สมการเชิงอนุพันธ์

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1) แบร์นูลลี

$$dy + \frac{1}{x} y\, dx = 3x^2 y^2\, dx$$

$$\frac{dy}{dx} + \frac{1}{x} y = 3x^2 y^2 \qquad \text{—— (1)}$$

form $\quad \frac{dy}{dx} + P(x) y = Q(x) y^n \longrightarrow n = 2$

ให้ $\quad v = y^{1-n} = y^{1-2} = y^{-1} \qquad \text{—— (2)}$

เปลี่ยนตัวแปร $\to y \to v$

$$\frac{dv}{dx} = \frac{d y^{-1}}{dx} = -1\, y^{-2} \frac{dy}{dx}$$

$$-\frac{dv}{dx} = y^{-2} \frac{dy}{dx} \qquad \text{—— (3)}$$

จาก (1) หาร y^2 ตลอด

$$y^{-2} \frac{dy}{dx} + \frac{1}{x} y^{-1} = 3x^2 \qquad \text{—— (4)}$$

(2) (3) แทนค่าใน (4)

$$-\frac{dv}{dx} + \frac{1}{x} v = 3x^2$$

$$\frac{dv}{dx} - \frac{1}{x} v = -3x^2$$

$$v = e^{-\int(-\frac{1}{x})dx}\left[\int e^{\int -\frac{1}{x}dx}(-3x^2)\,dx + c\right]$$

$$v = e^{\ln x}\left[\int e^{-\ln x}(-3x^2)\,dx + c\right]$$

$$v = x\left[\int \frac{1}{x}(-3x^2)\,dx + c\right]$$

$$v = x\left[\int -3x^1\,dx + c\right]$$

$$v = x\left[-3\int x\,dx + c\right]$$

$$v = x\left[-3\frac{x^2}{2} + c\right]$$

$$\cancel{v = -\frac{3}{2}x^3 + cx}$$

$$\boxed{xy\left(c - \frac{3}{2}x^2\right) = 1}$$

$$v = x\left[c - \frac{3}{2}x^2\right] \quad\text{———}\quad (5)$$

or $v = y^{-1} = \frac{1}{y}$ ——— (6)

$\cancel{y}$ 2 use 6 → 5

$$1 = yx\left[c - \frac{3}{2}x^2\right] \quad\text{———}\quad (7)$$

2.441 $\quad y = \dfrac{1}{Ce^{-3x} - \frac{1}{3}}$ $\quad$ ①

② พิจารณาสมการ

$$\frac{dy}{dx} = -2 - y + y^2 \quad \text{①} \qquad \longrightarrow y_1 = 2$$

เป็น form $\quad \dfrac{dy}{dx} = P(x) + Q(x)y + R(x)y^2$

$P(x) = -2$

$Q(x) = -1$

$R(x) = 1$

ให้ $\quad z = \dfrac{1}{y - y_1} = \dfrac{1}{y-2}$

$$y - 2 = \frac{1}{z}$$

$$y = \frac{1}{z} + 2 = \frac{1+2z}{z}$$

$$\frac{dy}{dx} = \frac{dz^{-1}}{dx} + \cancel{\frac{dz}{dx}}^{\,0}$$

$$\frac{dy}{dx} = -1\,z^{-2}\frac{dz}{dx} = -z^{-2}\frac{dz}{dx} \quad \text{③}$$

② ③ แทน ①

$$-z^{-2}\frac{dz}{dx} = -2 - \left(\frac{1}{z} + 2\right) + \left(\frac{1}{z} + 2\right)^2$$

$$-z^{-2}\frac{dz}{dx} = -2 - \frac{1}{z} - 2 + \left(\frac{1}{z^2} + \frac{4}{z} + 4\right)$$

$$-z^{-2}\frac{dz}{dx} = \cancel{-4} - \frac{1}{z} + \frac{1}{z^2} + \frac{4}{z} \cancel{+4}$$

$$-z^{-2}\frac{dz}{dx} = \frac{3}{z} + \frac{1}{z^2}$$

$$-\frac{1}{z^2}\frac{dz}{dx} = \frac{3}{z} + \frac{1}{z^2}$$

$-z^2$ ගුණනයෙන්

$$\frac{dz}{dx} = -3z - 1 = -3z - 1$$

$$\frac{dz}{dx} + 3z = -1$$ —— [illegible]

form $$\frac{dz}{dx} + p(x)z = Q(x)$$

$$z = e^{-\int p(x)dx}\left[\int e^{\int p(x)dx} Q(x)\,dx + C\right]$$

$$z = e^{-\int 3dx}\left[\int e^{\int 3dx}(-1)\,dx + C\right]$$

$$z = e^{-3x}\left[-\int e^{3x}dx + C\right]$$

$$z = e^{-3x}\left(-\frac{1}{3}e^{3x} + C\right)$$

$$z = -\frac{1}{3} