.52
2.53
2.54
2.55
2.56
2.57
2.58
2.59

0.9938
0.9940
0.9941
0.9943
0.9945
0.9946
0.9948
0.9949
0.9951
0.9952

0.0062
0.0060
0.0059
0.0057
0.0055
0.0054
0.0052
0.0051
0.0049
0.0048

0.0175
0.0171
0.0167
0.0163
0.0158
0.0155
0.0151
0.0147
0.0143
0.0139

0.9821
0.9826
0.9830
0.9834
0.9838
0.9842
0.9846
0.9850
0.9854
0.9857

0.0179
0.0174
0.0170
0.0166
0.0162
0.0158
0.0154
0.0150
0.0146
0.0143

0.0440
0.0431
0.0422
0.0413
0.0404
0.0396
0.0387
0.0379
0.0371
0.0363

2.60
2.61
2.62
2.63
2.64
2.65
2.66
2.67
2.68
2.69

0.9953
0.9955
0.9956
0.9957
0.9959
0.9960
0.9961
0.9962
0.9963
0.9964

0.0047
0.0045
0.0044
0.0043
0.0042
0.0040
0.0039
0.0038
0.0037
0.0036

0.0136
0.0132
0.0129
0.0126
0.0122
0.0119
0.0116
0.0113
0.0110
0.0107

2.20
2.21
2.22
2.23
2.24
2.25
2.26
2.27
2.28
2.29

0.9861
0.9865
0.9868
0.9871
0.9875
0.9878
0.9881
0.9884
0.9887
0.9890

0.0139
0.0135
0.0132
0.0129
0.0126
0.0122
0.0119
0.0116
0.0113
0.0110

0.0355
0.0347
0.0339
0.0332
0.0325
0.0317
0.0310
0.0303
0.0296
0.0290

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2.72
2.73
2.74
2.75
2.76
2.77
2.78
2.79

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0.9970
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0.0033
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0.0030
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0.0101
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0.0091
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2.35
2.36
2.37
2.38
2.39

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0.9898
0.9901
0.9904
0.9906
0.9909
0.9911
0.9913
0.9916

0.0107
0.0104
0.0102
0.0099
0.0096
0.0094
0.0091
0.0089
0.0087
0.0084

0.0283
0.0277
0.0271
0.0264
0.0258
0.0252
0.0246
0.0241
0.0235
0.0229

2.80
2.81
2.82
2.83
2.84
2.85
2.86
2.87
2.88
2.89

0.9974
0.9975
0.9976
0.9977
0.9977
0.9978
0.9979
0.9980
0.9980
0.9981

0.0026
0.0025
0.0024
0.0023
0.0023
0.0022
0.0021
0.0021
0.0020
0.0019

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0.0073
0.0071
0.0069
0.0067
0.0065
0.0063
0.0061

2.40
2.41
2.42
2.43
2.44
2.45
2.46
2.47
2.48
2.49

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0.9920
0.9922
0.9925
0.9927
0.9929
0.9930
0.9932
0.9934
0.9936

0.0082
0.0080
0.0078
0.0076
0.0073
0.0071
0.0069
0.0068
0.0066
0.0064

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0.0219
0.0213
0.0208
0.0203
0.0198
0.0194
0.0189
0.0184
0.0180

2.90
2.91
2.92
2.93
2.94
2.95
2.96
2.97
2.98
2.99

0.9981
0.9982
0.9982
0.9983
0.9984
0.9984
0.9985
0.9985
0.9986
0.9986

0.0019
0.0018
0.0018
0.0017
0.0016
0.0016
0.0015
0.0015
0.0014
0.0014

0.0060
0.0058
0.0056
0.0054
0.0053
0.0051
0.0050
0.0049
0.0047
0.0046

2.50

0.9938

0.0062

0.0175

3.00

0.9987

0.0014

0.0044

x

F (x)

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2.01
2.02
2.03
2.04
2.05
2.06
2.07
2.08
2.09

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0.9778
0.9783
0.9788
0.9793
0.9798
0.9803
0.9808
0.9812
0.9817

2.10
2.11
2.12
2.13
2.14
2.15
2.16
2.17
2.18
2.19

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1 − F (x)

1 − F (x)

f (x)


7.12.3 PERCENTAGE POINTS, STUDENT'S $t$-DISTRIBUTION

For a given value of $n$ this table gives the value of $t$ such that

$$F(t) = \int_\infty^t \frac{\Gamma((n+1)/2)}{\sqrt{n\pi}\Gamma(n/2)} \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2} dx$$

is a specified number. The $t$-distribution is symmetrical, so that $F(-t) = 1 - F(t)$.

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| 1   | 0.325   | 1.000   | 3.078  | 6.314  | 12.706 | 31.821 | 63.657 | 127.381 | 318.309 | 636.619  
| 2   | 0.289   | 0.816   | 1.886  | 2.920  | 4.303  | 6.965  | 9.210  | 12.706  | 16.282  | 18.309   
| 3   | 0.277   | 0.765   | 1.638  | 2.353  | 3.182  | 4.541  | 5.841  | 7.021  | 8.610  | 9.924    
| 4   | 0.271   | 0.741   | 1.533  | 2.132  | 2.776  | 3.747  | 4.604  | 5.576  | 6.866  | 7.977    
| 5   | 0.267   | 0.727   | 1.476  | 2.015  | 2.571  | 3.365  | 4.032  | 4.780  | 5.598  | 6.436    
| 6   | 0.265   | 0.718   | 1.440  | 1.943  | 2.447  | 3.143  | 3.707  | 4.297  | 5.041  | 5.708    
| 7   | 0.263   | 0.711   | 1.415  | 1.895  | 2.365  | 2.998  | 3.499  | 3.980  | 4.849  | 5.328    
| 8   | 0.262   | 0.706   | 1.397  | 1.860  | 2.306  | 2.869  | 3.355  | 3.745  | 4.501  | 5.041    
| 9   | 0.261   | 0.703   | 1.383  | 1.833  | 2.262  | 2.821  | 3.250  | 3.581  | 4.297  | 4.781    
| 10  | 0.260   | 0.700   | 1.372  | 1.812  | 2.228  | 2.764  | 3.169  | 3.440  | 4.115  | 4.587    
| 11  | 0.260   | 0.697   | 1.363  | 1.796  | 2.201  | 2.718  | 3.106  | 3.301  | 4.025  | 4.337    
| 12  | 0.259   | 0.695   | 1.356  | 1.782  | 2.179  | 2.681  | 3.055  | 3.192  | 3.930  | 4.172    
| 13  | 0.259   | 0.694   | 1.350  | 1.771  | 2.160  | 2.650  | 3.012  | 3.082  | 3.841  | 4.022    
| 14  | 0.258   | 0.692   | 1.345  | 1.761  | 2.145  | 2.624  | 2.977  | 2.980  | 3.756  | 3.944    
| 15  | 0.258   | 0.691   | 1.341  | 1.753  | 2.131  | 2.602  | 2.947  | 2.897  | 3.686  | 3.873    
| 16  | 0.258   | 0.690   | 1.337  | 1.746  | 2.120  | 2.583  | 2.921  | 2.828  | 3.626  | 3.807    
| 17  | 0.257   | 0.689   | 1.333  | 1.740  | 2.110  | 2.567  | 2.898  | 2.772  | 3.579  | 3.745    
| 18  | 0.257   | 0.688   | 1.330  | 1.734  | 2.101  | 2.552  | 2.878  | 2.724  | 3.531  | 3.690    
| 19  | 0.257   | 0.688   | 1.328  | 1.729  | 2.093  | 2.539  | 2.861  | 2.681  | 3.496  | 3.640    
| 20  | 0.257   | 0.687   | 1.325  | 1.725  | 2.086  | 2.528  | 2.845  | 2.640  | 3.467  | 3.601    
| 25  | 0.256   | 0.684   | 1.316  | 1.708  | 2.060  | 2.485  | 2.787  | 2.551  | 3.350  | 3.525    
| 50  | 0.255   | 0.679   | 1.299  | 1.676  | 2.009  | 2.403  | 2.678  | 2.440  | 3.261  | 3.469    
| 100 | 0.254   | 0.677   | 1.290  | 1.660  | 1.984  | 2.364  | 2.626  | 2.373  | 3.174  | 3.390    
| $\infty$ | 0.253   | 0.674   | 1.282  | 1.645  | 1.960  | 2.326  | 2.576  | 2.310  | 3.091  | 3.291    

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### 7.12.4 PERCENTAGE POINTS, CHI-SQUARE DISTRIBUTION

For a given value of $n$ this table gives the value of $\chi^2$ such that

$$F(\chi^2) = \int_0^{\chi^2} \frac{x^{(n-2)/2} e^{-x/2}}{2^{n/2} \Gamma(n/2)} \, dx$$

is a specified number.

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Given \( n \) and \( m \) this gives the value of \( f \) such that

\[
F(f) = \frac{\Gamma((n + m)/2)}{\Gamma(m/2)\Gamma(n/2)} m^{n/2}n^{m/2}x^{m/2-1}(n + mx)^{-(n+m)/2} \ dx = 0.9.
\]

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Given $n$ and $m$ this gives the value of $f$ such that

$$F(f) = \int_0^f \frac{\Gamma((n + m)/2)}{\Gamma(m/2)\Gamma(n/2)} m^{m/2} n^{n/2} x^{m/2 - 1} (n + mx)^{-(m+n)/2} \, dx = 0.95.$$ 

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Given \( n \) and \( m \) this gives the value of \( f \) such that

\[
F(f) = \int_0^f \frac{\Gamma((n + m)/2)}{\Gamma(m/2)\Gamma(n/2)} x^{m/2} \left(1 - \frac{x(n + 1)}{2}ight)^{(m + n)/2} dx = 0.99.
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$$F(f) = \int_0^f \frac{\Gamma((n+m)/2)}{\Gamma(m/2)\Gamma(n/2)} \frac{m^{m/2}n^{n/2}(n+mx)^{-(m+n)/2}}{x^{m/2-1}} dx = 0.995.$$
Given \( n \) and \( m \) this gives the value of \( f \) such that

\[
F(f) = \int_0^f \frac{\Gamma((n + m)/2)}{\Gamma(m/2)\Gamma(n/2)} m^{m/2} n^{n/2} x^{m/2-1} (n + mx)^{-(m+n)/2} \, dx = 0.999.
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7.12.6 CUMULATIVE TERMS, BINOMIAL DISTRIBUTION

\[ B(n, x; p) = \sum_{k=0}^{x} \binom{n}{k} p^k (1 - p)^{n-k}. \]

Note that \( B(n, x; p) = B(n, n-x; 1 - p) \).

If \( p \) is the probability of success, then \( B(n, x; p) \) is the probability of \( x \) or fewer successes in \( n \) independent trials. For example, if a biased coin has a probability \( p = 0.4 \) of being a head, and the coin is independently flipped 5 times, then there is a 68\% chance that there will be 2 or fewer heads (since \( B(5, 2; 0.4) = 0.6826 \)).

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7.12.7 CUMULATIVE TERMS, POISSON DISTRIBUTION

\[ F(x; \lambda) = \sum_{k=0}^{x} e^{-\lambda} \frac{\lambda^k}{k!}. \]

If \( \lambda \) is the rate of Poisson arrivals, then \( F(x; \lambda) \) is the probability of \( x \) or fewer arrivals occurring in a unit of time. For example, if customers arrive at the rate of \( \lambda = 0.5 \) customers per hour, then the probability of having no customers in any specified hour is 0.61 (the probability of one or fewer customers is 0.91).

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7.12.8 CRITICAL VALUES, KOLMOGOROV–SMIRNOV TEST

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Approximation for $n > 40$:

\[
\sqrt{n} \quad \sqrt{n} \quad \sqrt{n} \quad \sqrt{n} \quad \sqrt{n}
\]

7.12.9 CRITICAL VALUES, TWO SAMPLE KOLMOGOROV–SMIRNOV TEST

The value of $D$ listed below is so large that the hypothesis $H_0$, the two distributions are the same, is to be rejected at the indicated level of significance. Here, $n_1$ and $n_2$ are assumed to be large, and $D = \max |F_{n_1}(x) - F_{n_2}(x)|$. 

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7.12.10 CRITICAL VALUES, SPEARMAN’S RANK CORRELATION

Spearman’s coefficient of rank correlation, \( \rho_s \), measures the correspondence between two rankings. Let \( d_i \) be the difference between the ranks of the \( i \)th pair of a set of \( n \) pairs of elements. Then Spearman’s rho is defined as

\[
\rho_s = 1 - \frac{6 \sum_{i=1}^{n} d_i^2}{n^3 - n} = 1 - \frac{6S_r}{n^3 - n}
\]

where \( S_r = \sum_{i=1}^{n} d_i^2 \). The table below gives critical values for \( S_r \) when there is complete independence.

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7.13 SIGNAL PROCESSING

7.13.1 ESTIMATION

Let \( \{e_t\} \) be a white noise process (so that \( \mathbb{E}[e_t] = \mu \), variance \( (e_t) = \sigma^2 \), and covariance \( (e_t, e_s) = 0 \) for \( s \neq t \)). Suppose that \( \{X_t\} \) is a time series. A nonanticipating linear model presumes that \( \sum_{n=0}^{\infty} h_n X_{t-n} = e_t \), where the \( \{h_n\} \) are constants. This can be written \( H(z)X_t = e_t \) where \( H(z) = \sum_{n=0}^{\infty} h_n z^n \) and \( z^n X_t = X_{t-n} \). Alternately, \( X_t = H^{-1}(z)e_t \). In practice, several types of models are used:

- **AR(k), autoregressive model of order \( k \):** This assumes that
  \[
  H(z) = 1 + a_1 z + \cdots + a_k z^k
  \]
  and so
  \[
  X_t + a_1 X_{t-1} + \cdots + a_k X_{t-k} = e_t. \tag{7.13.1}
  
- **MA(l), moving average of order \( l \):** This assumes that
  \[
  H^{-1}(z) = 1 + b_1 z + \cdots + b_l z^l
  \]
  and so
  \[
  X_t = e_t + b_1 e_{t-1} + \cdots + b_l e_{t-l}. \tag{7.13.2}
  
- **ARMA(k, l), mixed autoregressive/moving average of order \( k, l \):** This assumes that
  \[
  H^{-1}(z) = \frac{1 + b_1 z + \cdots + b_l z^l}{1 + a_1 z + \cdots + a_k z^k}
  \]
  and so
  \[
  X_t + a_1 X_{t-1} + \cdots + a_k X_{t-k} = e_t + b_1 e_{t-1} + \cdots + b_l e_{t-l}. \tag{7.13.3}
  
7.13.2 FILTERS

1. **Butterworth filter of order \( n \):**
   \[
   |H_a(j\Omega)|^2 = \frac{1}{1 + (j\Omega/j\Omega_c)^{2n}}.
   
2. **Chebyshev filter of order \( n \):**
   \[
   |H_a(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 T_n(\Omega/\Omega_c)}.
   
7.13.3 MATCHED FILTERING (WEINER FILTER)

Let \( X(t) \) represent a signal to be recovered, let \( N(t) \) represent noise, and let \( Y(t) = X(t) + N(t) \) represent the observable signal. A prediction of the signal is

\[
X_p(t) = \int_0^{\infty} K(z)Y(t-z) \, dz,
\]

where \( K(z) \) is a filter. The mean square error is \( \mathbb{E}[(X(t) - X_p(t))^2] \); this is minimized by the optimal (Weiner) filter \( K_{opt}(z) \).
When $X$ and $Y$ are stationary, define their autocorrelation functions as $R_{XX}(t - s) = E[X(t)X(s)]$ and $R_{YY}(t - s) = E[Y(t)Y(s)]$. If $\mathcal{F}$ represents the Fourier transform (see page 530), then the optimal filter is given by

$$\mathcal{F}[K_{\text{opt}}(t)] = \frac{1}{2\pi} \mathcal{F}[R_{XX}(t)]$$

(7.13.4)

For example, if $X$ and $N$ are uncorrelated, then

$$\mathcal{F}[K_{\text{opt}}(t)] = \frac{1}{2\pi} \mathcal{F}[R_{XX}(t)] / [\mathcal{F}[R_{XX}(t)] + \mathcal{F}[R_{NN}(t)]]$$

(7.13.5)

In the case of no noise, $\mathcal{F}[K_{\text{opt}}(t)] = \frac{1}{2\pi}$, $K_{\text{opt}}(t) = \delta(t)$, and $S_p(t) = Y(t)$.

### 7.13.4 KALMAN FILTERING

Kalman filtering is a linear least squares recursive estimator. It is used when the state space has a higher dimension than the observation space. For example, in some airport radars the distance to aircraft is measured and the velocity of each aircraft is inferred. The general case covered by Kalman filtering is

$$x_n = F_n x_{n-1} + G_n w_n$$

$$y_n = H_n x_n + J_n v_n,$$

(7.13.6)

where $x_n$ is the state to be estimated and $y_n$ is the observable. Here, $F_n$, $G_n$, $H_n$, and $J_n$ are matrices ($F_n$ is required to be non-singular), $\{x_n\}$ is real-valued with $x_{-1} = 0$, and $\{w_n\}$ and $\{v_n\}$ are uncorrelated white Gaussian noise processes with $E[w_n] = 0$, $E[w_n w_n^T] = \sigma_w^2 I$, and $E[v_n v_n^T] = \sigma_v^2 I$.

Consider the simpler system

$$x_n = F x_{n-1} + G w_n$$

$$y_n = x_n + v_n.$$  

(7.13.7)

Here, $F$ and $G$ are constant matrices ($F$ is also nonsingular). If $\hat{x}_{n,m}$ is the predictor of $x_n$ using the values $\{y_0, \ldots, y_m\}$, then the Kalman predictor for $x_{n,n-1}$ is

$$\hat{x}_{n,n-1} = F [\hat{x}_{n-1,n-2} + K_{n-1} (y_{n-1} - \hat{x}_{n-1,n-2})]$$

(7.13.8)

and the Kalman predictor for $x_{n,n}$ (called the Kalman filter) is

$$\hat{x}_{n,n} = F \hat{x}_{n-1,n} + K_n (y_n - F \hat{x}_{n-1,n-1})$$

(7.13.9)

with $\hat{x}_{-1,-1} = 0$. Here $K_n$ is the Kalman gain matrix.

Define the error $\tilde{x}_n = \hat{x}_{n,n-1} - x_n$ and its covariance matrix $\epsilon_n = E[\tilde{x}_n \tilde{x}_n^T]$. Then

$$K_n = \epsilon_n (\epsilon_n + \sigma_w^2)^{-1}, \quad \epsilon_n = F \epsilon_{n-1} [I - K_{n-1}^T] F^T + G \sigma_w^2 G^T,$$

which can be solved simultaneously.

For example, consider the system $\{x_n = 0.9 x_{n-1} + w_n, y_n = x_n + v_n\}$ with $x_{-1} = 0$, $\sigma_w^2 = 0.19$, and $\sigma_v^2 = 1$. In this case $K_n = \epsilon_n / (1 + \epsilon_n)$, and $\epsilon_n = (0.19 + \epsilon_{n-1})/(1 + \epsilon_n)$. In the limit, $\epsilon_{\infty} = 0.436 \ldots$, and $K_{\infty} = 0.304 \ldots$. Hence, for large values of $n$, $\hat{x}_{n,n} = 0.626 \hat{x}_{n-1,n-1} + 0.304 y_n$. 

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7.13.5 WALSH FUNCTIONS

The Rademacher functions are defined by \( r_k(x) = \text{sgn} \sin \left( 2^{k+1} \pi x \right) \). If the binary expansion of \( n \) has the form \( n = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_m} \), then the Walsh function of order \( n \) is \( W_n(x) = r_{i_1}(x)r_{i_2}(x) \cdots r_{i_m}(x) \).
7.13.6 WAVELETS

The Haar wavelet is \( H(x) = 1 \) if \( 0 \leq x < 1/2 \), \(-1\) if \( 1/2 \leq x < 1 \), and 0 otherwise. Define \( H_{j,k}(x) = 2^{j/2} H(2^j x - k) \). Then the Haar system, \( \{H_{j,k}\}_{j,k=-\infty}^{+\infty} \), forms an orthonormal basis for the Hilbert space, \( L^2(\mathbb{R}) \), consisting of functions \( f \) with finite energy, i.e., \( \int_{-\infty}^{+\infty} |f(x)|^2 \, dx < \infty \).

Definitions

The construction of other wavelet orthonormal bases \( \{\psi_{j,k}\}_{j,k=-\infty}^{+\infty} \) begins by choosing real coefficients \( h_0, \ldots, h_n \) which satisfy the following conditions (we set \( h_k = 0 \) if \( k < 0 \) or \( k > n \)):

- **Normalization:** \( \sum_k h_k = \sqrt{2} \).
- **Orthogonality:** \( \sum_k h_k h_{k-2^j} = 1 \) if \( j = 0 \) and 0 if \( j \neq 0 \).
- **Accuracy** \( p \): \( \sum_k (-1)^k k^p h_k = 0 \) for \( j = 0, \ldots, p - 1 \) with \( p > 0 \).
- **Cohen–Lawton Criterion:** A technical condition only rarely violated by coefficients which satisfy the normalization, orthogonality, and accuracy \( p \) conditions.

The terms order of approximation or Strang–Fix conditions are often used in place of “accuracy”. The orthogonality condition implies that \( n \) is odd.

The four conditions above imply the existence of a solution \( \varphi \in L^2(\mathbb{R}) \), called the scaling function, to the following refinement equation:

\[
\varphi(x) = \sqrt{2} \sum_{k=0}^{n} h_k \varphi(2x - k).
\]

The scaling function has a nonvanishing integral which we normalize to \( \int \varphi(x) \, dx = 1 \). Then \( \varphi(x) \) is unique, and it vanishes outside of the interval \([0, n]\). The maximum possible accuracy is \( p = (n+1)/2 \). Thus, increasing the accuracy requires increasing the number of coefficients \( h_k \). High accuracy is desirable, as it implies that each of the polynomials \( 1, x, \ldots, x^{p-1} \) can be written as an infinite linear combination of the integer translates \( \varphi(x - k) \). In particular, \( \sum_k \varphi(x - k) = 1 \). Also, the smoothness of \( \varphi \) is limited by the accuracy; \( \varphi \) can have at most \( n - 2 \) derivatives, although in practice it usually has fewer.

For each fixed integer \( j \), let \( V_j \) be the closed subspace of \( L^2(\mathbb{R}) \) spanned by the functions, \( \{\varphi_{j,k}\}_{k=-\infty}^{+\infty} \), where \( \varphi_{j,k}(x) = 2^{j/2} \varphi(2^j x - k) \). The sequence of subspaces \( \{V_j\}_{j=-\infty}^{+\infty} \) forms a multiresolution analysis for \( L^2(\mathbb{R}) \), meaning that:

- The subspaces are nested: \( \cdots \subset V_{-1} \subset V_0 \subset V_1 \cdots \).
- They are obtained from each other by dilation by 2: \( v(x) \in V_j \iff v(2x) \in V_{j+1} \).
- \( V_0 \) is integer translation invariant: \( v(x) \in V_0 \iff v(x + 1) \in V_0 \).
- The \( V_j \) increase to all of \( L^2(\mathbb{R}) \) and decrease to zero: \( \cup V_j \) is dense in \( L^2(\mathbb{R}) \) and \( \cap V_j = \{0\} \).

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The set of integer translates \( \{\psi(x - k)\}_k \) forms an orthonormal basis for \( V_0 \).

The projection of \( f(x) \) onto the subspace \( V_j \) is an approximation at resolution level \( 2^{-j} \). It is given by \( f_j(x) = \sum_k c_{j,k} \varphi_{j,k}(x) \) with \( c_{j,k} = \langle f, \varphi_{j,k} \rangle = \int f(x) \varphi_{j,k}(x) \, dx \).

The wavelet \( \psi \) is derived from the scaling function \( \varphi \) by the formula,

\[
\psi(x) = \sqrt{2} \sum_{k=0}^{n} g_k \varphi(2x - k), \quad \text{where } g_k = (-1)^k h_{n-k}.
\]

The wavelet \( \psi \) has the same smoothness as \( \varphi \), and the accuracy \( p \) condition implies vanishing moments for \( \psi \): \( \int x^p \psi(x) \, dx = 0 \) for \( j = 0, \ldots, p - 1 \). The functions \( \psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k) \) are orthonormal, and the entire collection \( \{\psi_{j,k}\}_{j,k=-\infty}^{\infty} \) forms an orthonormal basis for \( L^2(\mathbb{R}) \).

With \( j \) fixed, let \( W_j \) be the closed subspace of \( L^2(\mathbb{R}) \) spanned by \( \psi(2^j x - k) \) for integer \( k \). Then \( V_j \) and \( W_j \) are orthogonal subspaces whose direct sum is \( V_{j+1} \). Let \( f \) be a function and let \( p_j = \sum_k d_{j,k} \psi_{j,k}(x) \) be its projection onto \( W_j \), where \( d_{j,k} = (f, \psi_{j,k}) \). Then the approximation \( f_{j+1} \) with resolution \( 2^{-(j+1)} \) is \( f_{j+1} = f_j + p_j \), the sum of the approximation \( f_j \) at resolution \( 2^{-j} \) and the additional fine details \( p_j \) needed to give the next higher resolution level.

The discrete wavelet transform is an algorithm for computing the coefficients \( c_{j,k} \) and \( d_{j,k} \) from the coefficients \( c_{j+1,k} \). It can also be interpreted as an algorithm dealing directly with discrete data, dividing data \( c_{j+1,k} \) into a low-pass part \( c_{j,k} \) and a high-pass part \( d_{j,k} \). Specifically,

\[
c_{j,l} = \sum_k h_{k-2l} c_{j+1,k} \quad \text{and} \quad d_{j,l} = \sum_k g_{k-2l} c_{j+1,k}.
\]

The inverse transform is

\[
c_{j+1,k} = \sum_l h_{k-2l} c_{j,l} + \sum_l g_{k-2l} d_{j,l}.
\]

The discrete wavelet transform is closely related to engineering techniques known as sub-band coding and quadrature mirror filtering.

Example: the Daubechies family. For each even integer \( N > 0 \), there is a unique set of coefficients \( h_0, \ldots, h_{N-1} \) which satisfy the normalization and orthogonality conditions with maximal accuracy \( p = N/2 \). The corresponding \( \varphi \) and \( \psi \) are the Daubechies scaling function \( D_N \) and Daubechies wavelet \( W_N \). The Haar wavelet \( H \) is the same as the Daubechies wavelet \( W_2 \). For the Haar wavelet, the coefficients are \( h_0 = h_1 = 1/\sqrt{2} \) and the subspace \( V_j \) consists of all functions which are piecewise constant on each interval \( [k2^{-j}, (k+1)2^{-j}) \). The coefficients for \( D_k \) are: \( h_0 = (1 + \sqrt{3})/(4\sqrt{2}), h_1 = (3 + \sqrt{3})/(4\sqrt{2}), h_2 = (3 - \sqrt{3})/(4\sqrt{2}), \) and \( h_3 = (1 - \sqrt{3})/(4\sqrt{2}) \).

Generalizations

For a given number of coefficients, the Daubechies wavelet has the highest accuracy. Other wavelets reduce the accuracy in exchange for other properties. In the Coiflet family the scaling function and the wavelet possesses vanishing moments,
leading to simple one-point quadrature formulas. The “least asymmetric” wavelets are close to being symmetric or antisymmetric (perfect symmetry is incompatible with orthogonality, except for Haar).

Wavelet packets are libraries of basis functions defined recursively from the scaling functions $\phi$ and $\psi$. The Walsh functions are wavelet packets based on the Haar wavelet.

Allowing infinitely many coefficients results in wavelets supported on the entire real line. The Meyer wavelet is band-limited, possesses infinitely many derivatives, and has accuracy $p = \infty$. The Battle–Lemarié wavelets are piecewise splines and have exponential decay.

$M$-band wavelets replace the ubiquitous dilation factor 2 by another integer $M$. Multiwavelets replace the coefficients $h_k$ by $r \times r$ matrices and the scaling function and wavelet by vector-valued functions $(\varphi_1, \ldots, \varphi_r)$ and $(\psi_1, \ldots, \psi_r)$, resulting in an orthonormal basis for $L^2(\mathbb{R})$ generated by the several wavelets $\varphi_1, \ldots, \varphi_r$.

For higher dimensions, a separable wavelet basis is constructed via a tensor product, with scaling function $\varphi(x)\varphi(y)$ and three wavelets $\varphi(x)\psi(y)$, $\psi(x)\varphi(y)$, $\psi(x)\psi(y)$. Nonseparable wavelets replace the dilation factor 2 by a dilation matrix.

Biorthogonal wavelets allow greater flexibility of design by relaxing the requirement that the wavelet system $\{\psi_{j,k}\}$ form an orthonormal basis to requiring only that it form a Riesz basis. The Cohen–Daubechies–Feauveau wavelets are symmetric and their coefficients $h_k$ are dyadic rationals. The Chui–Wang–Aldroubi–Unser semiorthogonal wavelets are splines with explicit analytic formulas.

The basis condition may be further relaxed by allowing $\{\psi_{j,k}\}$ to be a frame, an overcomplete system with basis-like properties. Introducing further redundancy, the continuous wavelet transform uses all possible dilates and translates $a^{1/2}\psi(ax + b)$.

**Wavelet coefficients and figures**

Coefficients for Daubechies scaling functions$^1$

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<th>$D_2$</th>
<th>$D_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_0$</td>
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</tr>
<tr>
<td>$h_1$</td>
<td>0.707106781187</td>
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<tr>
<td>$h_2$</td>
<td>0.630880767930</td>
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</tbody>
</table>

<table>
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<th>0.482962913145</th>
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</thead>
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<td>$h_1$</td>
<td>0.836516303738</td>
<td></td>
</tr>
<tr>
<td>$h_2$</td>
<td>0.224143868042</td>
<td></td>
</tr>
<tr>
<td>$h_3$</td>
<td>$-0.129409522551$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
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<th>$h_0$</th>
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</thead>
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</tr>
<tr>
<td>$h_2$</td>
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<td></td>
</tr>
<tr>
<td>$h_3$</td>
<td>$-0.135011020010$</td>
<td></td>
</tr>
<tr>
<td>$h_4$</td>
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<td></td>
</tr>
<tr>
<td>$h_5$</td>
<td>0.035226291886</td>
<td></td>
</tr>
</tbody>
</table>

$^1$The tables and figures are taken in part from Table 6.1 and Figure 6.3 of Ingrid Daubechies’ book *Ten Lectures on Wavelets*, published by SIAM Press and used with permission.
\[ D_{10} : \quad h_0: \ 0.160102397974 \quad D_{12} : \quad h_0: \ 0.111540743350 \]
\[ h_1: \ 0.603829269797 \quad h_1: \ 0.494623890398 \]
\[ h_2: \ 0.724308528438 \quad h_2: \ 0.751133908021 \]
\[ h_3: \ 0.138428145901 \quad h_3: \ 0.315250351709 \]
\[ h_4: \ -0.242294887066 \quad h_4: \ -0.226264693965 \]
\[ h_5: \ -0.032244869585 \quad h_5: \ -0.129766867567 \]
\[ h_6: \ 0.077571493840 \quad h_6: \ 0.097501605587 \]
\[ h_7: \ -0.006241490213 \quad h_7: \ 0.027522865530 \]
\[ h_8: \ -0.012580751999 \quad h_8: \ -0.031582039317 \]
\[ h_9: \ 0.003335725285 \quad h_9: \ 0.000553842201 \]
\[ h_{10}: \ 0.004777257511 \quad h_{11}: \ -0.001077301085 \]
References


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REFERENCES

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Numerical Analysis, Fifth Edition, Prindle, Weber & Schmidt (Boston) 1993, by R. L. Burden and J. D. Faires, was the primary reference for most of the information presented in this chapter.

8.1 BASIC NUMERICAL ANALYSIS

8.1.1 APPROXIMATIONS AND ERRORS

Numerical methods involve finding approximate solutions to mathematical problems. Errors of approximation can result from two sources: error inherent in the method or formula used and round off error. Round off error results when a calculator or computer is used to perform real-number calculations with a finite number of significant digits. All but the first specified number of digits are either chopped or rounded to that number of digits.

If $p^*$ is an approximation to $p$, the absolute error is defined to be $|p - p^*|$ and the relative error is $|p - p^*|/|p|$, provided that $p \neq 0$.

Iterative techniques often generate sequences that (ideally) converge to an exact solution. It is sometimes desirable to describe the rate of convergence.

DEFINITION 8.1.1

Suppose $\lim \beta_n = 0$ and $\lim \alpha_n = \alpha$. If a positive constant $K$ exists with $|\alpha_n - \alpha| < K |\beta_n|$ for large $n$, then $\{\alpha_n\}$ is said to converge to $\alpha$ with a rate of convergence $O(\beta_n)$. This is read “big oh of $\beta_n$” and written $\alpha_n = \alpha + O(\beta_n)$.

DEFINITION 8.1.2

Suppose $\{p_n\}$ is a sequence that converges to $p$. If positive constants $\lambda$ and $\alpha$ exist with $\lim_{n \to \infty} |p_{n+1} - p| = \lambda$, then $\{p_n\}$ is said to converge to $p$ of order $\alpha$, with asymptotic error constant $\lambda$.

In general, a higher order of convergence yields a more rapid rate of convergence. A sequence has linear convergence if $\alpha = 1$ and quadratic convergence if $\alpha = 2$.

Aitken’s $\Delta^2$ method

DEFINITION 8.1.3

Given the sequence $\{p_n\}$, define the forward difference $\Delta p_n$ by $\Delta p_n = p_{n+1} - p_n$. 

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for \( n \geq 0 \). Higher powers \( \Delta^k p_n \) are defined recursively by \( \Delta^k p_n = \Delta(\Delta^{k-1} p_n) \) for \( k \geq 2 \). In particular, \( \Delta^2 p_n = \Delta(p_{n+1} - p_n) = p_{n+2} - 2p_{n+1} + p_n \).

If a sequence \( \{p_n\} \) converges linearly to \( p \), the new sequence \( \{\hat{p}_n\} \) generated by

\[
\hat{p}_n = p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n}
\]

for all \( n \geq 0 \), called Aitken’s \( \Delta^2 \) method, satisfies \( \lim_{n \to \infty} \frac{\hat{p}_n - p}{p_n - p} = 0 \).

**Richardson’s extrapolation**

Improved accuracy can be achieved by combining extrapolation with a low-order formula. Suppose the unknown value \( M \) is approximated by a formula \( N(h) \) for which

\[
M = N(h) + K_1 h + K_2 h^2 + K_3 h^3 + \cdots
\]

for some unspecified constants \( K_1, K_2, K_3, \ldots \). To apply extrapolation, set \( N_1(h) = N(h) \), and generate new approximations \( N_j(h) \) by

\[
N_j(h) = N_{j-1}(h) \left( \frac{h}{2} \right)^j + \frac{N_{j-1}(\frac{h}{2}) - N_{j-1}(h)}{2^{j-1} - 1}.
\]

Then \( M = N_j(h) + O(h^j) \). A table of the following form is generated, one row at a time:

| \( N_1(h) \) | \( N_1(\frac{h}{2}) \) | \( N_2(h) \) |
| \( N_1(\frac{h}{2}) \) | \( N_2(\frac{h}{2}) \) | \( N_3(h) \) |
| \( N_1(\frac{h}{4}) \) | \( N_2(\frac{h}{4}) \) | \( N_3(\frac{h}{2}) \) | \( N_4(h) \) |

Extrapolation can be applied whenever the truncation error for a formula has the form \( \sum_{j=1}^{m-1} K_j h^{\alpha_j} + O(h^{\alpha_m}) \) for constants \( K_j \) and \( \alpha_1 < \alpha_2 < \cdots < \alpha_m \). In particular, if \( \alpha_j = 2j \), the following computation can be used:

\[
N_j(h) = N_{j-1}(h) \left( \frac{h}{2} \right)^j + \frac{N_{j-1}(\frac{h}{2}) - N_{j-1}(h)}{4^{j-1} - 1},
\]

where the entries in the \( j \)th column of the table have order \( O(h^{2j}) \).

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8.1.2 SOLUTION TO ALGEBRAIC EQUATIONS

Iterative methods generate sequences \( \{p_n\} \) that converge to a solution \( p \) of an equation.

**DEFINITION 8.1.4**

A solution \( p \) of \( f(x) = 0 \) is a zero of multiplicity \( m \) if \( f(x) \) can be written as \( f(x) = (x - p)^m q(x) \), for \( x \neq p \), where \( \lim_{x \to p} q(x) \neq 0 \). A zero is called simple if \( m = 1 \).

**Fixed point iteration**

A fixed point \( p \) for a function \( g \) satisfies \( g(p) = p \). Given \( p_0 \), generate \( \{p_n\} \) by

\[
p_{n+1} = g(p_n) \quad \text{for } n \geq 0.
\]

(8.1.5)

If \( \{p_n\} \) converges, then it will converge to a fixed point of \( g \) and the value \( p_n \) can be used as an approximation for \( p \). The following theorem gives conditions that guarantee convergence.

**THEOREM 8.1.1** *(Fixed point theorem)*

Let \( g \in C[a, b] \) and suppose that \( g(x) \in [a, b] \) for all \( x \) in \([a, b]\). Suppose, in addition, that \( g' \) exists on \((a, b)\) with \( |g'(x)| \leq k < 1 \), for all \( x \in (a, b) \). If \( p_0 \) is any number in \([a, b]\), then the sequence defined by Equation (8.1.5) converges to the (unique) fixed point \( p \) in \([a, b]\), and \( |p_n - p| \leq \frac{k^n}{1-k} |p_0 - p_1| \) for all \( n \geq 1 \).

The iteration sometimes converges even if all the conditions are not satisfied.

**THEOREM 8.1.2**

Suppose \( g \) is a function that satisfies the conditions of Theorem 8.1.1 and \( g' \) is also continuous on \((a, b)\). If \( g'(p) \neq 0 \), then for any number \( p_0 \) in \([a, b]\), the sequence generated by Equation (8.1.5) converges only linearly to the unique fixed point \( p \) in \([a, b]\).

**THEOREM 8.1.3**

Let \( p \) be a solution of the equation \( x = g(x) \). Suppose that \( g'(p) = 0 \) and \( g'' \) is continuous and strictly bounded by \( M \) on an open interval \( I \) containing \( p \). Then there exists a \( \delta > 0 \) such that, for \( p_0 \in [p - \delta, p + \delta] \), the sequence defined by Equation (8.1.5) converges at least quadratically to \( p \).

**Steffensen’s method**

For a linearly convergent fixed point iteration, convergence can be accelerated by applying Aitken’s \( \Delta^2 \) method. This is called Steffensen’s method. Define \( p_0^{(0)} = p_0 \).
compute $p_1^{(0)} = g(p_0^{(0)})$ and $p_2^{(0)} = g(p_1^{(0)})$. Set $p_0^{(1)} = \hat{p}_0$ which is computed using
Equation (8.1.1) applied to $p_0^{(0)}$, $p_1^{(0)}$ and $p_2^{(0)}$. Use fixed point iteration to compute
$p_1^{(1)}$ and $p_2^{(1)}$ and then Equation (8.1.1) to find $p_0^{(2)}$. Continuing, generate \{p^{(n)}_0\}.

THEOREM 8.1.4

Suppose that $x = g(x)$ has the solution $p$ with $g'(p) \neq 1$. If there exists a $\delta > 0$ such
that $g \in C^3[p - \delta, p + \delta]$, then Steffensen’s method gives quadratic convergence for
the sequence \{p^{(n)}_0\} for any $p_0 \in [p - \delta, p + \delta]$.

Newton–Raphson method (Newton’s method)

To solve $f(x) = 0$, given an initial approximation $p_0$, generate \{p_n\} using

$$p_{n+1} = p_n \frac{f(p_n)}{f'(p_n)}, \quad \text{for } n \geq 0 \hspace{1cm} (8.1.6)$$

Figure 8.1.1 describes the method geometrically. Each value $p_{n+1}$ represents the
$x$-intercept of the tangent line to the graph of $f(x)$ at the point $[p_n, f(p_n)]$.

THEOREM 8.1.5

Let $f \in C^2[a, b]$. If $p \in [a, b]$ is such that $f(p) = 0$ and $f'(p) \neq 0$, then there
exists a $\delta > 0$ such that Newton’s method generates a sequence \{p_n\} converging to
$p$ for any initial approximation $p_0 \in [p - \delta, p + \delta]$.

1From R.L. Burden and J.D. Faires, Numerical Analysis, 5th ed., Prindle, Weber & Schmidt, Boston,
Note:

1. Generally the conditions of the theorem cannot be checked. Therefore one usually generates the sequence \( \{p_n\} \) and observes whether or not it converges.

2. An obvious limitation is that the iteration terminates if \( f'(p_n) = 0 \).

3. For simple zeros of \( f' \), Theorem 8.1.3 implies that Newton’s method converges quadratically. Otherwise, the convergence is much slower.

**Modified Newton’s method**

Newton’s method converges only linearly if \( p \) has multiplicity larger that one. However, the function \( u(x) = \frac{f(x)}{f'(x)} \) has a simple zero at \( p \). Hence, the Newton iteration formula applied to \( u(x) \) yields quadratic convergence to a root of \( f(x) = 0 \). The iteration simplifies to

\[
p_{n+1} = p_n - \frac{f(p_n) f'(p_n)}{[f'(p_n)]^2 - f(p_n) f''(p_n)}, \quad \text{for } n \geq 0.
\]  

(8.1.7)

**Secant method**

To solve \( f(x) = 0 \), the secant method uses the \( x \)-intercept of the secant line passing through \((p_n, f(p_n))\) and \((p_{n-1}, f(p_{n-1}))\). The derivative of \( f \) is not needed. Given \( p_0 \) and \( p_1 \), generate the sequence with

\[
p_{n+1} = p_n - \frac{(p_n - p_{n-1})}{f(p_n) - f(p_{n-1})}, \quad \text{for } n \geq 1.
\]  

(8.1.8)

**Root bracketing methods**

Suppose \( f(x) \) is continuous on \([a, b]\) and \( f(a) f(b) < 0 \). The intermediate value theorem guarantees a number \( p \in [a, b] \) exists with \( f(p) = 0 \). A root bracketing method constructs a sequence of nested intervals \([a_n, b_n]\), each containing a solution of \( f(x) = 0 \). At each step, compute \( p_n \in [a_n, b_n] \) and proceed as follows:

If \( f(p_n) = 0 \), stop the iteration and \( p = p_n \).

Else,

if \( f(a_n) f(p_n) < 0 \), then set \( a_{n+1} = a_n, b_{n+1} = p_n \).

Else, set \( a_{n+1} = p_n, b_{n+1} = b_n \).

**Bisection method**

This is a special case of the root bracketing method. The values \( p_n \) are computed by

\[
p_n = a_n + \frac{b_n - a_n}{2} = \frac{a_n + b_n}{2}, \quad \text{for } n \geq 1.
\]  

(8.1.9)
Clearly, \( |p_n - p| \leq (b - a)/2^n \) for \( n \geq 1 \). The rate of convergence is \( O(2^{-n}) \). Although convergence is slow, the exact number of iterations for a specified accuracy \( \epsilon \) can be determined. To guarantee that \( |p_N - p| < \epsilon \), use

\[
N > \log_2 \left( \frac{b - a}{\epsilon} \right) = \frac{\ln(b - a) - \ln \epsilon}{\ln 2}.
\] (8.1.10)

False position (\textit{regula falsi})

\[
p_n = b_n - f(b_n) \frac{b_n - a_n}{f(b_n) - f(a_n)}, \quad \text{for } n \geq 1.
\] (8.1.11)

Both methods converge, provided the initial criteria are satisfied.

Horner’s method with deflation

If Newton’s method is used to solve for roots of the polynomial \( P(x) = 0 \), then the polynomials \( P \) and \( P' \) are repeatedly evaluated. Horner’s method efficiently evaluates a polynomial of degree \( n \) using only \( n \) multiplications and \( n \) additions.

Horner’s algorithm

To evaluate \( P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \) and its derivative at \( x_0 \) \( (y = P(x_0), z = P'(x_0)) \),

**INPUT:** degree \( n \), coefficients \( \{a_0, a_1, \ldots, a_n\} \); \( x_0 \),

**OUTPUT:** \( y = P(x_0); z = P'(x_0) \).

**Algorithm**

1. Set \( y = a_n; z = a_n \).
2. For \( j = n - 1, n - 2, \ldots, 1 \),
   set \( y = x_0 y + a_j; z = x_0 z + y \).
3. Set \( y = x_0 y + a_0 \).
4. OUTPUT \((y, z)\). STOP.

When satisfied with the approximation \( \hat{x}_1 \) for a root \( x_1 \) of \( P \), use synthetic division to compute \( Q_1(x) \) so that \( P(x) \approx (x - \hat{x}_1) Q_1(x) \). Estimate a root of \( Q_1(x) \) and write \( P(x) \approx (x - \hat{x}_1)(x - \hat{x}_2) Q_2(x) \), and so on. Eventually, \( Q_{n-2}(x) \) will be a quadratic, and the quadratic formula can be applied. This procedure, finding one root at a time, is called deflation.

Note: Care must be taken since \( \hat{x}_1 \) is an approximation for \( x_1 \). Some inaccuracy occurs when computing the coefficients of \( Q_1(x) \), etc. Although the estimate \( x_2 \) of a root of \( Q_2(x) \) can be very accurate, it may not be as accurate when estimating a root of \( P(x) \).
8.1.3 INTERPOLATION

Interpolation involves fitting a function to a set of data points \((x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\). The \(x_i\) are unique and the \(y_i\) may be regarded as the values of some function \(f(x)\), that is, \(y_i = f(x_i)\) for \(i = 0, 1, \ldots, n\). Polynomial interpolation methods are discussed below.

Lagrange interpolation

The **Lagrange interpolating polynomial**, denoted \(P_n(x)\), is the unique polynomial of degree at most \(n\) for which \(P(x_k) = f(x_k)\) for \(k = 0, 1, \ldots, n\). It is given by

\[
P(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x)\tag{8.1.12}
\]

where \(\{x_0, \ldots, x_n\}\) are called **node points**.

\[
L_{n,k}(x) = \frac{(x - x_0)(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)(x_k - x_1)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)},
\]

\[
= \prod_{i=0, i\neq k}^{n} \frac{(x - x_i)}{(x_k - x_i)}, \quad \text{for } k = 0, 1, \ldots, n.\tag{8.1.13}
\]

**THEOREM 8.1.6** (*Error formula*)

If \(x_0, x_1, \ldots, x_n\) are distinct numbers in \([a, b]\) and \(f \in C^{n+1}[a, b]\), then, for each \(x\) in \([a, b]\), a number \(\xi(x)\) in \((a, b)\) exists with

\[
f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n + 1)!} (x - x_0)(x - x_1)\cdots(x - x_n),
\]

where \(P\) is the interpolating polynomial given in Equation (8.1.12).

Although the Lagrange polynomial is unique, it can be expressed and evaluated in several ways. Equation (8.1.12) is tedious to evaluate, and including more nodes affects the entire expression. **Neville’s method** evaluates the Lagrange polynomial at a single point **without** explicitly finding the polynomial and adapts easily to the addition of nodes.

Neville’s method

Let \(P_{m_1, m_2, \ldots, m_k}\) denote the Lagrange polynomial using distinct nodes \(\{x_{m_1}, x_{m_2}, \ldots, x_{m_k}\}\). If \(P(x)\) denotes the Lagrange polynomial using nodes \(\{x_0, x_1, \ldots, x_k\}\) and \(x_i\) and \(x_j\) are two distinct numbers in this set, then

\[
P(x) = \frac{(x - x_j)P_{0,1,\ldots,j-1,i,j+1,\ldots,k}(x) - (x - x_i)P_{0,1,\ldots,i-1,j+1,\ldots,k}(x)}{(x_i - x_j)}. \tag{8.1.14}
\]
**Neville’s algorithm**

Generate a table of entries \( Q_{i,j} \) for \( j \geq 0 \) and \( 0 \leq i \leq j \) where

\[
Q_{i,j} = P_{i-j,i-j+1,...,i-1,i}.
\]

Calculations use Equation (8.1.14) for a specific value of \( x \) as shown:

\[
\begin{align*}
x_0 & \quad Q_{0,0} = P_0, \\
x_1 & \quad Q_{1,0} = P_1, \quad Q_{1,1} = P_{0,1}, \\
x_2 & \quad Q_{2,0} = P_2, \quad Q_{2,1} = P_{1,2}, \quad Q_{2,2} = P_{0,1,2}, \\
x_3 & \quad Q_{3,0} = P_3, \quad Q_{3,1} = P_{2,3}, \quad Q_{3,2} = P_{1,2,3}, \quad Q_{3,3} = P_{0,1,2,3}.
\end{align*}
\]

Note that \( P_k = P_k(x) = f(x_k) \) and \( \{Q_{i,j}\} \) represents successive estimates of \( f(x) \) using Lagrange polynomials. Nodes may be added until \( |Q_{i,i} - Q_{i-1,i-1}| < \epsilon \) as desired.

Some interpolation formulae involve divided differences. Given a sequence \( \{x_i\} \) and corresponding function values \( f(x_i) \), the zeroth divided difference is \( f[x_i] = f(x_i) \). The first divided difference is defined by

\[
f[x_i, x_{i+1}] = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}.
\] (8.1.15)

The \( k \)th divided difference is defined by

\[
f[x_{i_1}, x_{i_2}, \ldots, x_{i_k}, x_{i_k+1}] = \frac{f[x_{i_1}, x_{i_2}, \ldots, x_{i_k}] - f[x_{i_1}, x_{i_2}, \ldots, x_{i_k+1}]}{x_{i_k+1} - x_i}.
\] (8.1.16)

Divided differences are usually computed by forming a triangular table.

### Divided differences

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>First divided differences</th>
<th>Second divided differences</th>
<th>Third divided differences</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_0 )</td>
<td>( f[x_0] )</td>
<td>( f[x_0] )</td>
<td>( f[x_0, x_1] )</td>
<td>( f[x_0, x_1, x_2] )</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>( f[x_1] )</td>
<td>( f[x_1] )</td>
<td>( f[x_1, x_2] )</td>
<td>( f[x_1, x_2, x_3] )</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>( f[x_2] )</td>
<td>( f[x_2] )</td>
<td>( f[x_1, x_2, x_3] )</td>
<td>( f[x_0, x_1, x_2, x_3] )</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>( f[x_3] )</td>
<td>( f[x_2, x_3] )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Newton’s interpolatory divided-difference formula**

(also known as the Newton polynomial)

\[
P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x - x_0) \cdots (x - x_{k-1}).
\] (8.1.17)

Labeling the nodes as \( \{x_n, x_{n-1}, \ldots, x_0\} \), a formula similar to Equation (8.1.17) results in **Newton’s backward divided-difference formula**.
\[ P_n(x) = f[x_n] + f[x_{n-1}, x_n](x - x_n) \\
+ f[x_{n-2}, x_{n-1}, x_n](x - x_n)(x - x_{n-1}) \\
+ \cdots + f[x_0, x_1, \ldots, x_n](x - x_n) \cdots (x - x_1). \quad (8.1.18) \]

If the nodes are equally-spaced (that is, \( x_i - x_{i-1} = h \)), define the parameter \( s \) by the equation \( x = x_0 + sh \). The following formulae evaluate \( P_n(x) \) at a single point:

1. Newton’s interpolatory divided-difference formula,
\[
P_n(x) = P_n(x_0 + sh) = \sum_{k=0}^{n} \binom{s}{k} k! h^k f[x_0, x_1, \ldots, x_k]. \tag{8.1.19} \]

2. Newton’s forward-difference formula (Newton–Gregory),
\[
P_n(x) = P_n(x_0 + sh) = \sum_{k=0}^{n} \binom{s}{k} \Delta^k f(x_0). \tag{8.1.20} \]

3. Newton–Gregory backward formula (fits nodes \( x_{-n} \) to \( x_0 \)),
\[
P_n(x) = f(x_0) + \binom{s}{1} \Delta f(x_{-1}) + \binom{s+1}{2} \Delta^2 f(x_{-2}) + \cdots + \binom{s+n-1}{n} \Delta^n f(x_{-n}). \tag{8.1.21} \]

4. Newton’s backward-difference formula,
\[
P_n(x) = \sum_{k=0}^{n} (-1)^k \binom{-s}{k} \nabla^k f(x_n), \tag{8.1.22} \]

where \( \nabla^k f(x_n) \) is the \( k \)th backward difference defined by \( \nabla p_n = p_n - p_{n-1} \) for \( n \geq 1 \). Higher powers are defined recursively by \( \nabla^k p_n = \nabla(\nabla^{k-1} p_n) \) for \( k \geq 2 \).

5. Stirling’s formula (for equally-spaced nodes \( x_{-m}, \ldots, x_{-1}, x_0, x_1, \ldots, x_m \)),
\[
P_n(x) = P_{2m+1}(x) = f[x_0] + \frac{sh}{2} (f[x_{-1}, x_0] + f[x_0, x_1]) + \\
\frac{s^2 h^2}{2} f[x_{-1}, x_0, x_1] + \frac{s(s^2 - 1)h^3}{2} (f[x_{-1}, x_0, x_1, x_2] + f[x_{-2}, x_{-1}, x_0, x_1]) \\
+ \cdots + \frac{s^2(s^2 - 1)(s^2 - 4) \cdots (s^2 - (m-1)^2)h^{2m} f[x_{-m}, \ldots, x_m]}{2} \\
+ \frac{s(s^2 - 1) \cdots (s^2 - m^2)h^{2m+1}}{2} (f[x_{-m}, \ldots, x_{m+1}] + f[x_{m-1}, \ldots, x_m]). \]

Use the entire formula if \( n = 2m + 1 \) is odd, and omit the last term if \( n = 2m \) is even. The following table identifies the desired divided differences used in Stirling’s formula:
Inverse interpolation

Any method of interpolation which does not require the nodes to be equally-spaced may be applied by simply interchanging the nodes (x values) and the function values (y values).

Hermite interpolation

Given node points \( \{x_0, x_1, \ldots, x_n\} \), the Hermite interpolating polynomial for a function \( f \) is the unique polynomial \( H(x) \) of degree at most \( 2n + 1 \) that satisfies \( H(x_i) = f(x_i) \) and \( H'(x_i) = f'(x_i) \) for each \( i = 0, 1, \ldots, n \).

A technique and formula similar to Equation (8.1.17) can be used. For distinct nodes \( \{x_0, x_1, \ldots, x_n\} \), define \( \{z_0, z_1, \ldots, z_{2n+1}\} \) by \( z_{2i} = z_{2i+1} = x_i \) for \( i = 0, 1, \ldots, n \). Construct a divided difference table for the ordered pairs \( \{z_i, f(z_i)\} \) using \( f'(x_i) \) in place of \( f[z_i, z_{i+1}] \), which would be undefined. Denote the Hermite polynomial by \( H_{2n+1}(x) \).

Hermite interpolating polynomial

\[
H_{2n+1}(x) = f[z_0] + \sum_{k=1}^{2n+1} f[z_0, z_1, \ldots, z_k](x - z_0) \cdots (x - z_{k-1})
= f[z_0] + f[z_0, z_1](x - x_0) + f[z_0, z_1, z_2](x - x_0)^2
+ f[z_0, z_1, z_2, z_3](x - x_0)^2(x - x_1)
+ \cdots + f[z_0, \ldots, z_{2n+1}](x - x_0)^2 \cdots (x - x_{n-1})^2(x - x_n).\] (8.1.23)

THEOREM 8.1.7 (Error formula)

If \( f \in C^{(2n+2)}[a, b] \), then

\[
f(x) = H_{2n+1}(x) + \frac{f^{(2n+2)}(\xi(x))}{(2n + 2)!} (x - x_0)^2 \cdots (x - x_n)^2
\]

for some \( \xi \in (a, b) \) and where \( x_i \in [a, b] \) for each \( i = 0, 1, \ldots, n \).
8.1.4 FITTING EQUATIONS TO DATA

Piecewise polynomial approximation

An interpolating polynomial has large degree and tends to oscillate greatly for large data sets. **Piecewise polynomial approximation** divides the interval into a collection of subintervals and constructs an approximating polynomial on each subinterval. **Piecewise linear interpolation** consists of simply joining the data points with line segments. This collection is continuous but not differentiable at the node points. **Hermite polynomials** would require derivative values. **Cubic spline interpolation** is popular since no derivative information is needed.

**DEFINITION 8.1.5**

Given a function \( f \) defined on \([a, b]\) and a set of numbers \( a = x_0 < x_1 < \cdots < x_n = b \), a cubic spline interpolant, \( S \), for \( f \) is a function that satisfies

1. \( S \) is a piecewise cubic polynomial; denoted \( S_j \) on \([x_j, x_{j+1}]\) for each \( j = 0, 1, \ldots, n - 1 \).
2. \( S(x_j) = f(x_j) \) for each \( j = 0, 1, \ldots, n \).
3. \( S_{j+1}(x_{j+1}) = S_j(x_{j+1}) \) for each \( j = 0, 1, \ldots, n - 2 \).
4. \( S'_{j+1}(x_{j+1}) = S'_j(x_{j+1}) \) for each \( j = 0, 1, \ldots, n - 2 \).
5. \( S''_{j+1}(x_{j+1}) = S''_j(x_{j+1}) \) for each \( j = 0, 1, \ldots, n - 2 \).
6. One of the following sets of boundary conditions is satisfied:
   - \( S''(x_0) = S''(x_n) = 0 \) (free or natural boundary),
   - \( S'(x_0) = f'(x_0) \) and \( S'(x_n) = f'(x_n) \) (clamped boundary).

If a function \( f \) is defined at all node points, then \( f \) has a unique natural spline interpolant. If, in addition, \( f \) is differentiable at \( a \) and \( b \), then \( f \) has a unique clamped spline interpolant. To construct a cubic spline, set

\[
S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3
\]

for each \( j = 0, 1, \ldots, n - 1 \). The constants \( \{a_j, b_j, c_j, d_j\} \) are found by solving a triadiagonal system of linear equations which is included in the following algorithms.

**Algorithm for natural cubic splines**

**INPUT:** \( n \), \( \{x_0, x_1, \ldots, x_n\} \),
\[ a_0 = f(x_0), a_1 = f(x_1), \ldots, a_n = f(x_n). \]

**OUTPUT:** \( \{a_j, b_j, c_j, d_j\} \) for \( j = 0, 1, \ldots, n - 1 \).

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Algorithm for clamped cubic splines

INPUT: \( n, \{x_0, x_1, \ldots, x_n\} \),
\[ a_0 = f(x_0), a_1 = f(x_1), \ldots, a_n = f(x_n), \]
\[ F_0 = f'(x_0), F_n = f'(x_n). \]
OUTPUT: \( (a_j, b_j, c_j, d_j) \) for \( j = 0, 1, \ldots, n - 1 \).

Algorithm

1. For \( i = 0, 1, \ldots, n - 1 \), set \( h_i = x_{i+1} - x_i \).
2. For \( i = 1, 2, \ldots, n - 1 \), \( \alpha_i = \frac{3}{h_i} (a_{i+1} - a_i) - \frac{1}{h_{i-1}} (a_i - a_{i-1}) \).
3. Set \( l_0 = 1, \mu_0 = 0, z_0 = 0 \).
4. For \( i = 1, 2, \ldots, n - 1 \),
   \[ l_i = 2(x_{i+1} - x_i) - h_{i-1} \mu_{i-1}, \]
   \[ \mu_i = \frac{h_i}{l_i}, \ z_i = \frac{\alpha_i - h_{i-1} z_{i-1}}{h_i}. \]
5. Set \( l_n = 1, z_n = 0, c_n = 0 \).
6. For \( j = n - 1, n - 2, \ldots, 0 \),
   \[ c_j = z_j - \mu_j c_{j+1}, \ b_j = \frac{(a_{j+1} - a_j)}{h_j} - \frac{h_j (c_{j+1} + 2 c_j)}{3}, \]
   \[ d_j = \frac{(c_{j+1} - c_j)}{3 h_j}. \]
7. OUTPUT \( (a_j, b_j, c_j, d_j) \) for \( j = 0, 1, \ldots, n - 1 \). STOP.

Discrete least squares approximation

Another approach to fit a function to a set of data points \( \{(x_i, y_i) \mid i = 1, 2, \ldots, m\} \) is least squares approximation. If a polynomial of degree \( n \) is used, the polynomial
\[ P_n(x) = \sum_{k=0}^{n} a_k x^k \]
is found that minimizes the least squares error
\[ E = \sum_{i=1}^{m} (y_i - P_n(x_i))^2. \]
To find \( \{a_0, a_1, \ldots, a_n\} \), solve the linear system, called the normal equations, created by setting partial derivatives of \( E \) taken with respect to each \( a_k \) equal to zero.
The coefficient of \( a_0 \) in the first equation is actually the number of data points \( m \).
Normal equations

\[
\begin{align*}
\sum_{i=1}^{m} a_0 x_i^0 &= \sum_{i=1}^{m} y_i, \\
\sum_{i=1}^{m} a_1 x_i &= \sum_{i=1}^{m} y_i x_i, \\
\sum_{i=1}^{m} a_2 x_i^2 &= \sum_{i=1}^{m} y_i x_i, \\
&\vdots \\
\sum_{i=1}^{m} a_n x_i^n &= \sum_{i=1}^{m} y_i x_i^n.
\end{align*}
\]

Note: \( P_n(x) \) can be replaced by a function \( f \) of specified form. Unfortunately, to minimize \( E \), the resulting system is generally not linear. Although these systems can be solved, one technique is to “linearize” the data. For example, if \( y = f(x) = be^{ax} \), then \( \ln y = \ln b + ax \). The least squares method applied to the data points \((x_i, \ln y_i)\) produces a linear system. Note that this technique does not find the least squares approximation for the original problem but, instead, approximates the “linearized” data.

**Best fit line**

Given the points \( P_1(x_1, y_1), P_2(x_1, y_1), \ldots P_n(x_1, y_1) \) the line of best fit is given by \( y - \bar{y} = m(x - \bar{x}) \) where

\[
\begin{align*}
\bar{x} &= \frac{1}{n} \sum_{i=1}^{n} x_i = \frac{x_1 + x_2 + \cdots + x_n}{n}, \\
\bar{y} &= \frac{1}{n} \sum_{i=1}^{n} y_i = \frac{y_1 + y_2 + \cdots + y_n}{n}, \\
m &= \frac{(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n) - n \bar{x} \bar{y}}{(x_1^2 + x_2^2 + \cdots + x_n^2) - n \bar{x}^2} = \frac{\bar{xy} - \bar{x} \bar{y}}{\bar{x}^2 - \bar{x}^2}.
\end{align*}
\]

# 8.2 NUMERICAL LINEAR ALGEBRA

## 8.2.1 SOLVING LINEAR SYSTEMS

The solution of systems of linear equations using Gaussian elimination with backward substitution is described in Section 8.2.2. The algorithm is highly sensitive to round off error. Pivoting strategies can reduce round off error when solving an \( n \times n \) system.
For a linear system $Ax = b$, assume that the equivalent matrix equation $A^{(k)}x = b^{(k)}$ has been constructed. Call the entry, $a_{k,k}^{(k)}$, the pivot element.

### 8.2.2 GAUSSIAN ELIMINATION

To solve the system $Ax = b$, Gaussian elimination creates the augmented matrix

$A' = [A : b] = \begin{bmatrix} a_{11} & \ldots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \ldots & a_{nn} & b_n \end{bmatrix}$.

This matrix is turned into an upper triangular matrix by a sequence of (1) row permutations, and (2) subtracting a multiple of one row from another. The result is a matrix of the form (the primes denote that the quantities have been modified)

$\begin{bmatrix} a'_{11} & a'_{12} & \ldots & a'_{1n} & b'_1 \\ 0 & a'_{22} & \ldots & a'_{2n} & b'_2 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & a'_{nn} & b'_n \end{bmatrix}$.

This matrix represents a system of linear equations (just as $A'$ did), equivalent to the original system. Back substitution can be used to determine, successively, $\{x_n, x_{n-1}, \ldots\}$.

### 8.2.3 GAUSSIAN ELIMINATION ALGORITHM

**INPUT:** number of unknowns and equations $n$, matrix $A$, and vector $b$.

**OUTPUT:** solution to linear system $x = (x_1, \ldots, x_n) = A^{-1}b$, or message that system does not have a unique solution.

**Algorithm**


2. For $i = 1, 2, \ldots, n - 1$ do the following: *(Elimination process)*
   
   (a) Let $p$ be the least integer with $i \leq p \leq n$ and $a'_{pi} \neq 0$
   
   If no integer can be found, then
   
   OUTPUT("no unique solution exists"). STOP.

   (b) If $p \neq i$ interchange rows $p$ and $i$ in $A'$. Call the new matrix $A'$.

   (c) For $j = i + 1, \ldots, n$ do the following:

   i. Set $m_{ij} = a'_{ij}/a'_{ii}$.

   ii. Subtract from row $j$ the quantity ($m_{ij}$ times row $i$).

3. If $a'_{nn} = 0$ then OUTPUT("no unique solution exists"). STOP.

4. Set $x_n = a'_{n,n+1}/a'_{nn}$. *(Start backward substitution)*
5. For $i = n - 1, \ldots, 2, 1$ set $x_i = \left[ a'_{i,n+1} - \sum_{j=i+1}^{n} a'_{i,j} x_j \right] / a'_{i,i}$.

6. OUTPUT $(x_1, \ldots, x_n)$. *(Procedure completed successfully)*. STOP.

---

### 8.2.4 PIVOTING

#### Maximal column pivoting

Maximal column pivoting (often called partial pivoting) finds, at each step, the element in the same column as the pivot element that lies on or below the main diagonal having the largest magnitude and moves it to the pivot position. Determine the least $p \geq k$ such that $|a^{(k)}_{p,k}| = \max_{k \leq i \leq n} |a^{(k)}_{i,k}|$ and interchange the $k^{th}$ equation with the $p^{th}$ equation before performing the elimination step.

#### Scaled-column pivoting

Scaled-column pivoting sometimes produces better results, especially when the elements of $A$ differ greatly in magnitude. The desired pivot element is chosen to have the largest magnitude relative to the other values in its row. For each row define a scale factor $s_i$ by $s_i = \max_{1 \leq j \leq n} |a_{i,j}|$. The desired pivot element at the $k^{th}$ step is determined by choosing the smallest integer $p$ with $\frac{|a^{(k)}_{p,k}|}{s_p} = \max_{k \leq j \leq n} \frac{|a^{(k)}_{j,k}|}{S_j}$.

#### Maximal (or complete) pivoting

The desired pivot element at the $k^{th}$ step is the entry of largest magnitude among $\{a_{i,j}\}$ with $i = k, k+1, \ldots, n$ and $j = k, k+1, \ldots, n$. Both row and column interchanges are necessary and additional comparisons are required, resulting in additional execution time.

---

### 8.2.5 EIGENVALUE COMPUTATION

#### Power method

Assume that the $n \times n$ matrix $A$ has $n$ eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ with independent eigenvectors $\{v^{(1)}, v^{(2)}, \ldots, v^{(n)}\}$. Assume further that $A$ has a unique dominant eigenvector $\lambda_1$, where $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$. For any $x \in \mathbb{R}^n$, $x = \sum_{j=1}^{n} a_j v^{(j)}$.

The algorithm is called the power method because powers of the input matrix are taken: $\lim_{k \to \infty} A^k x = \lim_{k \to \infty} \lambda_1^k \alpha_1 v^{(1)}$. However, this sequence converges to zero, if $|\lambda_1| < 1$, and diverges, if $|\lambda_1| \geq 1$, provided $\alpha_1 \neq 0$. Appropriate scaling of $A^k x$ is necessary to obtain a meaningful limit. Begin by choosing a unit vector $x^{(0)}$ having a component $x_{p_0}$ so that $x_{p_0}^{(0)} = 1 = \|x^{(0)}\|_{\infty}$. ©1996 CRC Press LLC
The algorithm inductively constructs sequences of vectors \( \{x^{(m)}\}_{m=0}^{\infty} \) and \( \{y^{(m)}\}_{m=0}^{\infty} \) and a sequence of scalars \( \{\mu^{(m)}\}_{m=1}^{\infty} \) by

\[
y^{(m)} = Ax^{(m-1)}, \quad \mu^{(m)} = y^{(m)}_{p_m}, \quad x^{(m)} = \frac{y^{(m)}}{y^{(m)}_{p_m}},
\]

(8.2.1)

where, at each step, \( p_m \) represents the least integer for which \( |y^{(m)}_{p_m}| = \|y^{(m)}\|_\infty \).

The sequence of scalars satisfies \( \lim_{m \to \infty} \mu^{(m)} = \lambda_1 \), provided \( \alpha_1 \neq 0 \), and the sequence of vectors \( \{x^{(m)}\}_{m=0}^{\infty} \) converges to an eigenvector of \( L_\infty \) associated with \( \lambda_1 \).

**Power method algorithm**

INPUT: dimension \( n \), matrix \( A \), vector \( x \), tolerance \( TOL \), and maximum iterations \( N \).

OUTPUT: approximate eigenvalue \( \mu \),

approximate eigenvector \( x \) (with \( \|x\|_\infty = 1 \)),

or a message that the maximum number of iterations was exceeded.

Algorithm

1. Set \( k = 1 \).
2. Find an integer \( p \) with \( 1 \leq p \leq n \) and \( |x_p| = \|x\|_\infty \).
3. Set \( x = \frac{1}{y_p} x \).
4. While \( (k \leq N) \) do the following:
   (a) Set \( y = Ax \).
   (b) Set \( \mu = y_p \).
   (c) Find an integer \( p \) with \( 1 \leq p \leq n \) and \( |y_p| = \|y\|_\infty \).
   (d) If \( y_p = 0 \) then OUTPUT (“Eigenvector”, \( x \ ));
       OUTPUT (“corresponds to eigenvalue 0;
        select a new vector \( x \) and restart”). STOP.
   (e) Set \( ERR = \left\| x - \frac{1}{y_p} y \right\|_\infty \); \( x = \frac{1}{y_p} y \).
   (f) If \( ERR < TOL \) then OUTPUT \( (\mu, x) \)
       (procedure successful). STOP.
   (g) Set \( k = k + 1 \).
5. OUTPUT (“Maximum number of iterations exceeded”). STOP.

Notes:

1. The method does not really require that \( \lambda_1 \) be unique. If the multiplicity is greater than one, the eigenvector obtained depends on the choice of \( x^{(0)} \).
2. The sequence constructed converges linearly, so that Aitken’s \( \Delta^2 \) method (Equation (8.1.1)) can be applied to accelerate convergence.
Inverse power method

The inverse power method modifies the power method to yield faster convergence by finding the eigenvalue of $A$ that is closest to a specified number $q$. Assume that $A$ satisfies the conditions as before. If $q \neq \lambda_i$, for $i = 1, 2, \ldots, n$, the eigenvalues of $(A - qI)^{-1}$ are $\frac{1}{\lambda_i - q}, \frac{1}{\lambda_2 - q}, \ldots, \frac{1}{\lambda_n - q}$ with the same eigenvectors $v^{(1)}, \ldots, v^{(n)}$.

Apply the power method to $(A - qI)^{-1}$. At each step, $y^{(m)} = (A - qI)^{-1}x^{(m-1)}$. Generally, $y^{(m)}$ is found by solving $(A - qI)y^{(m)} = x^{(m-1)}$ using Gaussian elimination with pivoting. Choose the value $q$ from an initial approximation to the eigenvector $x^{(0)}$ by $q = \frac{x^{(0)T}A x^{(0)}}{x^{(0)T}x^{(0)}}$.

The only changes in the algorithm for the power method (see page 685) are in setting an initial value $q$ as described (do this prior to step 1), determining $y^{(m)}$ by solving the linear system $(A - qI)y^{(m)} = x^{(m-1)}$ in step (4a), deleting step (4d), and replacing step (4f) with

$$\text{if } ERR < TOL \text{ then set } \mu = \frac{1}{\mu} + q; \quad \text{OUTPUT}(\mu, x); \quad \text{STOP}.$$  

Wielandt deflation

Once the dominant eigenvalue has been found, remaining eigenvalues can be found by using deflation techniques. A new matrix $B$ is formed having the same eigenvalues as $A$, except that the dominant eigenvalue of $A$ is replaced by 0. One method is Wielandt deflation which defines $x = \frac{1}{v_{(1)}^{(i)}}[a_1, a_2, \ldots, a_n]^T$, where $v_{(1)}^{(i)}$ is a coordinate of $v^{(1)}$ that is nonzero, and the values $\{a_1, a_2, \ldots, a_n\}$ are the entries in the $i$th row of $A$. Then the matrix $B = A - \lambda_1 v^{(1)}x^T$ has eigenvalues $0, \lambda_2, \lambda_3, \ldots, \lambda_n$ with associated eigenvectors $\{v^{(1)}, w^{(2)}, w^{(3)}$, $\ldots, w^{(n)}\}$, where

$$v^{(i)} = (\lambda_i - \lambda_1)w^{(i)} + \lambda_1 (x^T w^{(i)})v^{(1)}$$

for $i = 2, 3, \ldots, n$. The $i$th row of $B$ consists entirely of zero entries and $B$ may be replaced with an $(n - 1) \times (n - 1)$ matrix $B'$ obtained by deleting the $i$th row and $i$th column of $B$. The power method can be applied to $B'$ to find its dominant eigenvalue and so on.

8.2.6 HOUSEHOLDER’S METHOD

**DEFINITION 8.2.1**

Two $n \times n$ matrices $A$ and $B$ are said to be similar if a nonsingular matrix $S$ exists with $A = S^{-1}BS$.

Householder’s method constructs a symmetric tridiagonal matrix $B$ that is similar to a given symmetric matrix $A$. After applying this method, the QR algorithm can be used efficiently to approximate the eigenvalues of the resulting symmetric tridiagonal matrix.

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Algorithm for Householder’s method

To construct a symmetric tridiagonal matrix $A^{(n-1)}$ similar to the symmetric matrix $A = A^{(1)}$, construct matrices $A^{(2)}, A^{(3)}, \ldots, A^{(n-1)}$, where $A^{(k)} = (a_{ij}^{(k)})$ for $k = 1, 2, \ldots, n-1$.

INPUT: dimension $n$, matrix $A$.
OUTPUT: $A^{(n-1)}$. (At each step, $A$ can be overwritten.)

Algorithm

1. For $k = 1, 2, \ldots, n-2$ do the following steps:
   (a) Set $q = \sum_{j=k+1}^{n} (a_{jk}^{(k)})^2$.
   (b) If $a_{k+1,k}^{(k)} = 0$, then set $\alpha = -q^{\frac{1}{2}}$; else, set $\alpha = -q^{\frac{1}{2}} a_{k+1,k}^{(k)} / |a_{k+1,k}^{(k)}|$.
   (c) Set $RSQ = \alpha^2 - a_{k+1,k}^{(k)}$.
   (d) Set $v_k = 0$. (Note: $v_1 = \cdots = v_{k-1} = 0$, but are not needed.)
      Set $v_{k+1} = a_{k+1,k}^{(k)} - \alpha$. For $j = k + 1, \ldots, n$, set $v_j = a_{j,k}^{(k)}$.
   (e) For $j = k, k+1, \ldots, n$, set $u_j = \frac{1}{RSQ} \sum_{i=k+1}^{n} a_{ij}^{(k)} v_i$.
   (f) Set $PROD = \sum_{i=k+1}^{n} v_i u_i$.
   (g) For $j = k, k+1, \ldots, n$, set $z_j = u_j - (\frac{PROD}{RSQ}) v_j$.
   (h) For $l = k + 1, k + 2, \ldots, n-1$ do the following steps:
      i. For $j = l + 1, \ldots, n$, set $a_{ij}^{(k+1)} = a_{ij}^{(k)} - v_l z_j - v_j z_l$;
         $a_{ij}^{(k+1)} = a_{ij}^{(k+1)}$.
      ii. Set $a_{il}^{(k+1)} = a_{il}^{(k)} - 2 v_l z_l$.
   (i) Set $a_{nn}^{(k+1)} = a_{nn}^{(k)} - 2 v_n z_n$.
   (j) For $j = k + 2, \ldots, n$, set $a_{kj}^{(k+1)} = a_{kj}^{(k+1)} = 0$.
   (k) Set $a_{k+1,k}^{(k+1)} = a_{k+1,k}^{(k)} - v_{k+1} z_k$; $a_{k+1,k}^{(k+1)} = a_{k+1,k}^{(k+1)}$.
      (Note: The other elements of $A^{(k+1)}$ are the same as $A^{(k)}$.)

2. OUTPUT $A^{(n-1)}$. STOP.
   ($A^{(n-1)}$ is symmetric, tridiagonal, similar to $A$.)

8.2.7 QR ALGORITHM

The QR algorithm is generally used (instead of deflation) to determine all of the eigenvalues of a symmetric matrix. The matrix must be symmetric and in tridiagonal form. If necessary, first apply Householder’s method. Suppose the matrix $A$ has the form
If \( b_2 = 0 \) or \( b_n = 0 \), then \( A \) has the eigenvalue \( a_1 \) or \( a_n \), respectively. If \( b_j = 0 \) for some \( j, 2 < j < n \), the problem is reduced to considering, instead of \( A \), the smaller matrices,

\[
\begin{bmatrix}
  a_1 & b_2 & 0 & \cdots & 0 & 0 & 0 \\
  b_2 & a_2 & b_3 & 0 & \cdots & 0 & 0 \\
  0 & b_3 & a_3 & 0 & \cdots & 0 & 0 \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  0 & 0 & 0 & a_{n-2} & b_{n-1} & 0 \\
  0 & 0 & 0 & b_{n-1} & a_{n-1} & b_n \\
  0 & 0 & 0 & \cdots & 0 & b_n & a_n
\end{bmatrix}
\] and

\[
\begin{bmatrix}
  a_1 & b_2 & 0 & \cdots & 0 \\
  b_2 & a_2 & b_3 & 0 & \cdots & 0 \\
  0 & b_3 & a_3 & 0 & \cdots & 0 \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  0 & 0 & a_{j-1} & b_{j-1} \\
  0 & 0 & b_{j-1} & a_{j-1}
\end{bmatrix}
\] and

\[
\begin{bmatrix}
  a_1 & b_2 & 0 & \cdots & 0 & 0 & 0 \\
  b_2 & a_2 & b_3 & 0 & \cdots & 0 & 0 \\
  0 & b_3 & a_3 & 0 & \cdots & 0 & 0 \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  0 & 0 & 0 & a_{n-2} & b_{n-1} & 0 \\
  0 & 0 & 0 & b_{n-1} & a_{n-1} & b_n \\
  0 & 0 & 0 & \cdots & 0 & b_n & a_n
\end{bmatrix}
\]

If none of the \( b_j \) are zero, the QR algorithm forms \( \{A^{(1)} = A, A^{(2)}, A^{(3)}, \ldots \} \) as follows:

1. \( A^{(1)} = A \) is factored as \( A^{(1)} = Q^{(1)}R^{(1)} \), with \( Q^{(1)} \) orthogonal and \( R^{(1)} \) upper triangular.
2. \( A^{(2)} = R^{(1)}Q^{(1)} \) and is factored as \( A^{(2)} = Q^{(2)}R^{(2)} \), with \( Q^{(2)} \) orthogonal and \( R^{(2)} \) upper triangular.

In general, \( A^{(i+1)} = R^{(i)}Q^{(i)} = (Q^{(i)T}A^{(i)})Q^{(i)} = Q^{(i)T}A^{(i)}Q^{(i)} \). Note that \( A^{(i+1)} \) is symmetric and tridiagonal with the same eigenvalues as \( A^{(i)} \) and, hence, has the same eigenvalues as \( A \).

**Algorithm for QR**

To obtain eigenvalues of the symmetric tridiagonal \( n \times n \) matrix

\[
A \equiv A_1 = \begin{bmatrix}
  a_1^{(1)} & b_2^{(1)} & 0 & \cdots & 0 & 0 & 0 \\
  b_2^{(1)} & a_2^{(1)} & b_3^{(1)} & 0 & \cdots & 0 & 0 \\
  0 & b_3^{(1)} & a_3^{(1)} & 0 & \cdots & 0 & 0 \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  0 & 0 & 0 & a_{n-2}^{(1)} & b_{n-1}^{(1)} & 0 \\
  0 & 0 & 0 & b_{n-1}^{(1)} & a_{n-1}^{(1)} & b_n^{(1)} \\
  0 & 0 & 0 & \cdots & 0 & b_n^{(1)} & a_n^{(1)}
\end{bmatrix}
\]

**INPUT:** \( n; a_1^{(1)}, \ldots, a_n^{(1)}, b_2^{(1)}, \ldots, b_n^{(1)} \), tolerance \( TOL \), and maximum iterations \( M \).
OUTPUT: eigenvalues of \( A \), or recommended splitting of \( A \),
or a message that the maximum number of iterations was exceeded.

Algorithm

1. Set \( k = 1 \), \( SHIFT = 0 \). (Accumulated shift)
2. While \( k \leq M \), do steps 3–8.
3. Test for success
   (a) If \( |b_n^{(k)}| \leq TOL \), then set \( \lambda = a_n^{(k)} + SHIFT \);
       OUTPUT \( \lambda \); set \( n = n - 1 \);
   (b) If \( |b_j^{(k)}| \leq TOL \) for \( 3 \leq j \leq n - 1 \), then
       OUTPUT (”split into”, \( \{a_1^{(k)}, \ldots, a_{j-1}^{(k)}, b_2^{(k)}, \ldots, b_{j-1}^{(k)}\} \),
       and \( \{a_j^{(k)}, \ldots, a_n^{(k)}, b_{j+1}^{(k)}, \ldots, b_n^{(k)}\} \), \( SHIFT \); STOP.
   (c) If \( |b_2^{(k)}| \leq TOL \) then, set \( \lambda = a_1^{(k)} + SHIFT \);
       OUTPUT (\( \lambda \));
       Set \( n = n - 1 \); \( a_1^{(k)} = a_2^{(k)} \);
       For \( j = 2, \ldots, n \), set \( a_j^{(k)} = a_{j+1}^{(k)}; b_j^{(k)} = b_{j+1}^{(k)} \).
   • Set \( b = -(a_{n-1}^{(k)} + a_n^{(k)}), c = a_n^{(k)} a_{n-1}^{(k)} - [b_n^{(k)}]^2 \), and
     \( d = (b^2 - 4c)^{\frac{1}{2}} \).
   • If \( b > 0 \), then set \( \mu_1 = \frac{-2c}{b + d}, \mu_2 = \frac{-b + d}{2} \).
   Else, set \( \mu_1 = \frac{d - b}{2} \) and \( \mu_2 = \frac{2c}{d - b} \).
   • If \( n = 2 \), then set \( \lambda_1 = \mu_1 + SHIFT \),
     \( \lambda_2 = \mu_2 + SHIFT \); OUTPUT (\( \lambda_1, \lambda_2 \) \). STOP.
   • Choose \( s \) so that \( |s - a_n^{(k)}| = \min \{|\mu_1 - a_n^{(k)}|, |\mu_2 - a_n^{(k)}|\} \).
5. Accumulate shift. Set \( SHIFT = SHIFT + s \).
6. Perform shift. For \( j = 1, \ldots, n \), set \( d_j = a_j^{(k)} - s \).
7. Compute \( R^{(k)} \).
   (a) Set \( x_1 = d_1; y_1 = b_2 \);
   (b) For \( j = 2, \ldots, n \),
     set \( z_{j-1} = (x_{j-1}^2 + [b_j^{(k)}]^2)^{\frac{1}{2}}, c_j = \frac{z_{j-1}}{z_{j-2}}, \)
     Set \( S_j = \frac{b_j^{(k)}}{c_{j-1}}, q_{j-1} = c_j y_{j-1} + S_j d_j \), and
     set \( x_j = -S_j y_{j-1} + c_j d_j \).
   (c) If \( j \neq n \) then, set \( r_j-1 = S_j b_{j+1}^{(k)}, y_j = c_j b_{j+1}^{(k)} \).
     \( A_j^{(k)} = P_j A_{j-1}^{(k)} \) has been computed (\( P_j \) is a rotation matrix) and
     \( R^{(k)} = A_n^{(k)} \).
8. Compute \( A^{(k+1)} \).
   (a) Set \( z_n = x_n, a_1^{(k+1)} = s_2 q_1 + c_2 z_1, b_2^{(k+1)} = s_2 z_2 \).
(b) For $j = 2, 3, \ldots, n - 1$, set
\[ a_j^{(k+1)} = s_{j+1} q_j + c_j c_{j+1} z_j, \]
and
\[ b_j^{(k+1)} = s_{j+1} z_{j+1}. \]
(c) Set $a_n^{(k+1)} = c_n z_n$, $k = k + 1$.

9. OUTPUT (“Maximum iterations exceeded”); (Procedure unsuccessful.) STOP.

8.2.8 NONLINEAR SYSTEMS AND NUMERICAL OPTIMIZATION

Newton’s method

Many iterative methods exist for solving systems of nonlinear equations. Newton’s method is a natural extension of his method for solving a single equation in one variable. Convergence is generally quadratic but usually requires an initial approximation that is near the true solution. Assume $F(x) = 0$ where $x$ is an $n$-dimensional vector, $F : \mathbb{R}^n \to \mathbb{R}^n$, and $0$ is the zero vector, that is,

\[ F(x_1, x_2, \ldots, x_n) = (f_1(x_1, x_2, \ldots, x_n), \ldots, f_n(x_1, x_2, \ldots, x_n))^T. \]

A fixed point iteration is performed on $G(x) = x - (J(x))^{-1}F(x)$ where $J(x)$ is the Jacobian matrix,

\[
J(x) = \begin{bmatrix}
\frac{\partial f_1(X)}{\partial x_1} & \frac{\partial f_1(X)}{\partial x_2} & \cdots & \frac{\partial f_1(X)}{\partial x_n} \\
\frac{\partial f_2(X)}{\partial x_1} & \frac{\partial f_2(X)}{\partial x_2} & \cdots & \frac{\partial f_2(X)}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n(X)}{\partial x_1} & \frac{\partial f_n(X)}{\partial x_2} & \cdots & \frac{\partial f_n(X)}{\partial x_n}
\end{bmatrix}.
\] (8.2.2)

The iteration is given by

\[ x^{(k)} = G(x^{(k-1)}) = x^{(k-1)} - [J(x^{(k-1)})]^{-1}F(x^{(k-1)}), \] (8.2.3)

The algorithm avoids calculating $J(x)^{-1}$ at each step. Instead, it finds a vector $y$ so that $J(x^{(k-1)})y = -F(x^{(k-1)})$, and then it sets $x^{(k)} = x^{(k-1)} + y$.

For the special case of a two-dimensional system (the equations $f(x, y) = 0$ and $g(x, y) = 0$ are to be satisfied), Newton’s iteration becomes:

\[
\begin{align*}
x_{n+1} &= x_n - \left. \begin{vmatrix}
f g_y - f y g_x \\
f x g_y - f y g_x
\end{vmatrix} \right|_{x=x_n, y=y_n}, \\
y_{n+1} &= y_n - \left. \begin{vmatrix}
fg_y - f y g_x \\
f x g_y - f y g_x
\end{vmatrix} \right|_{x=x_n, y=y_n}.
\end{align*}
\]
Method of steepest descent

The method of steepest descent determines the local minimum for a function of the form $g: \mathbb{R}^n \to \mathbb{R}$. It can also be used to solve a system $\{f_i\}$ of nonlinear equations. The system has a solution $x = (x_1, x_2, \ldots, x_n)^T$ when the function,

$$g(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} [f_i(x_1, x_2, \ldots, x_n)]^2,$$

has the minimal value zero.

This method converges only linearly to the solution but it usually converges even for poor initial approximations. It can be used to locate initial approximations that are close enough so that Newton’s method will converge. Intuitively, a local minimum for a function $g: \mathbb{R}^n \to \mathbb{R}$ can be found as follows:

1. Evaluate $g$ at an initial approximation $x^{(0)} = (x_1^{(0)}, \ldots, x_n^{(0)})^T$.
2. Determine a direction from $x^{(0)}$ that results in a decrease in the value of $g$.
3. Move an appropriate distance in this direction and call the new vector $x^{(1)}$.
4. Repeat steps 1 through 3 with $x^{(0)}$ replaced by $x^{(1)}$.

The direction of greatest decrease in the value of $g$ at $x$ is the direction given by $-\nabla g(x)$ where $\nabla g(x)$ is the gradient of $g$.

**DEFINITION 8.2.2**

If $g: \mathbb{R}^n \to \mathbb{R}$, the gradient of $g$ at $x = (x_1, x_2, \ldots, x_n)$, denoted $\nabla g(x)$, is

$$\nabla g(x) = \left( \frac{\partial g}{\partial x_1}(x), \frac{\partial g}{\partial x_2}(x), \ldots, \frac{\partial g}{\partial x_n}(x) \right).$$

Thus, set $x^{(1)} = x^{(0)} - \alpha \nabla g(x^{(0)})$ for some constant $\alpha > 0$. Ideally the value of $\alpha$ minimizes the function $h(\alpha) = g(x^{(0)} - \alpha \nabla g(x^{(0)}))$. Instead of tedious direct calculation, the method interpolates $h$ using a quadratic polynomial $P$ and three numbers $\alpha_1$, $\alpha_2$, and $\alpha_3$ that are hopefully close to the minimum value of $h$.

**Algorithm for steepest descent**

To approximate a solution $p$ to the minimization problem $g(p) = \min_{x \in \mathbb{R}^n} g(x)$, given an initial approximation $x$,

**INPUT:** number $n$ of variables, initial $x = (x_1, x_2, \ldots, x_n)^T$, tolerance $TOL$, and maximum iterations $N$.

**OUTPUT:** approximate solution $x = (x_1, x_2, \ldots, x_n)^T$ or a message of failure.
Algorithm

1. Set $k = 0$.

2. While $(k \leq N)$, do the following steps:
   (a) Set: $g_1 = g(x_1, \ldots, x_n)$ (Note: $g_1 = g(x^{(k)})$);
       $z = \nabla g(x_1, \ldots, x_n)$ (Note: $z = \nabla g(x^{(k)})$);
       $z_0 = \|z\|_2$.
   (b) If $z_0 = 0$ then OUTPUT ("Zero gradient");
       OUTPUT $(x_1, \ldots, x_n, g_1)$
       (procedure completed, may have a minimum). STOP.
   (c) Set $z = \frac{z_0}{z_0}$. Make $z$ a unit vector.
       Set $\alpha_1 = 0$, $\alpha_3 = 1$, $g_3 = g(x - \alpha_3 z)$.
   (d) While $(g_3 \geq g_1)$, do the following steps:
       i. Set $\alpha_3 = \frac{\alpha_3}{2}$, $g_3 = g(x - \alpha_3 z)$.
       ii. If $\alpha_3 < TOL$, then
           OUTPUT ("no likely improvement");
           OUTPUT $(x_1, \ldots, x_n, g_1)$
           (procedure completed, may not have a minimum). STOP.
   (e) Set $\alpha_2 = \frac{\alpha_2}{2}$, $g_2 = g(x - \alpha_2 z)$.
   (f) Set $h_1 = \frac{(a_2 - a_3)}{a_2}$, $h_2 = \frac{(a_3 - a_2)}{(a_3 - a_2)}$, $h_3 = \frac{(h_2 - h_3)}{a_3}$.
   (g) Set $\alpha_0 = \frac{1}{2}(\alpha_2 - \frac{h_1}{h_3})$ (critical point occurs at $\alpha_0$).
       set $g_0 = g(x - \alpha_0 z)$.
   (h) Find $\alpha$ from $\{\alpha_0, \alpha_3\}$ so that $g = g(x - \alpha z) = \min\{g_0, g_3\}$.
       i. Set $x = x - \alpha z$.
       j. If $|g - g_1| < TOL$ then OUTPUT $(x_1, \ldots, x_n, g)$
          (procedure completed successfully). STOP.
   (k) Set $k = k + 1$.

3. OUTPUT ("maximum iterations exceeded");
   (procedure unsuccessful). STOP.
8.3 NUMERICAL INTEGRATION AND DIFFERENTIATION

8.3.1 NUMERICAL INTEGRATION

Numerical quadrature involves estimating \( \int_a^b f(x) \, dx \) using a formula of the form

\[
\int_a^b f(x) \, dx \approx \sum_{i=0}^n c_i f(x_i).
\]  

(8.3.1)

Newton–Cotes formulae

A closed Newton–Cotes formula uses nodes \( x_i = x_0 + i h \) for \( i = 0, 1, \ldots, n \), where \( h = (b - a)/n \). Note that \( x_0 = a \) and \( x_n = b \).

An open Newton–Cotes formula uses nodes \( x_i = x_0 + i h \) for \( i = 0, 1, \ldots, n \), where \( h = (b - a)/(n + 2) \). Here \( x_0 = a + h \) and \( x_n = b - h \). Set \( x_{-1} = a \) and \( x_{n+1} = b \). The nodes actually used lie in the open interval \((a, b)\).

In all formulae, \( \xi \) is a number for which \( a < \xi < b \) and \( f_i \) denotes \( f(x_i) \).

Closed Newton–Cotes formulae

1. \((n = 1)\) trapezoidal rule

\[
\int_a^b f(x) \, dx = \frac{h}{2} [f(x_0) + f(x_1)] - \frac{h^3}{12} f''(\xi).
\]

2. \((n = 2)\) Simpson’s rule

\[
\int_a^b f(x) \, dx = \frac{h}{3} [f(x_0) + 4 f(x_1) + f(x_2)] - \frac{h^5}{90} f^{(4)}(\xi).
\]

3. \((n = 3)\) Simpson’s three-eighths rule

\[
\int_a^b f(x) \, dx = \frac{3h}{8} [f(x_0) + 3 f(x_1) + 3 f(x_2) + f(x_3)] - \frac{3h^5}{80} f^{(4)}(\xi).
\]

4. \((n = 4)\) Milne’s rule

\[
\int_a^b f(x) \, dx = \frac{2h}{45} [7 f_0 + 32 f_1 + 12 f_2 + 32 f_3 + 7 f_4] - \frac{8h^7}{945} f^{(6)}(\xi).
\]

5. \((n = 5)\)

\[
\int_a^b f(x) \, dx = \frac{5h}{288} [19 f_0 + 75 f_1 + 50 f_2 + 50 f_3 + 75 f_4 + 19 f_5] - \frac{275h^7}{12096} f^{(6)}(\xi).
\]

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6. \( (n = 6) \) Weddle’s rule
\[
\int_a^b f(x) \, dx = \frac{h}{140} \left[ 41f_0 + 216f_1 + 27f_2 + 272f_3 + 27f_4 + 216f_5 + 41f_6 \right]
- \frac{9h^9}{1400} f^{(8)}(\xi).
\]

Open Newton–Cotes formulae
1. \( (n = 0) \) midpoint rule
\[
\int_a^b f(x) \, dx = 2hf(x_0) + \frac{h^3}{3} f''(\xi).
\]
2. \( (n = 1) \)
\[
\int_a^b f(x) \, dx = \frac{3h}{2}[f(x_0) + f(x_1)] + \frac{3h^3}{4} f''(\xi).
\]
3. \( (n = 2) \)
\[
\int_a^b f(x) \, dx = \frac{4h}{3}[2f(x_0) - f(x_1) + 2f(x_2)] + \frac{14h^5}{45} f^{(4)}(\xi).
\]
4. \( (n = 3) \)
\[
\int_a^b f(x) \, dx = \frac{5h}{24}[11f(x_0) + f(x_1) + f(x_2) + 11f(x_3)] + \frac{95h^5}{144} f^{(4)}(\xi).
\]

Composite rules
Some Newton–Cotes formulae extend to composite formulae. In the following note that \( a < \mu < b \).

1. Composite trapezoidal rule for \( n \) subintervals: If \( f \in C^2[a, b] \), \( h = \frac{b-a}{n} \), and \( x_j = a + jh \), for \( j = 0, 1, \ldots, n \), then
\[
\int_a^b f(x) \, dx = \frac{h}{2} \left[ f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right] - \frac{b-a}{12} h^2 f''(\mu).
\]

2. Composite Simpson’s rule for \( n \) subintervals: If \( f \in C^4[a, b] \), \( n \) is even, \( h = \frac{b-a}{n} \), and \( x_j = a + jh \), for \( j = 0, 1, \ldots, n \), then
\[
\int_a^b f(x) \, dx = \frac{h}{3} \left[ f(a) + 2 \sum_{j=1}^{(n/2)-1} f(x_{2j}) + 4 \sum_{j=1}^{n/2} f(x_{2j-1}) + f(b) \right]
- \frac{b-a}{180} h^4 f^{(4)}(\mu).
\]
3. Composite midpoint rule for \( n^2 + 1 \) subintervals: If \( f \in C^2[a, b] \), \( n \) is even, \( h = \frac{b-a}{n+2} \), and \( x_j = a + (j+1)h \), for \( j = -1, 0, 1, \ldots, n+1 \), then
\[
\int_a^b f(x) \, dx = 2h \sum_{j=0}^{n^2} f(x_{2j}) + \frac{b-a}{6} h^2 f''(\mu).
\]

**Method H.2.1.2 Romberg integration**

Romberg integration uses the composite trapezoidal rule beginning with \( h_1 = b - a \) and \( h_k = \frac{h_{k-1}}{2} \), for \( k = 1, 2, \ldots \), to give preliminary estimates for \( \int_a^b f(x) \, dx \) and improves the estimates with Richardson’s extrapolation. However, since many function evaluations would be repeated, the first column of the extrapolation table (with entries \( R_{i,j} \)) can be more efficiently determined by the following recursion formula:

\[
R_{1,1} = \frac{h_1}{2} \left[ f(a) + f(b) \right] = \frac{b-a}{2} \left[ f(a) + f(b) \right]
\]
\[
R_{k,1} = \frac{1}{2} \left[ R_{k-1,1} + h_{k-1} \sum_{i=1}^{2^{k-2}} f(a + (2i-1)h_k) \right], \quad (8.3.2)
\]

for \( k = 2, 3, \ldots \). Now apply Equation (8.1.4) to complete the extrapolation table.

**Gaussian quadrature**

A quadrature formula, whose nodes (abscissas) \( x_i \) and coefficients \( w_i \) are chosen to achieve a maximum order of accuracy, is called a Gaussian quadrature formula. The integrand usually involves a weight function \( w \). An integral in \( r \) on an interval \((a, b)\) must be converted into an integral in \( x \) over the interval \((\alpha, \beta)\) specified for the weight function involved. This can be accomplished by the transformation \( x = \frac{(b-\alpha)t + (\beta-\alpha)}{(b-a)} \), for \( \alpha < \xi < \beta \). Gaussian quadrature formulae generally take the form
\[
\int_\alpha^\beta w(x) f(x) \, dx = \sum_i w_i f(x_i) + E_n \quad (8.3.3)
\]

where \( E_n = K_n f^{(2n)}(\xi) \) for some \( \alpha < \xi < \beta \). Many popular weight functions and their associated intervals are summarized in the following table:
<table>
<thead>
<tr>
<th>$w(x)$</th>
<th>Interval $(\alpha, \beta)$</th>
<th>Abcissas are zeros of $x_i$</th>
<th>$K_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$(-1, 1)$</td>
<td>$P_n(x)$</td>
<td>$\frac{2^{2n+1}(n!)^4}{(2n+1)[(2n)!]^2}$</td>
</tr>
<tr>
<td>$e^{-x}$</td>
<td>$(0, \infty)$</td>
<td>$L_n(x)$</td>
<td>$\frac{(n!)^2 x_i}{(n+1)^2 L_n^{(2)}(x_i)}$</td>
</tr>
<tr>
<td>$e^{-x^2}$</td>
<td>$(-\infty, \infty)$</td>
<td>$H_n(x)$</td>
<td>$\frac{2^{n-1}n!\sqrt{\pi}}{n^2 H_n^{(2)}(x_i)}$</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{1-x^2}}$</td>
<td>$(-1, 1)$</td>
<td>$T_n(x)$</td>
<td>$\frac{\cos\left(\frac{(2i-1)\pi}{2n}\right)}{2n}$</td>
</tr>
<tr>
<td>$\sqrt{1-x^2}$</td>
<td>$(-1, 1)$</td>
<td>$U_n(x)$</td>
<td>$\frac{\cos\left(\frac{i\pi}{n+1}\right)}{n+1} \sin^2\left(\frac{i\pi}{n+1}\right)$</td>
</tr>
<tr>
<td>$\sqrt{\frac{x}{1-x}}$</td>
<td>$(0, 1)$</td>
<td>$T_{2n+1}(\sqrt{x})$</td>
<td>$\frac{2\pi}{2n+1} \cos^2\left(\frac{(2i-1)\pi}{4n+2}\right)$</td>
</tr>
<tr>
<td>$\sqrt{\frac{1-x}{1+x}}$</td>
<td>$(0, 1)$</td>
<td>$J_n\left(x, \frac{1}{2}, \frac{1}{2}\right)$</td>
<td>$\frac{4\pi}{2n+1} \sin^2\left(\frac{i\pi}{2n+1}\right)$</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{x}}$</td>
<td>$(0, 1)$</td>
<td>$P_n(\sqrt{x})$</td>
<td>$(x_i^*)^2$</td>
</tr>
<tr>
<td>$\sqrt{x}$</td>
<td>$(0, 1)$</td>
<td>$P_{2n+1}(\sqrt{x})$</td>
<td>$(x_i^*)^2$</td>
</tr>
</tbody>
</table>

In this table, $P_n$, $L_n$, $H_n$, $T_n$, $U_n$, and $J_n$ denote the $n^{th}$ Legendre, Laguerre, Hermite, Chebyshev (first kind $T_n$, second kind $U_n$), and Jacobi polynomials, respectively. Also, $x_i^*$ denotes the $i^{th}$ positive root of $P_{2n+1}(x)$ and $h_i$ denotes the corresponding weight for $x_i^*$ in the Gauss–Legendre formula ($w(x) = 1$).

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The following tables give abscissas and weights for selected formulae. If some $x_i$ are specified (such as one or both end points), then the formulae of Radau and Lobatto may be used.

**Gauss–Legendre quadrature**

Weight function is $w(x) = 1$.

$$\int_{-1}^{1} f(x) \, dx \approx \sum_{i=1}^{n} w_i f(x_i).$$

<table>
<thead>
<tr>
<th>$n$</th>
<th>Nodes ${\pm x_i}$</th>
<th>Weights ${w_i}$</th>
<th>$n$</th>
<th>Nodes ${\pm x_i}$</th>
<th>Weights ${w_i}$</th>
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</table>
Gauss–Laguerre quadrature

Weight function is \( w(x) = e^{-x} \).

\[
\int_{0}^{\infty} e^{-x} f(x) \, dx \approx \sum_{i=1}^{n} w_i f(x_i).
\]

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Gauss–Hermite quadrature

Weight function is \( w(x) = e^{-x^2} \).

\[
\int_{-\infty}^{\infty} e^{-x^2} f(x) \, dx \approx \sum_{i=1}^{n} w_i f(x_i).
\]

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Radau quadrature

\[ \int_{-1}^{1} f(x) \, dx \approx \frac{2}{n(n - 1)} \left[ f(-1) + f(1) \right] + \sum_{i=1}^{n-1} w_i f(x_i) + \frac{2^{2n-1} n(n-1)!^4}{(2n-1)!^3} f^{(2n-1)}(\xi), \]

where \( x_i \) is the \( i \)th root of \( \frac{P_{n-1}(x)+P_n(x)}{x+1} \) and \( w_i = \frac{1-x_i}{n(P_{n-1}(x_i))} \) for \( i = 1, \ldots, n - 1 \).

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Lobatto quadrature

\[
\int_{-1}^{1} f(x) \, dx \approx w_1 f(-1) + w_n f(1) \\
+ \sum_{i=2}^{n-1} w_i f(x_i) - \frac{n(n-1)^{3}2^{2n-1}[n-2]!^4}{(2n-1)(2n-2)!^3} f^{(2n-2)}(\xi)
\]

where \(x_i\) is the \((i-1)\)th root of \(P_{n-1}^{(1)}(x)\) and \(w_i = \frac{2}{n(n-1)[P_{n-1}^{(1)}(x_i)]^2}\) for \(i = 1, \ldots, n-2\).

Note that \(w_1 = w_n = \frac{1}{n^2}\).

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Chebyshev quadrature

\[ \int_{-1}^{1} f(x) \, dx \approx \frac{2}{n} \sum_{i=1}^{n} f(x_i). \]

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Monte–Carlo methods

Monte–Carlo methods, in general, involve the generation of random numbers (actually pseudorandom when computer-generated) to represent independent, uniform random variables over \([0,1]\). A simulation can provide insight into the solutions of very complex problems. Refer to Section 7.5 for terminology and notation.

Monte–Carlo methods are generally not competitive with other numerical methods of this section. However, if the function fails to have continuous derivatives of moderate order, those methods may not be applicable. Monte–Carlo methods can be extended to multidimensional integrals quite easily although here only a few techniques for one-dimensional integrals \(I = \int_{a}^{b} g(x) \, dx\) are given.

Hit or miss method

Suppose \(0 \leq g(x) \leq c, a \leq x \leq b, \Omega = \{(x, y) \mid a \leq x \leq b, 0 \leq y \leq c\}\). If \((X, Y)\) is a random vector which is uniformly distributed over \(\Omega\), then the probability \(p\) that \((X, Y)\) lies in \(S\) (see Figure 8.3.2) is \(p = \frac{I}{c(b-a)}\).

If \(N\) independent random vectors \(\{(X_i, Y_i)\}_{i=1}^{N}\) are generated, the parameter \(p\) can be estimated by \(\hat{p} = \frac{N_H}{N}\) where \(N_H\) is the number of times \(Y_i \leq g(X_i), i = 1, 2, \ldots, N,\) called the number of “hits.” (Likewise \(N - N_H\) is the number

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of “misses.”) The value of $I$ is then estimated by the unbiased estimator $\theta_1 = c(b - a)N_H / N$.

**Hit or miss algorithm**

1. Generate a sequence $\{U_j\}_{j=1}^{2N}$ of $2N$ random numbers, uniformly distributed in $[0, 1)$.
2. Arrange the sequence into $N$ pairs $(U_1, U'_1), (U_2, U'_2), \ldots, (U_N, U'_N)$, so that each $U_j$ is used exactly once.
3. Compute $X_i = a + U_i(b - a)$ and $g(X_i)$ for $i = 1, 2, \ldots, N$.
4. Count the number of cases $N_H$ for which $g(X_i) > cU'_i$.
5. Compute $\theta_1 = c(b - a)N_H / N$. (Estimate $I$.)

The number of trials $N$ necessary for $P(|\theta_1 - I| < \epsilon) \geq \alpha$ is given by

$$N \geq \frac{(1 - p)p[c(b - a)]^2}{(1 - \alpha)\epsilon^2}. \quad (8.3.4)$$

Using the usual notation $z_\alpha$ for the value of the standard normal random variable $Z$ for which $P(Z > z_\alpha) = \alpha$, a confidence interval for $I$ with confidence level $1 - \alpha$ is

$$\theta_1 \pm z_\alpha \sqrt{\frac{p(1 - p)(b - a)c}{N}}. \quad (8.3.5)$$

**The sample-mean Monte–Carlo method**

Write the integral $I = \int_a^b g(x) \, dx$ as $\int_a^b \frac{g(x)}{\int_a^b f_X(x) \, dx} f_X(x) \, dx$. Then $I = E\left[\frac{g(X)}{\int_X f_X(x) \, dx}\right]$ where the random variable $X$ is distributed according to $f_X(x)$. Values from this distribution

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---

can be generated by the methods discussed in Section 7.5. For the case where \( f_X(x) \) is the uniform distribution on \([0, 1]\), \( I = (b - a)E[g(X)] \). An unbiased estimator of \( I \) is its sample mean

\[
\theta_2 = (b - a) \frac{1}{N} \sum_{i=1}^{N} g(X_i).
\]  

(8.3.6)

It follows that the variance of \( \theta_2 \) is less than or equal to the variance of \( \theta_1 \). In fact,

\[
\text{var} \theta_1 = \frac{1}{N} [c(b - a) - I], \quad \text{and} \quad \text{var} \theta_2 = \frac{1}{N} \left[ (b - a) \int_{a}^{b} g^2(x) \, dx - I^2 \right].
\]

Note that to estimate \( I \) with \( \theta_1 \) or \( \theta_2 \), \( g(x) \) is not needed explicitly. It is only necessary to evaluate \( g(x) \) at any point \( x \).

**Sample-mean algorithm**

1. Generate a sequence \( \{U_i\}_{i=1}^{N} \) of \( N \) random numbers, uniformly distributed in \([0, 1]\).
2. Compute \( X_i = a + U_i(b - a) \), for \( i = 1, 2, \ldots, N \).
3. Compute \( g(X_i) \) for \( i = 1, 2, \ldots, N \).
4. Compute \( \theta_2 \) according to Equation (8.3.6) (this is an estimate of \( I \)).

**Integration in the presence of noise**

Suppose \( g(x) \) is measured with some error: \( \tilde{g}(x_i) = g(x_i) + \epsilon_i \), for \( i = 1, 2, \ldots, N \), where \( \epsilon_i \) are independent identically distributed random variables with \( E[\epsilon_i] = 0 \), \( \text{var}(\epsilon) = \sigma^2 \), and \( |\epsilon_i| < k < \infty \).

If \((X, Y)\) is uniformly distributed on the rectangle \( a \leq x \leq b, 0 \leq y \leq c_1 \), where \( c_1 \geq g(x) + k \), set \( \tilde{\theta}_1 = c_1(b - a)N_H/N \) as in the hit or miss method. Similarly, set \( \tilde{\theta}_2 = \frac{1}{N} (b - a) \sum_{i=1}^{N} \tilde{g}(X_i) \) as in the sample-mean method. Then both \( \tilde{\theta}_1 \) and \( \tilde{\theta}_2 \) are unbiased and converge almost surely to \( I \). Again, \( \text{var} \tilde{\theta}_2 \leq \text{var} \tilde{\theta}_1 \).

**Weighted Monte–Carlo integration**

Estimate the integral \( I = \int_{0}^{1} g(x) \, dx \) according to the following algorithm:

1. Generate numbers \( \{U_1, U_2, \ldots, U_N\} \) from the uniform distribution on \([0, 1]\).
2. Arrange \( U_1, U_2, \ldots, U_N \) in the increasing order \( U_{(1)}, U_{(2)}, \ldots, U_{(N)} \).
3. Compute \( \theta_3 = \frac{1}{2} \left[ \sum_{i=0}^{N} (g(U_i)) + g(U_{(i+1)}) (U_{(i+1)} - U_{(i)}) \right] \), where \( U_{(0)} \equiv 0, U_{(N+1)} \equiv 1 \). This is an estimate of \( I \).

If \( g(x) \) has a continuous second derivative on \([0, 1]\), then the estimator \( \theta_3 \) satisfies variance \( \theta_3 = E(\theta_3 - I)^2 \leq k/N^4 \), where \( k \) is some positive constant.

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8.3.2 NUMERICAL DIFFERENTIATION

Derivative estimates

Selected formulae to estimate the derivative of a function at a single point, with error terms, are given. Nodes are equally spaced with \( x_i - x_{i-1} = h \); \( h \) may be positive or negative and, in the error formulae, \( \xi \) lies between the smallest and largest nodes. To shorten some of the formulae, \( f_j \) is used to denote \( f(x_0 + jh) \) and some error formulae are expressed as \( O(h^k) \).

1. Two-point formula for \( f'(x_0) \)

\[
 f'(x_0) = \frac{1}{h}(f(x_0 + h) - f(x_0)) - \frac{h}{2}f''(\xi) .
\] (8.3.7)

This is called a forward-difference formula if \( h > 0 \) and a backward-difference formula if \( h < 0 \).

2. Three-point formulae for \( f'(x_0) \)

\[
 f'(x_0) = \frac{1}{2h}[-3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h)] + \frac{h^2}{3}f^{(3)}(\xi) \\
= \frac{1}{2h}[f(x_0 + h) - f(x_0 - h)] - \frac{h^2}{6}f^{(3)}(\xi) .
\] (8.3.8)

3. Four-point formulae for \( f'(x_0) \)

\[
 f'(x_0) = \frac{1}{12h}[-25f_0 + 48f_1 - 36f_2 + 16f_3 - 3f_4] + \frac{h^4}{30}f^{(5)}(\xi) .
\] (8.3.9)

4. Five-point formulae for \( f'(x_0) \)

\[
 f'(x_0) = \frac{1}{12h}[-25f_0 + 48f_1 - 36f_2 + 16f_3 - 3f_4] + \frac{h^4}{5}f^{(5)}(\xi) .
\] (8.3.10)

5. Formulae for the second derivative

\[
 f''(x_0) = \frac{1}{h^2}[f_{-1} - 2f_0 + f_1] - \frac{h^2}{12}f^{(4)}(\xi), \\
= \frac{1}{h^2}[f_0 - 2f_1 + f_2] + \frac{h^2}{6}f^{(4)}(\xi_1) - hf^{(3)}(\xi_2) .
\] (8.3.11)

6. Formulae for the third derivative

\[
 f^{(3)}(x_0) = \frac{1}{h^3}[f_3 - 3f_2 + 3f_1 - f_0] + O(h), \\
= \frac{1}{2h^3}[f_2 - 2f_1 + 2f_{-1} - f_{-2}] + O(h^2). 
\] (8.3.12)
7. Formulae for the fourth derivative

\[ f^{(4)}(x_0) = \frac{1}{h^4} \left[ f_4 - 4f_3 + 6f_2 - 4f_1 + f_0 \right] + O(h), \]
\[ = \frac{1}{h^4} \left[ f_2 - 4f_1 + 6f_0 - 4f_{-1} + f_{-2} \right] + O(h^2). \] (8.3.13)

Richardson’s extrapolation can be applied to improve estimates. The error term of the formula must satisfy Equation (8.1.2) and an extrapolation procedure must be developed. As a special case, however, Equation (8.1.4) may be used when first-column entries are generated by Equation (8.3.8).

**Numerical solution of differential equations**

Numerical methods to solve differential equations depend on whether small changes in the statement of the problem cause small changes in the solution.

**DEFINITION 8.3.1**

The initial value problem,

\[ \frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha, \] (8.3.14)

is said to be well-posed if

1. A unique solution, \( y(t) \), to the problem exists.
2. For any \( \epsilon > 0 \), there exists a positive constant \( k(\epsilon) \) with the property that, whenever \( |\varepsilon_0| < \epsilon \) and \( \delta(t) \) is continuous with \( |\delta(t)| < \epsilon \) on \([a, b]\), a unique solution, \( z(t) \), to the problem,

\[ \frac{dz}{dt} = f(t, z) + \delta(t), \quad a \leq t \leq b, \quad z(a) = \alpha + \varepsilon_0, \]

exists with \( |z(t) - y(t)| < k(\epsilon)\epsilon \) for all \( a \leq t \leq b \).

This is called the **perturbed problem** associated with the original problem. Although other criteria exist, the following result gives conditions that are easy to check to guarantee that a problem is well-posed.

**THEOREM 8.3.1** (Well-posed condition)

Suppose that \( f \) and \( f_y \) (its first partial derivative with respect to \( y \)) are continuous for \( t \) in \([a, b]\). Then the initial value problem given by Equation (8.3.14) is well-posed.

Using Taylor’s theorem, numerical methods for solving the well-posed, first-order differential equation given by Equation (8.3.14) can be derived. Using equally-spaced mesh points \( t_i = a + ih \) (for \( i = 0, 1, 2, \ldots, N \)) and \( w_i \) to denote an approximation to \( y_i \equiv y(t_i) \), then the different methods use difference equations of the form
\[ w_0 = \alpha, \quad w_{i+1} = w_i + h\phi(t_i, w_i), \]

for each \( i = 0, 1, 2, \ldots, N - 1 \). The difference method has local truncation error given by

\[ \tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i), \]

for each \( i = 0, 1, 2, \ldots, N - 1 \). The following formulae are called Taylor methods. Each has local truncation error

\[ h^n \frac{n!}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i)) \]

for each \( i = 0, 1, 2, \ldots, N - 1 \), where \( \xi_i \in (t_i, t_{i+1}) \).

1. Euler’s method \((n = 1)\):

\[ w_{i+1} = w_i + hf(t_i, w_i). \tag{8.3.15} \]

2. Taylor method of order \( n \):

\[ w_{i+1} = w_i + hT^{(n)}(t_i, w_i), \tag{8.3.16} \]

where

\[ T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \cdots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, w_i). \]

The Runge–Kutta methods below are derived from the \( n \)th degree Taylor polynomial in two variables.

3. Midpoint method:

\[ w_{i+1} = w_i + h \left[ f \left( t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i) \right) \right]. \tag{8.3.17} \]

If all second-order partial derivatives of \( f \) are bounded, this method has local truncation error \( O(h^2) \), as do the following two methods.

4. Modified Euler method:

\[ w_{i+1} = w_i + \frac{h}{2} \left[ f(t_i, w_i) + f[t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)] \right]. \tag{8.3.18} \]

5. Heun’s method:

\[ w_{i+1} = w_i + \frac{h}{4} \left[ f(t_i, w_i) + 3 f \left[ t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i) \right] \right]. \tag{8.3.19} \]

6. Runge–Kutta method of order four:

\[ w_{i+1} = w_i + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4), \tag{8.3.20} \]

where

\[ k_1 = hf(t_i, w_i), \]
\[ k_2 = hf \left( t_i + \frac{h}{2}, w_i + \frac{1}{2} k_1 \right), \]
\[ k_3 = hf \left( t_i + \frac{h}{2}, w_i + \frac{1}{2} k_2 \right), \]
\[ k_4 = hf \left( t_{i+1}, w_i + k_3 \right). \]

The local truncation error is \( O(h^4) \) if the solution \( y(t) \) has five continuous derivatives.

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Multistep methods and predictor-corrector methods

A multistep method is a technique whose difference equation to compute \( w_{i+1} \) involves more prior values than just \( w_i \). An explicit method is one in which the computation of \( w_{i+1} \) does not depend on \( f(t_{i+1}, w_{i+1}) \) whereas an implicit method does involve \( f(t_{i+1}, w_{i+1}) \). For each formula, \( i = n - 1, n, \ldots, N - 1 \).

Adams–Bashforth \( n \)-step (explicit) methods

1. \( (n = 2) \):
   \[
   w_0 = \alpha, \ w_1 = \alpha_1, \ w_{i+1} = w_i + \frac{h}{2} [3f(t_i, w_i) - f(t_{i-1}, w_{i-1})].
   \]
   The local truncation error is \( \tau_{i+1}(h) = \frac{5}{24} y^{(4)}(\mu_i) h^3 \), for some \( \mu_i \in (t_{i-1}, t_{i+1}) \).

2. \( (n = 3) \):
   \[
   w_0 = \alpha, \ w_1 = \alpha_1, \ w_2 = \alpha_2, \ w_{i+1} = w_i + \frac{h}{24} [23f(t_i, w_i) - 16f(t_{i-1}, w_{i-1}) + 5f(t_{i-2}, w_{i-2})].
   \]
   The local truncation error is \( \tau_{i+1}(h) = \frac{5}{12} y^{(3)}(\mu_i) h^2 \), for some \( \mu_i \in (t_{i-1}, t_{i+1}) \).

3. \( (n = 4) \):
   \[
   w_0 = \alpha, \ w_1 = \alpha_1, \ w_2 = \alpha_2, \ w_3 = \alpha_3, \ w_{i+1} = w_i + \frac{h}{24} [55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})].
   \]
   The local truncation error is \( \tau_{i+1}(h) = \frac{23}{720} y^{(4)}(\mu_i) h^3 \), for some \( \mu_i \in (t_{i-2}, t_{i+1}) \).

4. \( (n = 5) \):
   \[
   w_0 = \alpha, \ w_1 = \alpha_1, \ w_2 = \alpha_2, \ w_3 = \alpha_3, \ w_4 = \alpha_4, \ w_{i+1} = w_i + \frac{h}{720} [95f(t_i, w_i) - 2774f(t_{i-1}, w_{i-1}) + 2616f(t_{i-2}, w_{i-2}) - 1274f(t_{i-3}, w_{i-3}) + 251f(t_{i-4}, w_{i-4})].
   \]
   The local truncation error is \( \tau_{i+1}(h) = \frac{95}{720} y^{(6)}(\mu_i) h^5 \), for some \( \mu_i \in (t_{i-4}, t_{i+1}) \).

Adams–Moulton \( n \)-step (implicit) methods

1. \( (n = 2) \):
   \[
   w_0 = \alpha, \ w_1 = \alpha_1, \ w_{i+1} = w_i + \frac{h}{24} [5f(t_{i+1}, w_{i+1}) + 8f(t_i, w_i) - f(t_{i-1}, w_{i-1})].
   \]
   The local truncation error is \( \tau_{i+1}(h) = -\frac{1}{24} y^{(4)}(\mu_i) h^3 \), for some \( \mu_i \in (t_{i-1}, t_{i+1}) \).

2. \( (n = 3) \):
   \[
   w_0 = \alpha, \ w_1 = \alpha_1, \ w_2 = \alpha_2, \ w_{i+1} = w_i + \frac{h}{24} [9f(t_{i+1}, w_{i+1}) + 19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})].
   \]
   The local truncation error is \( \tau_{i+1}(h) = y^{(5)}(\mu_i) h^4 \), for some \( \mu_i \in (t_{i-1}, t_{i+1}) \).

3. \( (n = 4) \):
   \[
   w_0 = \alpha, \ w_1 = \alpha_1, \ w_2 = \alpha_2, \ w_3 = \alpha_3, \ w_{i+1} = w_i + \frac{h}{240} [251f(t_{i+1}, w_{i+1}) + 646f(t_i, w_i) - 264f(t_{i-1}, w_{i-1}) + 106f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})].
   \]
   The local truncation error is \( \tau_{i+1}(h) = -\frac{1}{140} y^{(6)}(\mu_i) h^5 \), for some \( \mu_i \in (t_{i-3}, t_{i+1}) \).

In practice, implicit methods are not used by themselves. They are used to improve approximations obtained by explicit methods. An explicit method predicts an approximation and the implicit method corrects this prediction. The combination is called a predictor-corrector method. For example, the Adams–Bashforth method with \( n = 4 \) might be used with the Adams–Moulton method with \( n = 3 \) since both have comparable errors. Initial values may be computed, say, by the Runge–Kutta method of order four, Equation (8.3.20).
Higher order differential equations and systems

An \( m \)-th order system of first-order initial value problems can be expressed in the form

\[
\begin{align*}
\frac{du_1}{dt} &= f_1(t, u_1, u_2, \ldots, u_m), \\
\frac{du_2}{dt} &= f_2(t, u_1, u_2, \ldots, u_m), \\
&\vdots \\
\frac{du_m}{dt} &= f_m(t, u_1, u_2, \ldots, u_m).
\end{align*}
\]

Generalizations of methods for solving first-order equations can be used to solve such systems. An example here uses the Runge-Kutta method of order four.

Partition \([a, b]\) as before, and let \( w_{i,j} \) denote the approximation to \( u_i(t_j) \) for \( j = 0, 1, \ldots, N \) and \( i = 1, 2, \ldots, m \). For the initial conditions, set \( w_{1,0} = \alpha_1, w_{2,0} = \alpha_2, \ldots, w_{m,0} = \alpha_m \). From the values \( \{w_{1,j}, w_{2,j}, \ldots, w_{m,j}\} \) previously computed, obtain \( \{w_{1,j+1}, w_{2,j+1}, \ldots, w_{m,j+1}\} \) from

\[
\begin{align*}
k_{1,i} &= hf_i(t_j, w_{1,j}, w_{2,j}, \ldots, w_{m,j}), \\
k_{2,i} &= hf_i\left(t_j + \frac{h}{2}, w_{1,j} + \frac{1}{2}k_{1,1}, w_{2,j} + \frac{1}{2}k_{1,2}, \ldots, w_{m,j} + \frac{1}{2}k_{1,m}\right), \\
k_{3,i} &= hf_i\left(t_j + \frac{h}{2}, w_{1,j} + \frac{1}{2}k_{2,1}, w_{2,j} + \frac{1}{2}k_{2,2}, \ldots, w_{m,j} + \frac{1}{2}k_{2,m}\right), \\
k_{4,i} &= hf_i\left(t_j + h, w_{1,j} + k_{3,1}, w_{2,j} + k_{3,2}, \ldots, w_{m,j} + k_{3,m}\right), \\
w_{i,j+1} &= w_{i,j} + \frac{1}{6}[k_{1,i} + 2k_{2,i} + 2k_{3,i} + k_{4,i}],
\end{align*}
\]

where \( i = 1, 2, \ldots, m \) for each of the above.

A differential equation of high order can be converted into a system of first-order equations. Suppose that a single differential equation has the form

\[
y^{(m)} = f(t, y, y', y'', \ldots, y^{(m-1)}), \quad a \leq t \leq b
\]

with initial conditions \( y(a) = \alpha_1, y'(a) = \alpha_2, \ldots, y^{(m-1)}(a) = \alpha_m \). All derivatives are with respect to \( t \). That is, \( y^{(k)} = \frac{d^k y}{dt^k} \). Define \( u_1(t) = y(t), u_2(t) = y'(t), \ldots, u_m(t) = y^{(m-1)}(t) \). This yields first-order equations

\[
\begin{align*}
\frac{du_1}{dt} &= u_2, \quad \frac{du_2}{dt} = u_3, \quad \ldots \quad \frac{du_{m-1}}{dt} = u_m, \quad \frac{du_m}{dt} = f(t, u_1, u_2, \ldots, u_m),
\end{align*}
\]

with initial conditions \( u_1(a) = \alpha_1, \ldots, u_m(a) = \alpha_m \).
Partial differential equations

To develop difference equations for partial differential equations, one needs to estimate the partial derivatives of a function, say, $u(x, y)$. For example,

$$\frac{\partial u}{\partial x} (x, y) = \frac{u(x + h, y) - u(x, y)}{h} - \frac{h}{2} \frac{\partial^2 u(\xi, y)}{\partial x^2} \quad \text{for } \xi \in (x, x + h), \quad (8.3.21)$$

$$\frac{\partial^2 u}{\partial x^2} (x, y) = \frac{1}{h^2} [u(x + h, y) - 2u(x, y) + u(x - h, y)] - \frac{h^2}{12} \frac{\partial^4 u(\xi, y)}{\partial x^4}, \quad (8.3.22)$$

for $\xi \in (x - h, x + h)$.

Notes:

1. Equation (8.3.21) is simply Equation (8.3.7) applied to estimate the partial derivative. It is given here to emphasize its application for forming difference equations for partial differential equations. A similar formula applies for $\frac{\partial u}{\partial y}$, and others could follow from the formulae in Section 8.3.2.

2. An estimate of $\frac{\partial^2 u}{\partial y^2}$ is similar. A formula for $\frac{\partial^2 u}{\partial x \partial y}$ could be given. However, in practice, a change of variables is generally used to eliminate this mixed second partial derivative from the problem.

If a partial differential equation involves partials with respect to only one of the variables, the methods described for ordinary differential equations can be used. If, however, the equation involves partial derivatives with respect to both variables, the approximation of the partial derivatives require increments in both variables. The corresponding difference equations form a system of linear equations that must be solved.

Two specific forms of partial differential equations with popular methods of solution are given. The domains are assumed to be rectangular. Otherwise, additional considerations must be made for the boundary conditions.

Poisson equation

This elliptic partial differential equation has the form

$$\nabla^2 u(x, y) = \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = f(x, y) \quad (8.3.23)$$

for $(x, y) \in R = \{(x, y) \mid a < x < b, c < y < d\}$, with $u(x, y) = g(x, y)$ for $(x, y) \in S$, where $S = \partial R$.

Partition $[a, b]$ and $[c, d]$ by first choosing integers $n$ and $m$, define step sizes $h = (b - a)/n$ and $k = (d - c)/m$, and set $x_i = a + ih$ for $i = 0, 1, \ldots, n$ and $y_j = c + jk$ for $j = 0, 1, \ldots, m$. The lines $x = x_i$, $y = y_j$, are called grid lines and their intersections are called mesh points. Estimates $w_{i,j}$ for $u(x_i, y_j)$ can be generated using Equation (8.3.22) to estimate $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$.

Central difference method

Start with the values

$$w_{0,j} = g(x_0, y_j), \quad w_{n,j} = g(x_n, y_j), \quad w_{i,0} = g(x_i, y_0), \quad w_{i,m} = g(x_i, y_m),$$

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and then solve the system of linear algebraic equations

\[
2 \left[ \left( \frac{h}{k} \right)^2 + 1 \right] w_{i,j} - (w_{i+1,j} + w_{i-1,j}) - \left( \frac{h}{k} \right)^2 (w_{i,j+1} + w_{i,j-1}) = -h^2 f(x_i, y_j),
\]

for \( i = 1, 2, \ldots, n - 1 \) and \( j = 1, 2, \ldots, m - 1 \). The local truncation error is \( O(h^2+k^2) \).

If the interior mesh points are relabeled \( P_l = (x_l, y_l) \) where \( l = i + (m-1-j)(n-1), i = 1, 2, \ldots, n-1 \) and \( j = 1, 2, \ldots, m-1 \), then the two-dimensional array of values is now a one-dimensional array. This results in a banded linear system. The case \( n = m = 4 \) yields \( l = (n-1)(m-1) = 9 \). Using the relabeled grid points, \( f_\ell = f(P_\ell) \), the equations at the points \( P_l \) are

\[
\begin{align*}
P_1: \quad & 4w_1 - w_2 - w_4 = w_{1,4} - h^2 f_1, \\
P_2: \quad & 4w_2 - w_3 - w_5 = w_{2,4} - h^2 f_2, \\
P_3: \quad & 4w_3 - w_2 - w_6 = w_{4,3} + w_{3,4} - h^2 f_3, \\
P_4: \quad & 4w_4 - w_5 - w_1 = w_{0,2} - h^2 f_4, \\
P_5: \quad & 4w_5 - w_6 - w_4 - w_2 - w_8 = 0 - h^2 f_5, \\
P_6: \quad & 4w_6 - w_5 - w_3 - w_9 = w_{4,2} - h^2 f_6, \\
P_7: \quad & 4w_7 - w_8 - w_4 = w_{0,1} + w_{1,0} - h^2 f_7, \\
P_8: \quad & 4w_8 - w_9 - w_7 - w_5 = w_{2,0} - h^2 f_8, \\
P_9: \quad & 4w_9 - w_8 - w_6 = w_{3,0} + w_{4,1} - h^2 f_9,
\end{align*}
\]

where the right-hand sides of the equations are obtained from the boundary conditions.

**Heat or diffusion equation**

This parabolic partial differential equation has the form

\[
\frac{\partial u}{\partial t}(x,t) = \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t), \quad 0 < x < \ell, \quad t > 0.
\] (8.3.24)

One common set of initial and boundary conditions is \( u(0, t) = 0 \) and \( u(\ell, t) = 0 \) for \( t > 0 \), and \( u(x, 0) = f(x) \) for \( 0 \leq x \leq \ell \).

Select mesh constants \( h \) and \( k \) so that \( m = \ell / h \) is an integer. The difference equation for the Crank–Nicolson method is

\[
\frac{w_{i,j+1} - w_{i,j}}{k} = \frac{\alpha^2}{2} \left[ \frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{h^2} + \frac{w_{i,j+1} - 2w_{i,j+1} + w_{i-1,j+1}}{h^2} \right] = 0.
\]

This has local truncation error \( O(k^2+h^2) \). The difference equations can be represented in the matrix form \( Aw^{(j+1)} = Bw^{(j)} \), for each \( j = 0, 1, 2, \ldots \), where \( \lambda = \alpha^2 k / h^2 \), \( w^{(j)} = (w_{1,j}, w_{2,j}, \ldots, w_{m-1,j})^T \), and the matrices \( A \) and \( B \) are given by

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$$A = \begin{bmatrix}
(1 + \lambda) & -\frac{1}{2} & 0 & 0 & \ldots & 0 & 0 \\
-\frac{1}{2} & (1 + \lambda) & -\frac{1}{2} & 0 & \ldots & 0 & 0 \\
0 & -\frac{1}{2} & (1 + \lambda) & -\frac{1}{2} & \ldots & 0 & 0 \\
0 & 0 & -\frac{1}{2} & (1 + \lambda) & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & (1 + \lambda) & -\frac{1}{2} \\
0 & 0 & 0 & 0 & \ldots & -\frac{1}{2} & (1 + \lambda)
\end{bmatrix}$$

$$B = \begin{bmatrix}
(1 - \lambda) & \frac{1}{2} & 0 & 0 & \ldots & 0 & 0 \\
\frac{1}{2} & (1 - \lambda) & \frac{1}{2} & 0 & \ldots & 0 & 0 \\
0 & \frac{1}{2} & (1 - \lambda) & \frac{1}{2} & \ldots & 0 & 0 \\
0 & 0 & \frac{1}{2} & (1 - \lambda) & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & (1 - \lambda) & \frac{1}{2} \\
0 & 0 & 0 & 0 & \ldots & \frac{1}{2} & (1 - \lambda)
\end{bmatrix}$$

### 8.3.3 SCHEMES FOR THE ODE: $$y' = f(x, y)$$

- **Adams–Bashforth, order 2**: 
  $$v_n - v_{n-1} = \frac{1}{2} h \left[3 f_{n-1} - f_{n-2}\right]$$
- **Adams–Bashforth, order 4**: 
  $$v_n - v_{n-1} = \frac{1}{24} h \left[55 f_{n-1} - 59 f_{n-2} + 37 f_{n-3} - 9 f_{n-4}\right]$$
- **Adams–Moulton, order 4**: 
  $$v_n - v_{n-1} = \frac{1}{24} h \left[9 f_n + 19 f_{n-1} - 5 f_{n-2} + f_{n-3}\right]$$
- **Backward Euler**: 
  $$v_n - v_{n-1} = h f_n$$
- **Euler’s method**: 
  $$v_n - v_{n-1} = h f_n$$
- **Explicit leapfrog**: 
  $$v_{n+1} - v_{n-1} = h f_n$$
- **Implicit leapfrog**: 
  $$v_n - v_{n-1} = \frac{1}{2} h (f_n + f_{n-1})$$
- **Simpson’s rule**: 
  $$v_n - v_{n-2} = \frac{1}{3} h (f_n + 4 f_{n-1} + f_{n-2})$$
- **Trapezoidal rule**: 
  $$v_n - v_{n-1} = \frac{1}{2} h (f_n + f_{n-1})$$

---

<sup>a</sup> Also known as Milne’s method.

<sup>b</sup> Also known as Heun’s method and as the Adams–Moulton method of order 2.
8.3.4 EXPLICIT SCHEMES FOR THE PDE: $au_x + u_t = 0$

Below are explicit difference formulas for the PDE, $au_x + u_t = 0$. Here, $h$ is the uniform $x$ spacing, and $k$ is the uniform $t$ spacing. The approximation to $u(x_n, t_j) = u(x_0 + nh, t_0 + jk)$ is represented by $u_{n,j}$.

Forward in time, forward in space (FTFS):

$$a \frac{u_{n+1,j} - u_{n,j}}{h} + \frac{u_{n+1,j} - u_{n,j}}{k} = 0.$$  

Forward in time, centered in space (FTCS) (unstable):

$$a \frac{u_{n+1,j} - u_{n-1,j}}{2h} + \frac{u_{n+1,j} - u_{n,j}}{k} = 0.$$  

Forward in time, backward in space (FTBS):

$$a \frac{u_{n+1,j} - u_{n-1,j}}{h} + \frac{u_{n+1,j} - u_{n,j}}{k} = 0.$$  

Lax–Friedrichs method:

$$a \frac{u_{n+1,j} - u_{n-1,j}}{2h} + \frac{u_{n+1,j} - \frac{1}{2}(u_{n-1,j} - u_{n+1,j})}{k} = 0.$$  

Lax–Wendroff method:

$$u_{n+1,j} = u_{n,j} - \frac{ak}{2h} (u_{n+1,j} - u_{n-1,j}) + \frac{a^2k^2}{2h^2} (u_{n-1,j} - 2u_{n,j} + u_{n+1,j}).$$

8.3.5 IMPLICIT SCHEMES FOR THE PDE: $au_x + u_t = S(x, t)$

Below are implicit difference formulas for the PDE, $au_x + u_t = S(x, t)$. Here, $h$ is the uniform $x$ spacing, and $k$ is the uniform $t$ spacing. The approximation to $u(x_n, t_j) = u(x_0 + nh, t_0 + jk)$ is represented by $u_{n,j}$, and $S_{n,j}$ is used to represent $S(x_n, t_j)$.

Backward in time, backward in space (BTBS):

$$a \frac{u_{n+1,j} - u_{n,j}}{h} + \frac{u_{n+1,j} - u_{n,j}}{k} = S_{n+1,j+1}.$$  

Backward in time, centered in space (BTCS):

$$a \frac{u_{n+1,j} - u_{n-1,j}}{2h} + \frac{u_{n+1,j} - u_{n,j}}{k} = S_{n,j+1}.$$  

Crank–Nicolson:

$$\frac{1}{2} \left( a \frac{u_{n+1,j} - u_{n-1,j}}{2h} + a \frac{u_{n+1,j} - u_{n-1,j}}{2h} \right) + \frac{u_{n+1,j} - u_{n,j}}{k} = S_{n,j+1/2}.$$  

Wendroff method:

$$\frac{1}{2} \left( a \frac{u_{n+1,j} - u_{n+1,j}}{h} + a \frac{u_{n+1,j} - u_{n,j}}{h} \right) + \frac{1}{2} \left( a \frac{u_{n+1,j} - u_{n+1,j}}{h} + a \frac{u_{n+1,j} - u_{n,j}}{h} \right) = S_{n+1/2,j+1/2}.$$  

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8.3.6 SCHEMES FOR THE PDE: \( F(u)_x + u_t = 0 \)

Below are difference formulas for the PDE, \( F(u)_x + u_t = 0 \). Here, \( h \) is the uniform \( x \) spacing, \( k \) is the uniform \( t \) spacing, and the ratio of these is \( s = k/h \). The approximation to \( u(x_n, t_j) = u(x_0 + nh, t_0 + jk) \) is represented by \( u_{n,j} \) and \( F_{m,n} = F(u_{m,n}) \). A star superscript indicates an intermediate result, and \( F^*_n = F(u^*_n) \). Finally, \( a_n = F'_n = F'(u_n) \).

Centered in time, centered in space (unstable):
\[
 u_{n,j+1} = u_{n,j} - \frac{1}{2s} (F_{n+1,j} - F_{n-1,j}) .
\]

Lax–Friedrichs method:
\[
 u_{n,j+1} = \frac{1}{2} (u_{n+1,j} + u_{n-1,j}) - \frac{1}{2} s (F_{n+1,j} + F_{n-1,j}) .
\]

Lax–Wendroff method:
\[
 u_{n,j+1} = u_{n,j} - \frac{1}{2} s (F_{n+1,j} - F_{n-1,j}) \\
\text{ } + \frac{1}{2} s^2 \left[ a_{n+1/2,j} (F_{n+1,j} - F_{n,j}) - a_{n-1/2,j} (F_{n,j} - F_{n-1,j}) \right] .
\]

Richtmeyer method:
\[
 u^*_{n+1/2} = \frac{1}{2} (u_{n+1,j} + u_{n,j}) - \frac{1}{2} (F_{n+1,j} - F_{n,j}) \\
 u_{n,j+1} = u_{n,j} - s (F^*_n - F^*_n-1) .
\]

MacCormack method:
\[
 u^*_n = u_{n,j} - s (F_{n+1,j} - F_{n,j}) \\
 u_{n,j+1} = \frac{1}{2} \left[ u_{n,j} + u^*_n - s (F^*_n - F^*_n-1) \right] .
\]

FTBS upwind method (use when \( F'(u) > 0 \)):
\[
 u_{n,j+1} = u_{n,j} + s (F_{n-1,j} - F_{n,j}) .
\]

FTFS upwind method (use when \( F'(u) < 0 \)):
\[
 u_{n,j+1} = u_{n,j} - s (F_{n+1,j} - F_{n,j}) .
\]

8.3.7 SCHEMES FOR THE PDE: \( u_x = u_t \)

Below are difference formulas for the PDE, \( u_x = u_t \). Here, \( h \) is the uniform \( x \) spacing, \( k \) is the uniform \( t \) spacing, and \( \rho \) is defined to be \( \rho = h/k^2 \). The approximation to \( u(x_n, t_j) = u(x_0 + nh, t_0 + jk) \) is represented by \( u_{n,j} \).
Classic explicit approximation:
\[ u_{n+1,j} = (1 - 2\rho)u_{n,j} + \rho \left( u_{n,j+1} + u_{n,j-1} \right). \]
DuFort–Frankel explicit approximation:
\[ (1 + 2\rho)u_{n+1,j} = 2\rho \left( u_{n,j+1} + u_{n,j-1} \right) + (1 - 2\rho)u_{n-1,j}. \]
Richardson explicit approximation:
\[ u_{n+1,j} - u_{n-1,j} - 2\rho \left( u_{n,j+1} + u_{n,j-1} \right) + 4\rho u_{n,j} = 0. \]
Backward implicit approximation:
\[ (1 + 2\rho)u_{n+1,j} - \rho \left( u_{n+1,j+1} + u_{n+1,j-1} \right) = u_{n,j}. \]
Crank–Nicolson implicit approximation:
\[ 2(\rho + 1)u_{n+1,j} - \rho \left( u_{n+1,j+1} + u_{n+1,j-1} \right) = 2(1 - \rho)u_{n,j} + \rho \left( u_{n,j+1} + u_{n,j-1} \right). \]
Variable weighted implicit approximation (with \( 0 \leq \theta \leq 1 \)):
\[ (1 + 2\rho\theta)u_{n+1,j} = \rho (1 - \theta) \left( u_{n,j+1} + u_{n,j-1} \right) + \rho \theta \left( u_{n+1,j+1} + u_{n+1,j-1} \right) + [1 - 2\rho(1 - \theta)]u_{n,j}. \]

8.3.8 NUMERICAL SUMMATION

A sum of the form \( \sum_{j=0}^{n} f(x_0 + jh) \) (\( n \) may be infinite) can be approximated by the Euler–Maclaurin sum formula,
\[
\sum_{j=0}^{n} f(x_0 + jh) = \frac{1}{h} \int_{x_0}^{x_0 + nh} f(y) \, dy + \frac{1}{2} [ f(x_0 + nh) + f(x_0) ] \\
+ \sum_{k=1}^{m} B_{2k} (2k)! h^{2k-1} \left[ f^{(2k-1)}(x_0 + nh) - f^{(2k-1)}(x_0) \right] + E_m \tag{8.3.25}
\]
where \( E_m = \frac{2h^{2m+2}B_{2m+2}}{(2m+3)!} f^{(2m+2)}(\xi) \), and \( x_0 < \xi < x_0 + nh \). The \( B_n \) here are Bernoulli numbers (see Section 1.2.8).

The above formula is useful even when \( n \) is infinite, although the error can no longer be expressed in this form. A useful error estimate (which also holds when \( n \) is finite) is that the error is less than the magnitude of the first neglected term in the summation on the right-hand side of Equation (8.3.25) if \( f^{(2m+2)}(x) \) and \( f^{(2m+4)}(x) \) do not change sign and are of the same sign for \( x_0 < x < x_0 + nh \). If just \( f^{(2m+2)}(x) \) does not change sign in the interval, then the error is less than twice the first neglected term.

Quadrature formulae result from Equation (8.3.25) using estimates for the derivatives.
Gregory’s formula

Using $f_j$ to represent $f(x_0 + jh)$,

$$
\int_{x_0}^{x_0+nh} f(y) \, dy = h\left(\frac{1}{2} f_0 + f_1 + \cdots + f_{n-1} + \frac{1}{2} f_n\right)
+ \frac{h}{12} (\Delta f_0 - \Delta f_{n-1})
- \frac{h}{24} (\Delta^2 f_0 + \Delta^2 f_{n-2})
+ \frac{19h}{720} (\Delta^3 f_0 - \Delta^3 f_{n-3})
- \frac{3h}{160} (\Delta^4 f_0 + \Delta^4 f_{n-4}) + \ldots
$$

(8.3.26)

where $\Delta$ represents forward differences. The first expression on the right in Equation (8.3.26) is the composite trapezoidal rule, and additional terms provide improved approximations. Care must be taken not to carry this process too far because Gregory’s formula is only asymptotically convergent in general and round off error can be significant when computing higher differences.

8.4 PROGRAMMING TECHNIQUES

Efficiency and accuracy is the ultimate goal when solving any problem. Listed here are several suggestions to consider when developing algorithms and computer programs.

1. **Every algorithm must have an effective stopping rule.** For example, popular stopping rules for iteration methods described in Section 8.1.2 are based on the estimate of the absolute error, relative error, or function value. One might choose to stop when a combination of the following conditions are satisfied:

$$
|p_n - p_{n-1}| < \epsilon_1, \quad \frac{|p_n - p_{n-1}|}{|p_n|} < \epsilon_2, \quad |f(p_n)| < \epsilon_3,
$$

where each $\epsilon_i$ represents a prescribed tolerance. However, since some iterations are not guaranteed to converge, or converge very slowly, it is recommended that an upper bound, say $N$, is specified for the number of iterations to be performed (see algorithm on page 685). This will avoid infinite loops.

2. **Avoid the use of arrays whenever possible.** Subscripted values often do not require the use of an array. For example, in Newton’s method (see page 673) the calculations may be performed using $p = p_0 - \frac{f(p_0)}{f'(p_0)}$. Then check the stopping rule, say, if $|p - p_0| < \epsilon$, and update the current value by setting $p_0 = p$ before computing the next value of the sequence.

3. **Limit the use of arrays when forming tables.** A two-dimensional array can often be avoided. For example, a divided difference table can be formed and printed as a lower triangular matrix. The entries of any row depend only on the entries of the preceding row. Thus, one-dimensional arrays may be used to save the preceding row and the current row being calculated. It is important to note that usually the entire array need not be saved. For example, only special values in the table are needed for the coefficients of an interpolating polynomial.
4. **Avoid using formulae that may be highly susceptible to round off error.** Exercise caution when computing quotients of extremely small values as in Equation (8.3.7) with a very small value of $h$.

5. **Alter formulae** for iterations to obtain a “small correction” to an approximation. For example, writing $\frac{a+b}{2}$ as $a + \frac{b-a}{2}$ in the bisection method (see page 674) is recommended. Many of the iteration formulae in this chapter have this form.

6. **Pivoting strategies** are recommended when solving linear systems to reduce round off error.

7. **Eliminate unnecessary steps** that may increase execution time or round off error.

8. Some methods converge very rapidly, when they do converge, but rely on reasonably close initial approximations. A weaker, but reliable, method (such as the bisection method) to obtain such an approximation can be combined with a more powerful method (such as Newton’s method). The weaker method might converge slowly and, by itself, is not very efficient. The powerful method might not converge at all. The combination, however, might remedy both difficulties.

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**References**


Chapter 9

Financial Analysis

9.1 FINANCIAL FORMULAE

9.1.1 Definition of financial terms
9.1.2 Formulae connecting financial terms
9.1.3 Examples

9.2 FINANCIAL TABLES

9.3 OPTION PRICING

9.1 FINANCIAL FORMULAE

9.1.1 DEFINITION OF FINANCIAL TERMS

- $A$ amount that $P$ is worth, after $n$ time periods, with $i$ percent interest per period
- $B$ total amount borrowed
- $P$ principal to be invested (equivalently, present value)
- $a$ future value multiplier after one time period
- $i$ percent interest per time period (expressed as a decimal)
- $m$ amount to be paid each time period
- $n$ number of time periods

Note that the units of $A$, $B$, $P$, and $m$ must all be the same, for example, dollars.

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9.1.2 FORMULAE CONNECTING FINANCIAL TERMS

Interest: Let the principal amount \( P \) be invested at an interest rate of \( i\% \) per time period (expressed as a decimal), for \( n \) time periods. Let \( A \) be the amount that this is worth after \( n \) time periods. Then

- Simple interest:
  \[
  A = P(1 + ni) \quad \text{and} \quad \frac{A}{1 + ni} = \frac{1}{n} \left( \frac{A}{P} - 1 \right)
  \] (9.1.1)

- Compound interest (see tables beginning on page 723):
  \[
  A = P(1 + i)^n \quad \text{and} \quad \frac{A}{1 + i} = \left( \frac{A}{P} \right)^{1/n} - 1.
  \] (9.1.2)

When interest is compounded \( q \) times per time period for \( n \) time periods, it is equivalent to an interest rate of \( (i/q)\% \) per time period for \( nq \) time periods.

\[
A = P \left( \frac{1 + i}{q} \right)^{nq},
\] (9.1.3)

\[
P = A \left( \frac{1 + i}{q} \right)^{−nq},
\] (9.1.4)

\[
i = q \left[ \left( \frac{A}{P} \right)^{1/nq} - 1 \right].
\] (9.1.5)

Continuous compounding occurs when the interest is compounded infinitely often in each time period (i.e., \( q \to \infty \)). In this case: \( A = Pe^{in} \).

Present value: If \( A \) is to be received after \( n \) time periods of \( i\% \) interest per time period, then the present value \( P \) of such an investment is given by (from Equation (9.1.2)) \( P = A(1 + i)^{−n} \).

Annuities: Suppose that the amount \( B \) (in dollars) is borrowed, at a rate of \( i\% \) per time period, to be repaid at a rate of \( m \) (in dollars) per time period, for a total of \( n \) time periods. Then (see tables beginning on page 726):

\[
m = Bi \frac{(1 + i)^n}{(1 + i)^n - 1},
\] (9.1.6)

\[
B = \frac{m}{i} \left( 1 - \frac{1}{(1 + i)^n} \right).
\] (9.1.7)

Using \( a = 1 + i \), these equations can be written more compactly as

\[
m = Bi \frac{a^n}{a^n - 1} \quad \text{and} \quad B = \frac{m}{i} \left( 1 - \frac{1}{a^n} \right).
\] (9.1.8)

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9.1.3 EXAMPLES

1. **Question:** If $100 is invested at 5% per year, compounded for 10 years, what is the resulting amount?

   - **Analysis:** Using Equation (9.1.2), we identify
     
     - Principal invested, $P = 100$ (the units are dollars)
     - Time period, 1 year
     - Interest rate per time period, $i = 5\% = 0.05$
     - Number of time periods, $n = 10$

   - **Answer:** $A = P(1 + i)^n$ or $A = 100(1 + 0.05)^{10} = \$162.89$. (Or, see tables starting on page 726.)

2. **Question:** If $100 is invested at 5% per year and the interest is compounded 4 times a year for 10 years, what is the final amount?

   - **Analysis:** Using Equation (9.1.4) we identify
     
     - Principal invested, $P = 100$ (the units are dollars)
     - Time period, 1 year
     - Interest rate per time period, $i = 5\% = 0.05$
     - Number of time periods, $n = 10$
     - Number of compounding time periods, $q = 4$

   - **Answer:** $A = P \left(1 + \frac{i}{q}\right)^{nq}$ or $A = 100 \left(1 + \frac{0.05}{4}\right)^{4 \cdot 10} = 100(1.0125)^{40} = \$164.36$.

   - **Alternate analysis:** Using Equation (9.1.2), we identify
     
     - Principal invested, $P = 100$ (the units are dollars)
     - Time period, quarter of a year
     - Interest rate per time period, $i = \frac{5\%}{4} = 0.05 = 0.0125$
     - Number of time periods, $n = 10 \cdot 4 = 40$

   - **Alternate answer:** $A = P(1 + i)^n$ or $A = 100(1.0125)^{40} = \$164.36$. (Or, see tables starting on page 723.)

3. **Question:** If $100 is invested now, and we wish to have $200 at the end of 10 years, what yearly compound interest rate must we receive?
**Analysis:** Using Equation (9.1.2), we identify
- Principle invested, \( P = 100 \) (the units are dollars)
- Final amount, \( A = 200 \)
- Time period, 1 year
- Number of time periods, \( n = 10 \)

**Answer:**
\[ i = \left( \frac{A}{P} \right)^{1/n} - 1 \text{ or } i = \left( \frac{200}{100} \right)^{1/10} - 1 = 0.0717. \] (Or, see tables starting on page 724.) Hence, we must receive an annual interest rate of 7.2%.

4. **Question:** An investment returns $10,000 in 10 years time. If the interest rate will be 10% per year, what is the present value (that is, how much money would have to be invested now to obtain this amount in ten years)?

**Analysis:** Using Equation (9.1.2), we identify
- Final amount, \( A = 10,000 \) (the units are dollars)
- Time period, 10 years
- Interest rate per time period, \( i = 10\% = 0.1 \)
- Number of time periods, \( n = 10 \)

**Answer:**
\[ P = A(1 + i)^{-n} = 10000(1.1)^{-10} = 3855.43 \] (Or, see tables starting on page 723.) The present value of this investment is $3,855.43.

5. **Question:** A mortgage of $100,000 is obtained with which to buy a house. The mortgage will be repaid at an interest rate of 9% per year, compounded monthly, for 30 years. What is the monthly payment?

**Analysis:** Using Equation (9.1.8), we identify
- Amount borrowed, \( B = 100,000 \) (the units are dollars)
- Time period, 1 month
- Interest rate per time period, \( i = 0.09/12 = 0.0075 \)
- Number of time periods, \( n = 30 \cdot 12 = 360 \)

**Answer:**
\[ a = 1 + i = 1.0075 \text{ and } m = Bi \frac{a^n}{a - 1} = (100, 000)(.0075) \times \frac{(1.0075)^{360}}{(1.0075)^{360} - 1} = 755.63. \] (Or, see tables starting on page 726.) The monthly payment is $755.63.

6. **Question:** Suppose that interest rates on 15-year mortgages are currently 6%, compounded monthly. By spending $600 per month, what is the largest mortgage obtainable?

**Analysis:** Using Equation (9.1.8), we identify
- Time period, 1 month
- Payment amount, \( m = 600 \) (the units are dollars)
Interest rate per time period, \( i = 0.06/12 = 0.005 \)

Number of time periods, \( n = 15 \cdot 12 = 180 \)

- **Answer:** \( a = 1+i = 1.005 \) and \( B = m \left( 1 - \frac{1}{a^n} \right) / i = \frac{800}{0.005} \left( 1 - \frac{1}{1.005^{180}} \right) = 94802.81 \). (Or, see tables starting on page 726.) The largest mortgage amount obtainable is $94,802.81.

### 9.2 FINANCIAL TABLES

**Compound interest: find final value**

These tables use Equation (9.1.2) to determine the final value in dollars \( (A) \) when one dollar \( (P = 1) \) is invested at an interest rate of \( i \) per time period, the length of investment time being \( n \) time periods. For example, if $1 is invested at a return of 3% per time period, for 50 time periods, then the final value would be $4.38 (see following table). Analogously, if $10 had been invested, then the final value would be $43.84.
### Compound interest: find interest rate

These tables use Equation (9.1.2) to determine the compound interest rate \( i \) that must be obtained from an investment of one dollar \( P = 1 \) to yield a final value of \( A \) (in dollars) when the initial amount is invested for \( n \) time periods. For example, if $1 is invested for 50 time periods, and the final amount obtained is $4.00, then the actual interest rate has been 2.81% per time period (see following table). Analogously, if $100 had been invested, and the final amount was $400, then the interest rate would also be 2.81% per time period.

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Compound interest: find annuity

These tables use Equation (9.1.6) to determine the annuity (or mortgage) payment that must be paid each time period, for \( n \) time periods, at an interest rate of \( i\% \) per time period, to pay off a loan of one dollar (\( B = 1 \)). For example, if $1 is borrowed at 3% interest per time period, and the amount is to be paid back in equal amounts over 10 time periods, then the amount paid back per time period is $0.12 (see following table). Analogously, if $100 had been borrowed, then the mortgage amount would be $11.72.

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### 9.3 Option Pricing

Let $S$ represent the price of a share of stock, and presume that $S$ follows a geometric Brownian motion $dS = \mu S \, dt + \sigma S \, d\omega$, where $t$ is time, $\mu$ is a constant, and $\sigma$ is a constant called the volatility. Let $V(S, t)$ be the value of a derivative security whose payoff is solely a function of $S$ and $t$. We construct a portfolio consisting of $V$ and $\Delta$ shares of stock. The value $P$ of this portfolio is $P = V + S/\Delta$. The random component of the portfolio increment ($dP$) can be removed by choosing $\Delta = -\partial V/\partial S$. The concept of arbitrage says that $dP = rP \, dt$, where $r$ is the (constant) risk-free bank interest rate. Together this results in the classical Black–Scholes equation for option pricing (note that no transaction costs are included):

$$
\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0.
$$

(9.3.1)

If the asset pays a continuous dividend of $DS \, dt$ (i.e., this is proportional to the asset value $S$ during the time period $dt$) then the modified equation is

$$
\frac{\partial V}{\partial t} + (r - D)S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0.
$$

(9.3.2)

If $E$ is the exercise price of the option, and $T$ is the only date on which the option can be exercised ($T$ is called the expiry date), then the solution of the modified Black–Scholes equation is

$$
V(S, t) = e^{-D(T-t)} S \Phi(d_1) - E e^{-r(T-t)} \Phi(d_2)
$$

(9.3.3)

where

$$
d_1 = \frac{\log(S/E) + (r - D + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}, \quad d_2 = d_1 - \sigma \sqrt{T - t},
$$

and $\Phi$ is the cumulative probability distribution for the normal distribution.
Chapter 10

Miscellaneous

10.1 UNITS
   10.1.1 SI system of measurement
   10.1.2 Dimensional analysis/Buckingham pi
   10.1.3 Units of physical quantities
   10.1.4 Conversion: metric to English
   10.1.5 Conversion: English to metric
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10.2 CALENDAR COMPUTATIONS
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10.3 AMS CLASSIFICATION SCHEME

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10.8 FIELDS’ MEDALS

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REFERENCES

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10.1 UNITS

10.1.1 SI SYSTEM OF MEASUREMENT

SI, the abbreviation of the French words “Systeme Internationale d’Unites”, is the accepted abbreviation for the International Metric System. It has seven base units.

- Base units

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<td>Time</td>
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<tr>
<td>Electric current</td>
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<td>Temperature</td>
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<tr>
<td>Luminous intensity</td>
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- Derived units

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10.1.2 DIMENSIONAL ANALYSIS/BUCKINGHAM PI

The units of the parameters in a system constrain all the derivable quantities, regardless of the equations describing the system. In particular, all derived quantities are functions of dimensionless combinations of parameters. The number of dimensionless parameters and their forms are given by the Buckingham pi theorem.

In a system, the quantity \( u = f(W_1, W_2, \ldots, W_n) \) is to be determined in terms of the \( n \) measurable variables and parameters \( \{W_i\} \) where \( f \) is an unknown function. Let the quantities \( \{u, W_i\} \) involve \( m \) fundamental dimensions labeled by \( L_1, L_2, \ldots, L_m \) (such as length, mass, time, or charge). The dimensions of any of the \( \{u, W_i\} \) is given by a product of powers of the fundamental dimensions. For example, the dimensions of \( W_i \) are \( L_1^{b_{i1}} L_2^{b_{i2}} \cdots L_m^{b_{im}} \) where the \( \{b_{ij}\} \) are real and called the dimensional exponents. A quantity is called dimensionless if all of its dimensional exponents are zero. Let

\[
\mathbf{b}_i = \begin{bmatrix} b_{i1} & b_{i2} \cdots & b_{im} \end{bmatrix}^T
\]

be the dimension vector of \( W_i \) and let

\[
\mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}
\]

be the \( m \times n \) dimension matrix of the system. Let \( \mathbf{a} = \begin{bmatrix} a_1 & a_2 \cdots & a_m \end{bmatrix}^T \) be the dimension vector of \( u \) and let \( \mathbf{y} = \begin{bmatrix} y_1 & y_2 \cdots & y_n \end{bmatrix}^T \) represent a solution of \( \mathbf{B} \mathbf{y} = -\mathbf{a} \). Then,

1. The number of dimensionless quantities is \( k + 1 = n + 1 - \text{rank}(\mathbf{B}) \), and
2. The measurable quantity \( u \) can be expressed in terms of dimensionless parameters as

\[
u = W_1^{-\gamma_1} W_2^{-\gamma_2} \cdots W_n^{-\gamma_n} g(\pi_1, \pi_2, \ldots, \pi_k) \tag{10.1.1}\]

where \( g \) is an unknown function of its parameters and the \( \{\pi_i\} \) are the dimensionless quantities, \( \pi_i = W_1^{\gamma_{i1}} W_2^{\gamma_{i2}} \cdots W_n^{\gamma_{in}} \), and \( \mathbf{x}^{(i)} = \begin{bmatrix} x_{i1} & x_{i2} & \cdots & x_{in} \end{bmatrix}^T \) (for \( i = 1, 2, \ldots, k \)) represent the \( k = n - r(\mathbf{B}) \) linearly independent solutions of the system \( \mathbf{B} \mathbf{x} = \mathbf{0} \).

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10.1.3 UNITS OF PHYSICAL QUANTITIES

In the following, read “kilograms” for the mass $M$, “meters” for the length $L$, and “seconds” for the time $T$. For example, acceleration is measured in units of $LT^{-2}$, or meters per second squared.

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<td>$M$</td>
</tr>
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</tr>
<tr>
<td>Power</td>
<td>$ML^2T^{-3}$</td>
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<tr>
<td>Pressure</td>
<td>$ML^{-1}T^{-2}$</td>
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<td>Time</td>
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<tr>
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10.1.4 CONVERSION: METRIC TO ENGLISH

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<td>Liters</td>
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10.1.5 CONVERSION: ENGLISH TO METRIC

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<td>Pounds</td>
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10.1.6 TEMPERATURE CONVERSION

If $T_F$ is the temperature in degrees Fahrenheit and $T_C$ is the temperature in degrees Celsius, then

$$T_C = \frac{5}{9} (T_F - 32) \quad \text{and} \quad T_F = \frac{9}{5} T_C + 32. \quad (10.1.2)$$
10.1.7 MISCELLANEOUS CONVERSIONS

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<td>BTU</td>
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<td>BTU</td>
<td>Joules</td>
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<td>Horsepower-hours</td>
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<td>Kilowatt-hours</td>
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<td>Foot-pounds per minute</td>
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<td>Horsepower</td>
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<td>Feet</td>
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<td>Nautical miles</td>
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<td>Watts</td>
<td>BTU per minute</td>
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10.1.8 PHYSICAL CONSTANTS

Equatorial radius of the earth = 6378.388 km = 3963.34 statute miles.
Polar radius of the earth = 6356.912 km = 3949.99 statute miles.
1 degree of latitude at 40° = 6.9 miles.
1 international nautical mile = 1.15078 statute miles = 1852 m = 6076.115 ft.
Mean density of the earth = 5.522 g/cm³ = 344.7 lb/ft³.
G (gravitational constant) = \((6.673 \pm 0.003) \times 10^{-8}\) cm³/g·sec².
Acceleration, sea level, latitude 45° = 980.6194 cm/sec² = 32.1726 ft/sec².
Length of seconds pendulum, sea level, 45° = 99.3575 cm = 39.1171 in.
1 knot (international) = 101.269 ft/min = 1.6878 ft/sec = 1.1508 statute miles/hr.
1 micron = \(10^{-6}\) cm.
1 angstrom = \(10^{-8}\) cm.
Mass of hydrogen atom = \((1.67339 \pm 0.0031) \times 10^{-24}\) g.
Density of mercury, at 0°C = 13.5955 g/ml.
Density of water (maximum), at 3.98°C = 1.000000 g/ml.
Density of water, at 0°C = 0.999973 g/ml.
Density of dry air, at 0°C, 760 mm = 1.2929 g/liter.
Velocity of sound, dry air, at 0°C = 331.36 m/sec = 1087.1 ft/sec.
c (speed of light) = 299,792,458 m/sec (exact).
Heat of fusion of water, at 0°C = 79.71 cal/g.
Heat of vaporization of water, at 100°C = 539.55 cal/g.
Electrochemical equivalent of silver = 0.001118 g/sec international amp.
Absolute wavelength of red cadmium light, air at 15°C, 760 mm = 6438.4696 Å.
Wavelength of orange-red line of krypton 86 = 6057.802 Å.
e (charge of electron) = 1.602192 \times 10^{-19}\) coul.
Avogadro's number = 6.022169 \times 10^{23}.
1 astronomical unit = 1.495979 \times 10^{11} m.

10.2 CALENDAR COMPUTATIONS

10.2.1 LEAP YEARS

If a year is divisible by 4, then it will be a leap year, unless the year is divisible by 100 (when it will not be a leap year), unless the year is divisible by 400 (when it will be a leap year). Hence the list of leap years include \{..., 1896, 1904, 1908, ..., 1992, 1996, 2000, ...\} and the list of nonleap years includes \{..., 1900, ..., 1998, 1999, 2001, ...\}.

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### 10.2.2 DAY OF WEEK FOR ANY GIVEN DAY

The following formula gives the day of the week for the Gregorian calendar (i.e., for any date after 1582):

\[
W \equiv \left( k + \lfloor 2.6m - 0.2 \rfloor - 2C + Y + \left\lfloor \frac{Y}{4} \right\rfloor + \left\lfloor \frac{C}{4} \right\rfloor \right) \mod 7 \tag{10.2.1}
\]

where

- The “mod” function returns an nonnegative value.
- \( \lfloor \cdot \rfloor \) denotes the integer floor function.
- \( k \) is the day of the month (1 to 31).
- \( m \) is the month (1 = March, . . . , 10 = December, 11 = January, 12 = February).
  (January and February are treated as months of the preceding year).
- \( C \) is century (1997 has \( C = 19 \)).
- \( Y \) is the year (1997 has \( Y = 97 \) except \( Y = 96 \) for January and February).
- \( W \) is the day of the week (0 = Sunday, . . . , 6 = Saturday).

For example, consider the date 16 March 1997 (for which \( k = 16, m = 1, C = 19, \) and \( Y = 97 \)). From Equation (10.2.1), we compute

\[
W \equiv 16 + \lfloor 2.6 \rfloor - 38 + 97 + \left\lfloor \frac{97}{4} \right\rfloor + \left\lfloor \frac{19}{4} \right\rfloor \mod 7 \equiv 2 + 2 - 3 + 6 + 3 + 4 \mod 7 \equiv 0 \mod 7.
\]

So this date will be a Sunday.

Because 7 does not divide 400, January 1 occurs more frequently on some days than others! In a cycle of 400 years, January 1 and March 1 occur on the following days with the following frequencies:

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<th>Wed</th>
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### 10.2.3 NUMBER OF EACH DAY OF THE YEAR

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*In leap years, after February 28, add 1 to the tabulated number.*
### 10.3 AMS CLASSIFICATION SCHEME

<table>
<thead>
<tr>
<th>00</th>
<th>General</th>
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<tbody>
<tr>
<td>01</td>
<td>History and biography</td>
</tr>
<tr>
<td>03</td>
<td>Mathematical logic and foundations</td>
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<tr>
<td>04</td>
<td>Set theory</td>
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<tr>
<td>05</td>
<td>Combinatorics</td>
</tr>
<tr>
<td>06</td>
<td>Order, lattices, ordered algebraic structures</td>
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<td>08</td>
<td>General algebraic systems</td>
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<tr>
<td>11</td>
<td>Number theory</td>
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<tr>
<td>12</td>
<td>Field theory and polynomials</td>
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<tr>
<td>13</td>
<td>Commutative rings and algebras</td>
</tr>
<tr>
<td>14</td>
<td>Algebraic geometry</td>
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<tr>
<td>15</td>
<td>Linear and multilinear algebra; matrix theory</td>
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<tr>
<td>16</td>
<td>Associative rings and algebras</td>
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<tr>
<td>17</td>
<td>Nonassociative rings and algebras</td>
</tr>
<tr>
<td>18</td>
<td>Category theory, homological algebra</td>
</tr>
<tr>
<td>19</td>
<td>( K )-theory</td>
</tr>
<tr>
<td>20</td>
<td>Group theory and generalizations</td>
</tr>
<tr>
<td>22</td>
<td>Topological groups, Lie groups</td>
</tr>
<tr>
<td>26</td>
<td>Real functions</td>
</tr>
<tr>
<td>28</td>
<td>Measure and integration</td>
</tr>
<tr>
<td>30</td>
<td>Functions of a complex variable</td>
</tr>
<tr>
<td>31</td>
<td>Potential theory</td>
</tr>
<tr>
<td>32</td>
<td>Several complex variables and analytic spaces</td>
</tr>
<tr>
<td>33</td>
<td>Special functions</td>
</tr>
<tr>
<td>34</td>
<td>Ordinary differential equations</td>
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<tr>
<td>35</td>
<td>Partial differential equations</td>
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<tr>
<td>39</td>
<td>Finite differences and functional equations</td>
</tr>
<tr>
<td>40</td>
<td>Sequences, series, summability</td>
</tr>
<tr>
<td>41</td>
<td>Approximations and expansions</td>
</tr>
<tr>
<td>42</td>
<td>Fourier analysis</td>
</tr>
<tr>
<td>43</td>
<td>Abstract harmonic analysis</td>
</tr>
<tr>
<td>44</td>
<td>Integral transforms, operational calculus</td>
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<td>45</td>
<td>Integral equations</td>
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<tr>
<td>46</td>
<td>Functional analysis</td>
</tr>
<tr>
<td>47</td>
<td>Operator theory</td>
</tr>
<tr>
<td>49</td>
<td>Calculus of variations and optimal control; optimization</td>
</tr>
<tr>
<td>51</td>
<td>Geometry</td>
</tr>
<tr>
<td>52</td>
<td>Convex and discrete geometry</td>
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<td>53</td>
<td>Differential geometry</td>
</tr>
<tr>
<td>54</td>
<td>General topology</td>
</tr>
<tr>
<td>55</td>
<td>Algebraic topology</td>
</tr>
<tr>
<td>57</td>
<td>Manifolds and cell complexes</td>
</tr>
<tr>
<td>58</td>
<td>Global analysis, analysis on manifolds</td>
</tr>
<tr>
<td>60</td>
<td>Probability theory and stochastic processes</td>
</tr>
<tr>
<td>62</td>
<td>Statistics</td>
</tr>
<tr>
<td>65</td>
<td>Numerical analysis</td>
</tr>
<tr>
<td>68</td>
<td>Computer science</td>
</tr>
<tr>
<td>70</td>
<td>Mechanics of particles and systems</td>
</tr>
<tr>
<td>73</td>
<td>Mechanics of solids</td>
</tr>
<tr>
<td>76</td>
<td>Fluid mechanics</td>
</tr>
<tr>
<td>78</td>
<td>Optics, electromagnetic theory</td>
</tr>
<tr>
<td>80</td>
<td>Classical thermodynamics, heat transfer</td>
</tr>
<tr>
<td>81</td>
<td>Quantum theory</td>
</tr>
<tr>
<td>82</td>
<td>Statistical mechanics, structure of matter</td>
</tr>
<tr>
<td>83</td>
<td>Relativity and gravitational theory</td>
</tr>
<tr>
<td>85</td>
<td>Astronomy and astrophysics</td>
</tr>
<tr>
<td>86</td>
<td>Geophysics</td>
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<tr>
<td>90</td>
<td>Economics, operations research, programming, games</td>
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<tr>
<td>92</td>
<td>Biology and other natural sciences, behavioral sciences</td>
</tr>
<tr>
<td>93</td>
<td>Systems theory; control</td>
</tr>
<tr>
<td>94</td>
<td>Information and communication, circuits</td>
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</table>

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10.4 GREEK ALPHABET

For each Greek letter, we illustrate the form of the capital letter and the form of the lower case letter. In some cases, there is a popular variation of the lower case letter.

<table>
<thead>
<tr>
<th>Greek letter</th>
<th>Greek name</th>
<th>English equivalent</th>
<th>Greek letter</th>
<th>Greek name</th>
<th>English equivalent</th>
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<td>a</td>
<td>N ν</td>
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<td>B β</td>
<td>Beta</td>
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<td>Γ γ</td>
<td>Gamma</td>
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<td>Ω ω</td>
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<td>Δ δ</td>
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<td>Π π σρ</td>
<td>Pi</td>
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<td>e</td>
<td>Ρ ρ ϗ</td>
<td>Rho</td>
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<td>Σ σ ζ</td>
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<td>Υ υ</td>
<td>Upsilon</td>
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<td>I i</td>
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<td>i</td>
<td>Φ ϕ</td>
<td>Phi</td>
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<td>K κ</td>
<td>Kappa</td>
<td>k</td>
<td>Χ χ</td>
<td>Chi</td>
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<td>Lambda</td>
<td>l</td>
<td>Ψ ψ</td>
<td>Psi</td>
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<tr>
<td>M μ</td>
<td>Mu</td>
<td>m</td>
<td>Ω ω</td>
<td>Omega</td>
<td>o</td>
</tr>
</tbody>
</table>

10.5 PROFESSIONAL MATHEMATICAL ORGANIZATIONS

- **American Mathematical Society (AMS)**
  P.O. Box 6248, Providence, RI 02940-6248
  Telephone: 401/455-4000, 800/321-4AMS
  Electronic mail: ams@math.ams.org

- **American Mathematical Association of Two-Year Colleges**
  North Lake College, 5001 MacArthur Blvd, Irving, TX 75038-3899
  Telephone: 214/659-5328
  Electronic mail: memays@dccc.edu

- **American Statistical Association**
  1429 Duke Street, Alexandria, VA 22314-3402
  Telephone: 703/684-1221

- **Association for Symbolic Logic**
  Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61801
  Telephone: 217/244-7902
  Electronic mail: asl@symcom.math.uiuc.edu

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• Association for Women in Mathematics
  4114 Computer & Space Sciences Building, University of Maryland,
  College Park, MD 20742-2461
  Telephone: 301/405-7892
  Electronic mail: awm@math.umd.edu

• Canadian Applied Mathematics Society
  Department of Mathematics and Statistics, Simon Fraser University,
  Burnaby, British Columbia, Canada V5A 1S6
  Telephone: 604/291-3337, 604/291-3332
  Electronic mail: gac@cs.sfu.ca

• Canadian Mathematical Society
  577 King Edward, Suite 109, P.O. Box 450, Station A,
  Ottawa, Ontario, Canada K1N 6N5
  Telephone: 613/564-2223
  Electronic mail: exsmc@acadvm1.uottawa.ca

• Casualty Actuarial Society
  1100 North Glebe Road, Suite 600, Arlington, VA 22201
  Telephone: 703/276-3100

• Conference Board of the Mathematical Sciences
  1529 Eighteenth Street, N.W., Washington, DC 20036
  Telephone: 202/293-1170
  Electronic mail: ronrosier@guvax.georgetown.edu

• Consortium for Mathematics and Its Applications (COMAP)
  57 Bedford Street, Suite 210, Lexington, MA 02173
  Telephone: 617/862-7878
  Electronic mail: info@comap.com

• Council on Undergraduate Research: Mathematical and Computer
  Sciences Division
  University of North Carolina at Asheville, One University Heights,
  Asheville, NC 28804-3299
  Telephone: 704/251-6006

• Fibonacci Association
  Department of Computer Science, Box 2201,
  South Dakota State University, Brookings, SD 57007-0194
  Telephone: 605/688-5719

• Industrial Mathematics Society
  P.O. Box 159, Roseville, MI 48066
  Telephone: 313/771-0403

• Institute for Operations Research and the Management Sciences
  (INFORMS)
  940-A Elkridge Landing Road, Linthicum, MD 21090-2909
  Telephone: 410/850-0300, 800/4IN-FORMS
  Electronic mail: informs@jhuvms.hcf.jhu.edu

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<table>
<thead>
<tr>
<th>Organization</th>
<th>Address</th>
<th>Telephone</th>
<th>Electronic Mail</th>
</tr>
</thead>
<tbody>
<tr>
<td>Institute of Mathematical Statistics</td>
<td>3401 Investment Boulevard #7, Hayward, CA 94545-3819</td>
<td>510/783-8141</td>
<td><a href="mailto:ims@stat.berkeley.edu">ims@stat.berkeley.edu</a></td>
</tr>
<tr>
<td>International Mathematics Union (IMPA)</td>
<td>Estrada Dona Castorina, 110, Jardim Botânico, Rio de Janeiro – RJ, 22460 Brazil</td>
<td>55-21-294 9032, 55-21-5111749</td>
<td><a href="mailto:imu@impa.br">imu@impa.br</a></td>
</tr>
<tr>
<td>Joint Policy Board for Mathematics</td>
<td>1529 Eighteenth Street, N.W., Washington, DC 20036</td>
<td>202/234-9570</td>
<td><a href="mailto:jpbm@math.umd.edu">jpbm@math.umd.edu</a></td>
</tr>
<tr>
<td>Kappa Mu Epsilon (κµε)</td>
<td>National Mathematics Honor Society, Department of Mathematics, Niagara University, Niagara Falls, NY 14109</td>
<td></td>
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</tr>
<tr>
<td>Mathematical Association of America (MAA)</td>
<td>Dolciani Mathematical Center, 1529 Eighteenth Street, N.W., Washington, DC 20036</td>
<td>202/387-5200</td>
<td><a href="mailto:maahq@maa.org">maahq@maa.org</a></td>
</tr>
<tr>
<td>Mathematical Programming Society</td>
<td>Department of Mathematics and Computer Science, Eindhoven University of Technology, 5600 MB Eindhoven, The Netherlands</td>
<td>+31-40-474770</td>
<td><a href="mailto:jkl@win.tue.nl">jkl@win.tue.nl</a></td>
</tr>
<tr>
<td>Mu Alpha Theta (µαθ)</td>
<td>University of Oklahoma, 610 Elm Avenue, Room 423, Norman, OK 73019-0315</td>
<td>405/325-4489</td>
<td><a href="mailto:seliason@uoknor.edu">seliason@uoknor.edu</a></td>
</tr>
<tr>
<td>National Association of Mathematicians</td>
<td>Box 959, Elizabeth City State University, Elizabeth City, NC 27909</td>
<td>919/335-3326</td>
<td><a href="mailto:nam@ecsvax.uncecs.edu">nam@ecsvax.uncecs.edu</a></td>
</tr>
<tr>
<td>National Council of Teachers of Mathematics</td>
<td>1906 Association Drive, Reston, VA 22091</td>
<td>703/620-9840</td>
<td></td>
</tr>
<tr>
<td>ORSA (see INFORMS)</td>
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</tbody>
</table>
Pi Mu Epsilon ($\pi \mu \epsilon$)
National Mathematics Honorary Society, Department of Mathematics,
East Carolina University, Greenville, NC 27858
Telephone: 919/328-6414

Rocky Mountain Mathematics Consortium
Arizona State University, Box 871904, Tempe, AZ 85287-1904
Telephone: 602/965-3788

Society of Industrial and Applied Mathematics (SIAM)
3600 University City Science Center, Philadelphia, PA 19104-2688
Telephone: 215/382-9800
Electronic mail: siam@siam.org

Society for Mathematical Biology, Inc.
P. O. Box 11283, Boulder, CO 80301
Telephone: 303/499-0510

Society of Actuaries
475 North Martingale Road, Suite 800, Schaumburg, IL 60173-2226
Telephone: 708/706-3500

Statistical Society of Canada
Department of Mathematics and Statistics, University of Victoria,
Victoria, British Columbia, Canada V8W 3P4
Telephone: 604/721-7470
Electronic mail: roger@uvvm.uvic.ca

10.6 ELECTRONIC MATHEMATICAL RESOURCES

The following is a list of Uniform Resource Locators (URL). These are the addresses of Web sites, gopher pages, FTP sites, etc.

http://www.yahoo.com/Science/Mathematics/
A very large list of useful sites relating to mathematics. It is perhaps the best place to start researching an arbitrary mathematical question not covered elsewhere in this list.

http://daisy.uwaterloo.ca/~alopez-o/math-faq/math-faq.html
The FAQ (frequently asked questions) listing from the news group sci.math.

http://e-math.ams.org/
The American Mathematical Society home page, with information about AMS-$\TeX$, the Combined Membership List of the AMS, Math Reviews subject classifications, preprints, etc.

http://www.siam.org/
The Society for Industrial and Applied Mathematics.
http://netlib2.cs.utk.edu/master/
The master listing for Netlib, containing many standard programs, including linpack, eispack, hompack, SPARC packages, and ODEpack.

http://www.netlib.org/liblist.html
An alternative site for Netlib.

http://gams.cam.nist.gov/
The Guide to Available Mathematical Software.

http://www.nag.co.uk:70/
The home page of the Numerical Algorithms Group.

http://www.math.hmc.edu/codee/home.html
The Consortium of Ordinary Differential Equation Experiments, maintained by Harvey Mudd College.

http://www.ima.umn.edu/
The Institute for Mathematics and its Applications at the University of Minnesota.

http://cam.cornell.edu/~driscoll/research/drums.html
A famous question posed by mathematicat physicist M. Kac is, “Can one hear the shape of a drum?” In 1991, Gordon, Webb, and Wolpert found two different shaped drums with the same sequence of eigenvalues. This site contains some details of the calculations.

http://www.utm.edu:80/departments/math/largest.html
Information about primes, including largest known primes of various types.

http://www.geom.umn.edu/docs/snell/chance/sources.html
A listing of interesting sites, primarily related to statistics.

http://aleph0.clarku.edu/~djoyce/julia/explorer.html
Useful for exploring the Mandelbrot and Julia sets.

ftp://megrez.math.u-bordeaux.fr/pub/numberfields/
The Computational Number Theory group in Bordeaux has announced the availability (by anonymous ftp at the above URL) of extensive tables of number fields (almost 550000 number fields). For the number fields belonging to tables of reasonable length, this site contains the signature, the Galois group of the Galois closure of the field, the discriminant of the number field, the class number, the structure of the class group as a product of cyclic groups, an ideal in the class for each class generating these cyclic groups, the regulator, the number of roots of unity in the field, a generator of the torsion part of the unit group, and a system of fundamental units.

http://netlib.bell-labs.com/netlib/master/readme.html
The AT&T Bell Laboratories Math Archive.

gopher://ejde.math.unt.edu

http://nyjm.albany.edu:8000/nyjm.html
The New York Journal of Mathematics, the first electronic journal devoted to general mathematics. Also available via gopher at gopher nyjm.albany.edu 1070 and via FTP at nyjm.albany.edu in directory /pub/nyjm.

http://www.wavelet.org/wavelet/index.html
The Wavelet Digest is an electronic information service on the Internet at http://www.wavelet.org/wavelet/add.html, containing questions and answers and announcements of papers, books, journals, software, and conferences.

http://www.math.ohio-state.edu/JAT
The Journal of Approximation Theory.

http://www.combinatorics.org/
The World Combinatorics Exchange and The Electronic Journal of Combinatorics.

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http://www.cs.indiana.edu/cstr/search
Unified Computer Science TR Index.

http://www.c3.lanl.gov/laces
LACES is a database of preprints, papers, abstracts, and related documents in areas involving discrete mathematics. This includes combinatorics, graph theory, geometry, algorithms, etc. If you have preprints in any of these areas you are invited to send them to LACES. Submissions are made available to the public immediately. Submission instructions and facilities can be found at this site. Features of LACES include full search capabilities, including approximate search and relevance feedback, comment and reference links, revision, abstract and comment processing (anyone can add comments to documents in the database), and more. Documents are permanently archived, and copyrights remain with the authors.

http://rattler.cameron.edu/swjpam/swjpam.html

http://www.mag-browse.com/
Tables of contents of various magazines. As of the summer of 1995, 25 scientific magazines were included.

Sloane’s sequence identifier
To identify a sequence, send electronic mail to sequences@research.att.com, saying (for example) lookup 1 11 21 1211 111221.

10.7 COMPUTER LANGUAGES

The following is a sampling of the computer languages used by scientists and engineers:

- Numerical languages
  - Ada
  - APL
  - C
  - C++
  - Fortran
  - Lisp
  - Matlab
  - Pascal
- Statistical languages
  - SPSS
  - Minitab
- Optimization languages
  - GAMS (AMPL)
  - MINOS
  - MINTO
- Symbolic languages
  - Axiom
  - Derive
  - Macsyma
  - Maple
  - Mathematica
  - Reduce
10.7.1 CONTACT INFORMATION

1. AXIOM http://www.nag.co.uk/1h/symbolic/AX.html
2. Fortran http://www.nag.co.uk:70/CGI/Forms/Tools/imagemap.cgi/NAG
7. Mathematica http://www.wri.com

10.8 FIELDS’ MEDALS

The Fields’ medal is the most prestigious award that can be bestowed upon a mathematician. It is usually awarded to someone no more than 40 years of age; the ages of the recipients are listed below.

- 1936 Ahlfors, Lars 29 Harvard University
- 1936 Douglas, Jesse 39 MIT
- 1950 Schwartz, Laurent 35 Universite de Nancy
- 1950 Selberg, Atle 33 Princeton/Inst. for Advanced Studies
- 1954 Kodaira, Kunihiko 39 Princeton University
- 1954 Serre, Jean-Pierre 27 College de France
- 1958 Roth, Klaus 32 University of London
- 1958 Thom, Rene 35 University of Strasbourg
- 1962 Hormander, Lars 31 University of Stockholm
- 1962 Milnor, John 31 Princeton University
- 1966 Atiyah, Michael 37 Oxford University
- 1966 Cohen, Paul 32 Stanford University
- 1966 Grothendieck, Alexander 38 University of Paris
- 1966 Smale, Stephen 36 University of California at Berkeley
- 1970 Baker, Alan 31 Cambridge University
- 1970 Hironaka, Heisuke 39 Harvard University
- 1970 Novikov, Serge 32 Moscow University
- 1970 Thompson, John 37 University of Chicago
10.9 BIOGRAPHIES OF MATHEMATICIANS

Ah'mose (c. 1650 B.C.E.) was the scribe responsible for copying the Rhind Papyrus, the most detailed original document still extant on ancient Egyptian mathematics. The papyrus contains some 87 problems with solutions dealing with what we consider first-degree equations, arithmetic progressions, areas and volumes of rectangular and circular regions, proportions, and several other topics. It also contains a table of the results of the division of 2 by every odd number from 3 to 101.

Euclid (c. 300 B.C.E.) is responsible for the most famous mathematics text of all time, the Elements. Not only does this work deal with the standard results of plane geometry, but it also contains three chapters on number theory, one long chapter on irrational quantities, and three chapters on solid geometry, culminating with the construction of the five regular solids. The axiom-definition-theorem-proof style of Euclid’s work has become the standard for formal mathematical writing up to the present day.
Archimedes (287–212 B.C.E.) not only wrote several works on mathematical topics more advanced than Euclid, but also was the first mathematician to derive quantitative results from the creation of mathematical models of physical problems on earth. In several of his books, he described the reasoning process by which he arrived at his results in addition to giving formal proofs. For example, he showed how to calculate the areas of a segment of a parabola and the region bounded by one turn of a spiral, and the volume of a paraboloid of revolution.

Ptolemy (C. 100–178 C.E.) is most famous for the *Almagest*, a work in thirteen books, which contains a complete mathematical description of the Greek model of the universe with parameters for the various motions of the sun, moon, and planets. The first book provides the strictly mathematical material detailing the plane and spherical trigonometry, all based solely on the chord function, necessary for astronomical computations.

Hypatia (c. 370–415), the first woman mathematician on record, lived in Alexandria. She was given a very thorough education in mathematics and philosophy by her father Theon and was responsible for detailed commentaries on several important Greek works, including Ptolemy’s *Almagest*, Apollonius’s *Conics*, and Diophantus’s *Arithmetica*.

Brahmagupta (c. 598–670), from Rajasthan in India, is most famous for his *Brahmasphutasiddhanta* (*Correct Astronomical System of Brahma*), an astronomical work which contains many chapters on mathematics. Among the mathematical problems he considered and gave solution algorithms for were systems of linear congruences, quadratic equations, and special cases of the Pell equation $Dx^2 \pm 1 = y^2$. He also gave the earliest detailed treatment of rules for operating with positive and negative numbers.

Bhaskara (1114–1185), the most famous of medieval Indian mathematicians, gave a complete algorithmic solution to the Pell equation. In addition, he dealt with techniques of solving systems of linear equations with more unknowns than equations and was familiar with the basic combinatorial formulas, giving many examples, though no proofs, of their use.

Qin Jiushao (1202–1261), born in Sichuan, published a general procedure for solving systems of linear congruences—the Chinese remainder theorem—in his *Shushu jiuzhang* (*Mathematical Treatise in Nine Sections*) in 1247, a procedure which makes essential use of the Euclidean algorithm. He also gave a complete description of a method for solving numerically polynomial equations of any degree. Qin’s method was developed in China over a period of a thousand years or more and is very similar to what is now called the Horner method of solution, published by William Horner in 1819.

Muhammad al-Khwarizmi (c. 780–850), originally from Khwarizm in what is now Uzbekistan, was one of the first scholars called to the House of Wisdom in Baghdad by the caliph al-Ma’mun. He is best known for his algebra text, in which he gave a careful treatment of solution methods for quadratic equations. This Arabic text, after being translated into Latin in the twelfth century, provided Europeans with an introduction to algebra, a subject not considered by the ancient Greeks. Al-Khwarizmi’s book on arithmetic provided Europe with one of its earliest looks at the Hindu-Arabic number system.

Abu Ali ibn al-Haytham (965–1039), who spent much of his life in Egypt, is most famous for his work on optics, a work read and commented on for many centuries in Europe. In pure mathematics, he developed an inductive procedure for calculating formulas for the sums of integral powers of the first $n$ integers, and used the formula for fourth powers to calculate the volume of the solid formed by revolving a parabola about a line perpendicular to its axis.
Nasir al-Din al-Tusi (1201–1274) was the head of a large group of astronomers at the observatory in Maragha, in what is now Iran. He computed a new set of very accurate astronomical tables and developed some new ideas on planetary motion which may have influenced Copernicus in working out his heliocentric system. In pure mathematics, al-Tusi’s attempted proof of the parallel postulate was modified by his son and later published in Rome, where it influenced European work on non-Euclidean geometry. Al-Tusi also wrote the first systematic work on plane and spherical trigonometry, independent of astronomy, and gave the earliest proof of the theorem of sines.

Leonardo of Pisa (1170–1240), often known today as Fibonacci, is most famous for his Liber Abbaci (Book of Calculation), which contains the earliest publication of the Fibonacci numbers in the problem of how many pairs of rabbits can be bred in one year from one pair. Many of the sources of the book are in the Islamic world, where Leonardo spent much of his early life. The work contains the rules for computing with the new Hindu–Arabic numerals, many practical problems in such topics as calculation of profits and currency conversions, and topics now standard in algebra texts such as motion problems, mixture problems, and quadratic equations.

Levi ben Gerson (1288–1344) was a French rabbi and also an astronomer, philosopher, biblical commentator, and mathematician. His most famous mathematical work is the Maasei Hoshev (The Art of the Calculator), which contains detailed proofs of the standard combinatorial formulas, some of which use the principle of mathematical induction.

Gerolamo Cardano (1501–1576), a physician and gambler as well as a mathematician, wrote one of the earliest works containing systematic probability calculations, not all of which were correct. He is most famous, however, for his Ars Magna (The Great Art, 1545), an algebra text which contained the first publication of the rules for solving cubic equations algebraically. Some of the rules had been discovered earlier in the sixteenth century by Scipione del Ferro and Niccolò Tartaglia.

François Viète (1540–1603), a lawyer and advisor to two kings of France, was one of the earliest cryptanalysts and successfully decoded intercepted messages for his patrons. Although a mathematician only by avocation, he made important contributions to the development of algebra. In particular, he introduced letters to stand for numerical constants, thus enabling him to break away from the style of verbal algorithms of his predecessors and treat general examples by formulas rather than by giving rules for specific problems.

Simon Stevin (1548–1620) spent much of his life in the service of Maurice of Nassau, the Stadhouder of Holland, as a military engineer, advisor in finance and navigation, and quartermaster general of the Dutch army. In his book De Thiende (The Art of Tenths), Stevin introduced decimal fractions to Europe, although they had previously been used in the Islamic world. Stevin’s notation is different from our own, but he had a clear understanding of the advantage of decimals and advocated their use in all forms of measurement.

John Napier (1550–1617) was a Scottish laird who worked for years on the idea of producing a table which would enable one to multiply any desired numbers together by performing additions. These tables of logarithms first appeared in his 1614 book Mirifici Logarithmorum Canonis Descriptio (Description of the Wonderful Canon of Logarithms). Napier’s logarithms are different from, but related to, natural logarithms. His ideas were soon adapted by Henry Briggs, who eventually created the first table of common logarithms by 1628.
René Descartes (1596–1650) published the *Geometry* in 1637 as a supplement to his philosophical work, the *Discourse on the Method for Rightly Directing One’s Reason and Searching for Truth in the Sciences*. In it, he developed the principles of analytic geometry, showing how to derive algebraic equations which represented geometric curves. The *Geometry* also contained methods for solving polynomial equations, including the modern factor theorem and Descartes’ rule of signs.

Pierre de Fermat (1601–1665) was a French lawyer who spent his spare time doing mathematics. Not only was he a coinventor of analytic geometry, although his methods were somewhat different from those of Descartes, but he also was instrumental in the early development of probability theory and made many contributions to the theory of numbers. He is most remembered for the statement of his so-called “last theorem”, that the equation $x^n + y^n = z^n$ has no nontrivial integral solution if $n > 2$, a theorem whose proof was finally completed by Andrew Wiles in 1994.

Blaise Pascal (1623–1662) showed his mathematical precocity with his *Essay on Conics* of 1640 in which he stated his theorem that the opposite sides of a hexagon inscribed in a conic section always intersect in three collinear points. Pascal is better known, however, for his detailed study of what is now called Pascal’s triangle of binomial coefficients, the basic facts of which had been known in the Islamic and Chinese worlds for centuries. He also introduced the differential triangle in his *Treatise on the Sines of a Quadrant of a Circle*, an idea adopted by Leibniz in his calculus.

Isaac Newton (1642–1727), the central figure in the Scientific Revolution, is most famous for his *Philosophiae Naturalis Principia Mathematica* (*Mathematical Principles of Natural Philosophy*, 1687), in which he derived his system of the world based on his laws of motion and his law of universal gravitation. Over 20 years earlier, however, Newton had consolidated and generalized all the material on tangents and areas worked out by his predecessors into the magnificent problem solving tool of the calculus. He also developed the power series as a method of investigating various transcendental functions, stated the general binomial theorem, and, although never establishing his methods with the rigor of Greek geometry, did demonstrate an understanding of the concept of limit quite sufficient for him to apply the calculus to solve many important mathematical and physical problems.

Gottfried Wilhelm Leibniz (1646–1716), born in Leipzig, developed his version of the calculus some ten years after Isaac Newton, but published it much earlier. Leibniz based his calculus on the inverse relationship of sums and differences, generalized to infinitesimal quantities called differentials. By clever manipulation of differentials, based in part on the geometrical model of the differential triangle, Leibniz was able to derive all of the basic rules of the differential and integral calculus and apply them to solve physical problems expressible in terms of differential equations. Leibniz’s $d$ and $\int$ notation for differentials and integrals turned out to be much more flexible and useful than Newton’s dot notation and remains the notation of calculus to the present day.

Johann Bernoulli (1667–1748), one of a number of prominent mathematicians of his Swiss family, was one of the earliest proponents of Leibniz’s differential and integral calculus. Bernoulli helped to stimulate the development of the new techniques by proposing challenge problems to mathematicians, the most important probably being that of describing the brachistochrone, the curve representing the path of descent of a body between two given points in the shortest possible time. Many of the problems he posed required the solution of differential equations, and Bernoulli developed many techniques useful toward this end, including the calculus of the logarithmic and exponential functions.
Leonhard Euler (1707–1783), a student of Johann Bernoulli in Basel who became one of the earliest members of the St. Petersburg Academy of Sciences founded by Peter the Great of Russia, was the most prolific mathematician of all time. His series of analysis texts, *Introduction to Analysis of the Infinite*, *Methods of the Differential Calculus*, and *Methods of the Integral Calculus*, established many of the notations and methods still in use today. Among his numerous contributions to every area of mathematics and physics are his development of the calculus of the trigonometric functions, the establishment of the theory of surfaces in differential geometry, and the creation of the calculus of variations.

Maria Agnesi (1718–1799), the eldest child of a professor of mathematics at the University of Bologna, in 1748 published the clearest text on calculus up to that point. Based on the work of Leibniz and his followers, the work explained concepts lucidly and provided numerous examples, including some that have become standard in calculus texts to this day. Curiously, her name is often attached to a small item in her book not even original with her, a curve whose equation was \( y = \frac{\sqrt{x^2 - a^2}}{a} \). This curve was called *la versiera*, derived from the Latin meaning “to turn”; unfortunately the word also was the abbreviation of the Italian word meaning “wife of the devil” and so was translated into English as “witch.” The curve has ever since been known as the “witch of Agnesi.”

Joseph Lagrange (1736–1813), was born in Turin, becoming at age 19 a professor of mathematics at the Royal Artillery School there. He is most famous for his *Analytical Mechanics* (1788), a work which extended the mechanics of Newton and Euler, and demonstrated how problems in mechanics can generally be reduced to solutions of ordinary or partial differential equations. In 1797 he published his *Theory of Analytic Functions*, which attempted to reduce the ideas of calculus to those of algebraic analysis by assuming that every function could be represented as a power series. Although his central idea was incorrect, many of the proofs of basic theorems of calculus in this work were subsequently adapted by Cauchy into the forms still in use today.

Benjamin Banneker (1731–1806), the first American black to achieve distinction in science, taught himself sufficient mathematics and astronomy to publish a series of well-regarded almanacs in the 1790s. He also assisted Andrew Ellicott in the survey of the boundaries of the District of Columbia. He was fond of solving mathematical puzzles and problems and recorded many of these in his notebooks.

Augustin-Louis Cauchy (1789–1857), the most prolific mathematician of the nineteenth century, wrote several textbooks in analysis for use at the École Polytechnique, textbooks which became the model for calculus texts for the next hundred years. In his texts, Cauchy based the calculus on the notion of limit, using, for the first time, a definition which could be applied arithmetically to give proofs of some of the important results. Among numerous other subjects to which he contributed important ideas were complex analysis, in which he gave the first proof of the Cauchy integral theorem, the theory of matrices, in which he demonstrated that every symmetric matrix can be diagonalized by use of an orthogonal substitution, and the theory of permutations, in which he was the earliest to consider these from a functional point of view.

William Rowan Hamilton (1805–1865) became the Astronomer Royal of Ireland in 1827 because of his original work in optics accomplished during his undergraduate years at Trinity College, Dublin. In 1837, he showed how to introduce complex numbers into algebra axiomatically by considering \( a + ib \) as a pair \((a, b)\) of real numbers with appropriate computational rules. After many years of seeking an appropriate definition for multiplication rules for triples of numbers which could be applied to vector analysis in three-space, he discovered that it was in fact necessary to consider quadruplets of
numbers. It was out of the natural definition of multiplication of these quaternions that the modern notions of dot product and cross product of vectors evolved.

**Arthur Cayley (1821–1895)**, although graduating from Trinity College, Cambridge, as Senior Wrangler, became a lawyer because there was no suitable mathematics position available in England. He produced nearly 300 mathematical papers during his 14 years as a lawyer, however, and finally secured a professorship at Cambridge in 1863. Among his numerous mathematical achievements are the earliest abstract definition of a group in 1854, out of which he was able to calculate all possible groups of order up to eight and the basic rules for operating with matrices, including a statement (without rigorous proof) of the Cayley–Hamilton theorem that every matrix satisfies its characteristic equation.

**Richard Dedekind (1831–1916)** solved the problem of the lack of unique factorization in rings of algebraic integers by introducing ideals and their arithmetic and demonstrating that every ideal is either prime or can be expressed uniquely as a product of prime ideals. During his teaching at Zurich in the late 1850s, he realized that, although differential calculus deals with continuous magnitudes, there was no satisfactory definition available of what it means for the set of real numbers to be continuous. He therefore worked out a definition of irrational numbers through his idea of what is now called a Dedekind cut in the set of rational numbers. Somewhat later Dedekind also considered the basic ideas of set theory and gave a set theoretic characterization of the natural numbers.

**Carl Friedrich Gauss (1777–1855)** published his important work on number theory, the *Disquisitiones Arithmeticae*, when he was only 24, a work containing not only an extensive discussion of the theory of congruences, culminating in the quadratic reciprocity theorem, but also a detailed treatment of cyclotomic equations in which he showed how to construct regular $n$-gons by Euclidean techniques whenever $n$ is prime and $n - 1$ is a power of 2. Gauss also made fundamental contributions to the differential geometry of surfaces in his *General Investigations of Curved Surfaces* in 1827, as well as to complex analysis, astronomy, geodesy, and statistics during his long tenure as a professor at the University of Göttingen. Many ideas later published by others, including the basics of non-Euclidean geometry, were found in his notebooks after his death.

**Karl Weierstrass (1815–1897)** taught for many years at German gymnasia before producing a series of brilliant mathematical papers in the 1850s which resulted in his appointment to a professorship at the University of Berlin. It was in his lectures there that he insisted on defining every concept of analysis arithmetically, including such ideas as uniform convergence and uniform continuity, thus completing the transformation away from the use of terms such as “infinitely small”. Since he himself never published many of these ideas, his primary influence was through the work of his numerous students.

**Georg Bernhard Riemann (1826–1866)**, in his 1854 inaugural lecture at the University of Göttingen entitled “On the Hypotheses which Lie at the Foundation of Geometry”, discussed the general notion of an $n$-dimensional manifold, developed the idea of a metric relation on such a manifold, and gave criteria which would determine whether a three-dimensional manifold is Euclidean, or “flat”. This lecture had enormous influence on the development of geometry, including non-Euclidean geometry, as well as on the development of a new concept of our physical space ultimately necessary for the theory of general relativity. Among his other achievements, Riemann’s work on complex functions and their associated Riemann surfaces became one of the foundations of combinatorial topology.

**Sofia Kovalevskaya (1850–1891)** was the first European woman since the Renaissance to earn a Ph.D. in mathematics (1874), a degree based on her many new results in the
theory of partial differential equations. Because women were generally not permitted to study mathematics officially in European universities, Kovalevskaya had been forced to study privately with Weierstrass. Her mathematical talents eventually earned her a professorship at the University of Stockholm, an editorship of the journal *Acta Mathematica*, and the Prix Bordin of the French Academy of Sciences for her work on the revolution of a solid body about a fixed point. Unfortunately, her career was cut short by her untimely death from pneumonia at the age of 41.

**Henri Poincaré (1854–1912)**, one of the last of the universal mathematicians, contributed to virtually every area of mathematics, including physics and theoretical astronomy. Among his many contributions was the introduction of the idea of homology into topology, the creation of a model of Lobachevskian geometry which helped to convince mathematicians that this non-Euclidean geometry was as valid as Euclid’s, and a detailed study of the nonlinear partial differential equations governing planetary motion aimed at answering questions about the stability of the solar system. Toward the end of his life, Poincaré wrote several popular books emphasizing the importance of science and mathematics.

**David Hilbert (1862–1943)** is probably most famous for his lecture at the International Congress of Mathematicians in Paris in 1900 in which he presented a list of 23 problems which he felt would be of central importance for 20th century mathematics. Most of the problems have now been solved, while significant progress has been achieved in the remainder. Hilbert himself made notable contributions to the study of algebraic forms, algebraic number theory, the foundations of geometry, integral equations, theoretical physics, and the foundations of mathematics.

**Leonard Eugene Dickson (1874–1954)** was the first recipient of a doctorate in mathematics at the University of Chicago, where he ultimately spent most of his mathematical career. Dickson helped to develop the abstract approach to algebra by developing sets of axioms for such constructs as groups, fields, and algebras. Among his important books was his monumental three volume *History of the Theory of Numbers*, which traced the evolution of every important concept in that field.

**Emmy Noether (1882–1935)** received her doctorate from the University of Erlangen in 1908, a few years later moving to Göttingen to assist Hilbert in the study of general relativity. During her 18 years there, she was extremely influential in stimulating a new style of thinking in algebra by always emphasizing its structural rather than computational aspects. She is most famous for her work on what are now called Noetherian rings, but her inspiration of others is still evident in today’s textbooks in abstract algebra.

**Alan Turing (1912–1954)** developed the concept of a “Turing machine” in 1936 to answer the questions of what a computation is and whether a given computation can in fact be carried out. This notion today lies at the basis of the modern all-purpose computer, a machine which can be programmed to do any desired computation. During World War II, Turing led the successful effort in England to crack the German “Enigma” code, an effort central to the defeat of Nazi Germany.
10.10 ASCII CHARACTER CODES

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References

List of Notations

', derivative
'', derivative
!! double factorial
! factorial
\( a^n \) falling factorial
\( (a)_n \) shifted factorial
\( (r, s, \omega) \) type of a tensor
\( (v, k, \lambda) \) design nomenclature
\( (x, y, z) \) point in three-dimensional space
\( (x : y : t) \) homogeneous coordinates
\( (x : y : z : t) \) homogeneous coordinates
\( \begin{pmatrix} j_1 & j_2 & j \hfill \\ m_1 & m_2 & m \hfill \end{pmatrix} \) Clebsch–Gordan coefficients
\( \binom{n}{m} \) binomial coefficients
\( \binom{n}{n_1, n_2, \ldots, n_k} \) multinomial coefficient
\( \binom{a}{n} \) Jacobi symbol
\( \left( \begin{array}{c} \alpha \\ \mu \end{array} \right) \) Legendre symbol
\( \ldots \) group inverse
\( \langle a \rangle \) cyclic subgroup generated by \( a \)
\[ , \] commutator 1, 2
\[ \mathbf{[abc]} \] scalar triple product
\[ [n] \] Stirling numbers
\[ [r]_q \] Gaussian binomial coefficient
\[ [a_0, a_1, \ldots, a_n] \] continued fraction
\[ [j, k, \ell] \Rightarrow \]
\[ \mathbf{a} \cdot \mathbf{b} \]
\[ \mathbf{a} \times \mathbf{b} \]
\[ \nabla \]
\[ \Delta \]
\[ \cdot \]
\[ \cdot \equiv \]
\[ \emptyset \]
\[ \equiv \]
\[ \exists \]
\[ \forall \]
\[ \int \]
\[ \int^b_a \]
\[ \leftrightarrow \]
\[ \neg \]
\[ \oplus \]
\[ \odot \]
\[ \partial \]
\[ \preceq \]
\[ \preceq \]
\[ \sim \]
\[ \ast \]
\[ \ast \ast \]
\[ \ast \ast \ast \]
\[ \ast \ast \ast \ast \]
\[ \nabla \]
\[ \nabla \]
\[ \{ \binom{n}{m} \} \]
\[ b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \frac{a_4}{b_4 + \ldots}}}} \]

- Christoffel symbol of first kind
- vector inner product
- vector cross product
- backward difference
- forward difference
- inner product
- operation for multiplicative group
- group isomorphism 1, 2
- empty set
- congruence
- existential quantifier
- universal quantifier
- integration symbol
- definite integral
- if and only if
- logical not
- exclusive or
- factored graph notation
- Kronecker sum
- Kronecker product
- partial differentiation
- partial order
- logical implication
- asymptotic relation
- logical not
- vertex similarity
- binary operation
- convolution operation
- group operation
- reflection
- dual of a tensor
- operation for multiplicative group
- glide-reflection
- graph conjunction
- logical and
- wedge product
- Stirling cycle numbers 1, 2
- continued fraction
\( \overline{z} \) \hspace{1cm} \text{complex conjugate of } z
\times \hspace{1cm} \text{product}
\| \| \hspace{1cm} \text{determinant of a matrix}
\| \| \hspace{1cm} \text{graph order}
\| \| \hspace{1cm} \text{order of algebraic structure}
\| \| \hspace{1cm} \text{polynomial norm}
\| \| \hspace{1cm} \text{used in tensor notation}
\| \|_1 \hspace{1cm} L_1 \text{ norm}
\| \|_2 \hspace{1cm} L_2 \text{ norm}
\| \|_F \hspace{1cm} \text{Frobenius norm}
\| \|_\infty \hspace{1cm} \text{infinity norm}
(S, \preceq) \hspace{1cm} \text{poset notation}
I \hspace{1cm} \text{group identity}
A \hspace{1cm} \text{final amount}
A \hspace{1cm} \text{interarrival time}
A \hspace{1cm} \text{ampere}
A(n, d) \hspace{1cm} \text{number of codewords}
A(T) \hspace{1cm} \text{skew symmetric part of a tensor}
A^{-1} \hspace{1cm} \text{matrix inverse}
a.e. \hspace{1cm} \text{almost everywhere}
A/B/c/K/m/Z \hspace{1cm} \text{queue representation}
Ai \hspace{1cm} \text{Airy function}
a_i \hspace{1cm} \text{unit vector}
A_k \hspace{1cm} \text{radius of circumscribed circle}
a_k \hspace{1cm} \text{radius of inscribed circle}
\aleph_0 \hspace{1cm} \text{aleph null}
ALFS \hspace{1cm} \text{additive lagged-Fibonacci sequences}
\alpha \hspace{1cm} \text{one minus the confidence coefficient}
\alpha \hspace{1cm} \text{probability of type I error}
\alpha(G) \hspace{1cm} \text{graph independence number}
am \hspace{1cm} \text{amplitude}
A_n \hspace{1cm} \text{group of even permutations}
a_n \hspace{1cm} \text{Fourier coefficients}
a_n \hspace{1cm} \text{proportion of customers}
ANOVA \hspace{1cm} \text{analysis of variance}
AR(k) \hspace{1cm} \text{autoregressive model}
arg \hspace{1cm} \text{argument}
ARMA(k, l) \hspace{1cm} \text{mixed model}
Aut(G) \hspace{1cm} \text{graph automorphism group 1, 2}
B \hspace{1cm} \text{amount borrowed}
$B$  service time
$\mathcal{B}$  set of blocks
$b$  unit binormal vector
$B(p, q)$  beta function
B.C.E  (before the common era, B.C.)
$\beta$  probability of type II error
BFS  basic feasible solution
Bi  Airy function
BIBD  balanced incomplete block design
$B_n$  a block
$B_n$  Bell number
$B_n$  Bernoulli number
$B_n(x)$  Bernoulli polynomial
Bq  becquerel
C  coulomb
$C$  Roman numeral (100)
$C$  channel capacity
$\mathbb{C}$  complex numbers
$C$  integration contour 1, 2
$c$  number of identical servers
$c$  speed of light
$c(G)$  graph circumference
$C(m, r)$  $r$-combination 1, 2
$C(x)$  Fresnel integral
C.E.  (common era, A.D.)
cd  candela
$\chi' (G)$  chromatic index
$\chi (G)$  chromatic number
$\chi^2$  critical value
$\chi^2_n$  chi-square distributed
$C_l(z)$  cosine integral
$C_n$  Catalan numbers
$C_n$  type of graph 1, 2
$\mathbb{C}^n$  complex n element vectors
$c_n$  Fourier coefficients
$c_n(u, k)$  elliptic function
cof$_{ij}$$(A)$  cofactor of matrix $A$
cond$(A)$  condition number
$\cong$  group isomorphism 1, 2
$\cos$  trigonometric function
cot  trigonometric function
covers
$C^L$
dual code
$C^R(m, r)$
$r$-combination with replacement
csc
trigonometric function
$D$
constant service time
$D$
diagonal matrix
$D$
differentiation operator 1, 2
d
derivative operator
d
exterior derivative
d
minimum distance
$D$
Roman numeral (500)
d$(G)$
graph diameter
d$(u, v)$
distance between vertices 1, 2
$\Delta$
forward difference
$\Delta_c \text{ arg } f(z)$
change in the argument
$\delta$
designed distance
$\delta$
Feigenbaum's constant
$\Delta(G)$
maximum degree
$\delta(G)$
graph minimum degree
$\delta(x)$
delta function
det$(A)$
determinant of matrix $A$
$D_f$
region of convergence
DFT
discrete Fourier transform
d$_H(u, v)$
Hamming distance
diam$(G)$
graph diameter
DLG$(n; a, b)$
double loop graph
$D_n$
derangement
$D_n$
dihedral group
$\delta_n$
proportion of customers
$\delta n(u, k)$
elliptic function
$ds$
differential surface area
$dv$
differential volume
$dx$
fundamental differential
d$_x k(a)$
projection
$E$
edge set
$E$
event
e
vector of ones
e
algebraic identity
e
charge of electron
e
eccentricity
e
number
<table>
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<tr>
<th>Symbol</th>
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<tr>
<td>$E(u, v)$</td>
<td>first fundamental metric coefficient</td>
</tr>
<tr>
<td>$e(u, v)$</td>
<td>second fundamental metric coefficient</td>
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<td>$E[]$</td>
<td>expectation</td>
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<td>$ecc(v)$</td>
<td>eccentricity of a vertex 1, 2</td>
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<td>$\mathbf{e}_i$</td>
<td>unit vector</td>
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<tr>
<td>$e_{i_1\cdots i_n}$</td>
<td>permutation symbol</td>
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<tr>
<td>$\mathbf{E}_{ij}$</td>
<td>elementary matrix</td>
</tr>
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<td>Erlang-$k$ service time</td>
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<tr>
<td>$E_n$</td>
<td>Euler numbers</td>
</tr>
<tr>
<td>$E_n(x)$</td>
<td>Euler polynomial</td>
</tr>
<tr>
<td>$E_n(x)$</td>
<td>exponential integral</td>
</tr>
<tr>
<td>$\epsilon_{i_1\cdots i_n}$</td>
<td>Levi-Civita symbol</td>
</tr>
<tr>
<td>$\text{erf}$</td>
<td>error function</td>
</tr>
<tr>
<td>$\text{erfc}$</td>
<td>complementary error function</td>
</tr>
<tr>
<td>$\eta$</td>
<td>power of a test</td>
</tr>
<tr>
<td>$\eta(x, y)$</td>
<td>component of infinitesimal generator</td>
</tr>
<tr>
<td>$\text{exsec}$</td>
<td>trigonometric function</td>
</tr>
<tr>
<td>$\mathcal{F}$</td>
<td>Fourier transform</td>
</tr>
<tr>
<td>$F$</td>
<td>farad</td>
</tr>
<tr>
<td>$F(a, b; c; z)$</td>
<td>hypergeometric function</td>
</tr>
<tr>
<td>$F(u, v)$</td>
<td>first fundamental metric coefficient</td>
</tr>
<tr>
<td>$f(u, v)$</td>
<td>second fundamental metric coefficient</td>
</tr>
<tr>
<td>$F(x)$</td>
<td>Dawson's integral</td>
</tr>
<tr>
<td>$F(x)$</td>
<td>probability distribution function</td>
</tr>
<tr>
<td>$\hat{F}(x)$</td>
<td>sample distribution function</td>
</tr>
<tr>
<td>$\hat{f}(x)$</td>
<td>sample density function</td>
</tr>
<tr>
<td>$f(x)$</td>
<td>probability density function</td>
</tr>
<tr>
<td>$F_c$</td>
<td>Fourier cosine transform</td>
</tr>
<tr>
<td>FCFS</td>
<td>first come, first served</td>
</tr>
<tr>
<td>FFT</td>
<td>fast Fourier transform</td>
</tr>
<tr>
<td>FIFO</td>
<td>first in, first out</td>
</tr>
<tr>
<td>$F_n$</td>
<td>Fibonacci numbers</td>
</tr>
<tr>
<td>$\mathcal{F}_N$</td>
<td>discrete Fourier transform</td>
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<tr>
<td>$\mathcal{F}_n$</td>
<td>Farey sequence</td>
</tr>
<tr>
<td>$F_{p^n}$</td>
<td>Galois field</td>
</tr>
<tr>
<td>$F_{s}$</td>
<td>Fourier sine transform</td>
</tr>
<tr>
<td>$f_X(x)$</td>
<td>density function</td>
</tr>
<tr>
<td>$G$</td>
<td>general service time distribution</td>
</tr>
<tr>
<td>$G$</td>
<td>generating matrix</td>
</tr>
<tr>
<td>$G$</td>
<td>graph</td>
</tr>
<tr>
<td>$G$</td>
<td>gravitational constant</td>
</tr>
</tbody>
</table>
$G$  Green's function
$G$  primitive root
$G$  Catalan constant
$g$  determinant of the metric tensor
$g$  metric tensor
$g$  primitive root
$g(G)$  graph girth 1, 2
$G(k)$  Waring's problem
$g(k)$  Waring's problem
$G(s)$  generating function
$G(u, v)$  first fundamental metric coefficient
$g(x)$  generating polynomial
$G.M.$  geometric mean
$G[x_n]$  generating function
$\gamma$  Euler's constant
$\gamma(G)$  graph genus
$\Gamma(x)$  gamma function
$\gamma_1$  skewness
$\gamma_2$  excess
$\Gamma_{jk}$  Christoffel symbol of second kind
$\Gamma^i_{jk}$  connection coefficients
$GCD$  greatest common divisor
$g d x$  Gudermannian function
$GF(\mathbb{p}^n)$  Galois field
$G(x; z)$  Green's function
$GI$  general interarrival time
$g^{ij}$  covariant metric
$g_{ij}$  contravariant metric
$GL(n, \mathbb{C})$  matrix group
$GL(n, \mathbb{R})$  matrix group
$G_{n,m}$  isomorphism classes
$G_y$  gray
$H$  mean curvature
$H$  parity check matrix
$\mathcal{H}$  Hilbert transform
$H$  Hermitian conjugate
$H$  henry
$H(p_x)$  entropy
$H(x)$  Haar wavelet
$H(x)$  Heaviside function 1, 2
H.M. harmonic mean
$H_0$ null hypothesis
$H_1$ alternative hypothesis
$h_{i}$ metric coefficients
$H_k$ $k$-stage hyperexponential service time
$H_n(x)$ Hermite polynomials
$\mathcal{H}_v$ Hankel transform
$H_v^{(1)}(z)$ Hankel function
$H_v^{(2)}(z)$ Hankel function
Hz hertz
$I$ first fundamental form
$I$ identity matrix
$i$ unit vector
$i$ imaginary unit
$i$ interest rate
$I$ Roman numeral (1)
$I(X, Y)$ mutual information
$i$ unit vector
$II$ second fundamental form
iid independent and identically distributed
$\text{Im}$ imaginary part of a complex number
$I_n$ identity matrix
$\infty$ infinity
$\text{Inv}$ number of invariant elements
$(iv)$ fourth derivative
$\text{IVP}$ initial value problem
$J$ Jordan form
$j$ unit vector
$J$ joule
$J_c$ Julia set
$j$ unit vector
$j_n(z)$ half order Bessel function
$J_0(z)$ Bessel function
$j_{v,k}$ zero of Bessel function
$K$ Gaussian curvature
$K$ system capacity
$k$ curvature vector
$k$ unit vector
$k$ dimension of a code
\( K \)  
degrees Kelvin

\( k(x, t) \)  
kernel

\( k \)  
k-gon polygon

\( K_{1,n-1} \)  
star

\( \kappa(G) \)  
connectivity 1, 2

\( \kappa(s) \)  
curvature

\( \kappa_n \)  
cumulant

\( k_g \)  
geodesic curvature

\( k_g \)  
kilogram

\( \hat{k} \)  
unit vector

\( k_j \)  
block size

\( K_{m,n} \)  
complete bipartite graph 1, 2

\( K_n \)  
complete graph 1, 2

\( K_n \)  
Kalman gain matrix

\( k_n \)  
normal curvature vector

\( \bar{K}_n \)  
empty graph 1, 2

\( L \)  
length

\( L \)  
Roman numeral (50)

\( L \)  
arc length

\( L \)  
average number of customers

\( L \)  
period

\( \mathcal{L} \)  
Laplace transform

\( L_1 \)  
norm

\( L_2 \)  
norm

\( L_\infty \)  
space of measurable functions

\( \lambda \)  
average arrival rate

\( \lambda \)  
eigenvalue 1, 2, 3

\( \lambda \)  
number of blocks

\( \lambda(G) \)  
line connectivity 1, 2

\( \lambda(p) \)  
prime period lengths

\( \text{LCG} \)  
linear congruential generator

\( \text{LCL} \)  
lower control limit

\( \text{LCM} \)  
least common multiple

\( \text{Li}_1(z) \)  
logarithm

\( \text{Li}_2(z) \)  
dilogarithm

\( \text{Li}_n(z) \)  
polylogarithm

\( \text{li}(x) \)  
logarithmic integral

\( \text{LIFO} \)  
last in, first out

\( \text{lim} \)  
limits 1, 2

\( \text{Im} \)  
lumen

\( \text{In} \)  
logarithmic function
$L_n^{(a)}(x)$  Laguerre polynomials
$L_n(x)$  Laguerre polynomials
log  logarithmic function,
$log_b$  logarithm to base $b$
LP  linear programming
$L_Q$  average number of customers
$L_r$  Lie group
lx  lux
M  mass
M  Roman numeral (1000)
$M$  exponential service time
$M$  Mandelbrot set
$M$  number of codewords
$\mathcal{M}$  Mellin transform
$m$  mortgage amount
$m$  number in the source
m  meter
$M(P)$  measure of a polynomial
M.D.  mean deviation
MA(l)  moving average
mid  midrange
MLE  maximum likelihood estimator
$M/M/1$  queue
$M/M/c$  queue
$M_n$  Möbius ladder $1, 2$
mod  modular arithmetic
mol  mole
MOLS  mutually orthogonal Latin squares
MOM  method of moments
$\mu$  average service rate
$\mu$  mean
$\mu(n)$  Möbius function
$\mu_k$  centered moments
$\mu'_k$  moments
$N$  number of zeros
$N$  unit normal vector
$N$  natural numbers
$\mathcal{N}$  normal vector
$n$  principal normal unit vector
$\hat{n}$  unit normal vector
$n$  code length
$n$ number of time periods

$n$ order of a plane

$n$ \(n\) \(n\)th derivative

N newton

$N(A)$ null space

$N(\mu, \sigma)$ normal random variable

$\nabla$ gradient 1, 2

$\nabla$ linear connection

$\nabla^2$ Laplacian

$\nabla \cdot$ divergence

$\nabla \times$ curl

$N_q(n)$ number of monic irreducible polynomials

$v(G)$ graph crossing number 1, 2

$\overline{v}(G)$ rectilinear graph crossing number 1, 2

$O$ asymptotic function

$o$ asymptotic function

$O(n)$ matrix group

$\Omega$ asymptotic function

$\Omega$ ohm

$\omega(G)$ size of the largest clique 1, 2

$O_n$ odd graph

$P$ number of poles

$P$ principal

$P[\] $ Riemann P function

$p#$ product of prime numbers

$P(B \mid A)$ conditional probability

$P(m, r)$ \(r\)-permutation 1, 2

$p(n)$ partitions

$P(v, z)$ auxiliary function

$P(x, y)$ Markov transition function

Pa pascal

$P_G(x)$ chromatic polynomial 1, 2

$\phi$ Euler constant

$\phi$ golden ratio

$\phi$ incidence mapping

$\phi$ zenith

$\phi(n)$ totient function 1, 2

$\phi(t)$ characteristic function

$\Phi(x)$ normal distribution function

$\pi$ number

$\pi(x)$ prime counting function
\( \pi(x) \)  
probability distribution  

\( \text{PID} \)  
principal ideal domain  

\( p_k \)  
discrete probability  

\( p_k(n) \)  
partitions  

\( P_n^m(x) \)  
associated Legendre polynomials  

\( \oplus \)  
Kronecker sum  

\( + \)  
operation for additive group  

\( + \)  
vector addition  

\( + \)  
pseudo-inverse operator  

\( p_m(n) \)  
restricted partitions  

\( P_n \)  
path (type of graph)  

\( p_n \)  
proportion of time  

\( P_n^{(\alpha, \beta)}(x) \)  
Jacobi polynomials  

\( P_n(x) \)  
Lagrange interpolating polynomial  

\( P_n(x) \)  
Legendre function  

\( P_n(x) \)  
Legendre polynomials  

\( P^n(x, y) \)  
\( n \)-step Markov transition matrix  

\( P_v(z) \)  
Legendre function  

\( P^{R}(m, r) \)  
\( r \)-permutation with replacement  

\( \text{Per}(x_n) \)  
period of a sequence  

\( \text{PRI} \)  
priority service  

\( \text{PRNG} \)  
pseudorandom number generator  

\( \prod \)  
product symbol  

\( \psi(z) \)  
logarithmic derivative of the gamma function  

\( P_{X \times Y} \)  
joint probability distribution  

\( \mathbb{Q} \)  
rational numbers  

\( Q \)  
orthogonal matrix  

\( q \)  
none  

\( Q(v, z) \)  
auxiliary function  

\( Q_8 \)  
quaternion group  

\( Q_n \)  
cube (type of graph)  

\( Q_n(x) \)  
Legendre function  

\( Q_v(z) \)  
Legendre function  

\( R \)  
curvature tensor  

\( R \)  
radius (circumscribed circle)  

\( R \)  
range  

\( R \)  
rate of a code  

\( R \)  
Ricci tensor  

\( R \)  
Riemann tensor  

\( \mathbb{R} \)  
real numbers  

\( r \)  
distance in polar coordinates
$r$ modulus of a complex number
$r$ radius (inscribed circle) 1, 2
$r$ shearing factor
$R(A)$ range space
R.M.S. root mean square
rad radian
rad(G) radius of graph 1, 2
Re real part of a complex number
$\rho$ server utilization
$\rho(A)$ spectral radius
$\rho(s)$ radius of curvature
$\rho_{ij}$ correlation coefficient
$r_i$ replication number
$r_k(x)$ Rademacher functions
$\mathbb{R}^n$ real $n$ element vectors
$\mathbb{R}^{n \times m}$ real $n \times m$ matrices
RSS random service
$S$ sample space
$S$ torsion tensor
$s$ arc length parameter
$s$ sample standard deviation
$s$ semiperimeter
$s$ second
$S$ siemens
$S(T)$ symmetric part of a tensor
$S(x)$ Fresnel integral
$sec$ trigonometric function
$sgn$ signum function 1, 2
SI Systeme Internationale d'Unites
Si(z) sine integral
$\sigma$ standard deviation
$\sigma(n)$ sum of divisors
$\sigma^2$ variance
$\sigma_{ij}$ variance
$\sigma_{ij}$ covariance
$\sigma_k(n)$ sum of $k^{th}$ powers of divisors
$\sin$ trigonometric function
$S_k$ area of circumscribed polygon
$s_k$ area of inscribed polygon
$s_k$ elementary symmetric functions
$SL(n, \mathbb{C})$ matrix group
$SL(n, \mathbb{R})$ matrix group
$S_n$ permutation group
$S_n$ star (type of graph)
$S_n(r)$ surface area of a sphere
$s_n(u, k)$ elliptic function
$SO(2)$ matrix group
$SO(3)$ matrix group
SPRT sequential probability ratio test
sr steradian
SRS shift register sequence
$SU(n)$ matrix group
$\sum$ summation symbol
SVD singular value decomposition
T tesla
T time interval
$T_{n,m}$ isomorphism class of trees
$\tau$ transpose 1, 2
$\mathbf{t}$ unit tangent vector
$t$ $-(u, k, \lambda)$ design nomenclature
$t_a$ critical value
tan trigonometric function
$\tau$ Ramanujan function
$\tau(n)$ number of divisors
$\tau(s)$ torsion
$\Theta$ asymptotic function
$\theta$ angle in polar coordinates
$\theta$ argument of a complex number
$\theta$ azimuth
$\theta(G)$ graph thickness
TN$(w, s)$ Toeplitz network
$T_n(x)$ Tschebyshev polynomials
$T_{n,k}$ Turán graph
tr $(A)$ trace of matrix $A$
t$_{x,y}$ transition probabilities
$U$ universe
$\mu$ traffic intensity
$U(n)$ matrix group
$\mu(n)$ step impulse
$U(a, b)$ uniform random variable
UCL upper control limit
UFD unique factorization domain
$u_i$ distance
UMVU type of estimator
$U_n(x)$ Tschebyseff polynomials
$\gamma(G)$ graph arboricity
URL Uniform Resource Locators 1, 2
$\forall$ Roman numeral (5)
$\forall$ volt
$\forall$ vertex set
$\forall$ fifth derivative
$\forall$ vector operation
$\forall$ logical or
$\forall$ pseudoscalar product
vers trigonometric function
$V_n(r)$ volume of a sphere
$W$ average time
$W$ watt
$W(y_1, y_2)$ Wronskian
$W_b$ weber
$W_N$ root of unity
$W_n$ wheel (type of graph) 1, 2
$W_n(x)$ Walsh functions
$W_Q$ average time
$X$ infinitesimal generator
$X$ set of points
$\bar{x}$ arithmetic mean
$X$ Roman numeral (10)
$X^{(1)}$ first prolongation
$X^{(2)}$ second prolongation
$x_{(i)}$ $i^{th}$ order statistic
$\bar{x}_\alpha$ trimmed mean
$x_i$ rectangular coordinates
$\xi(x, y)$ component of infinitesimal generator
$\xi_p$ quantile of order $p$
$\tilde{x}_{n,m}$ predictor of $x_n$
$y_h(x)$ homogeneous solution
$y_n(z)$ half order Bessel function
$Y_\nu(z)$ Bessel function
$Y_{\nu,k}$ zero of Bessel function
$y_p(x)$ particular solution
$Z$ queue discipline
z complex number
$Z(G)$ center of a graph 1, 2
$z_{\alpha}$ critical value
$\mathcal{Z}$ $Z$-transform
$\mathcal{Z}$ integers
$\mathbb{Z}_{n}$ a group
$\mathbb{Z}_{p}$ integers modulo $p$
$\mathbf{0}$ null vector
$\zeta(k)$ Riemann zeta function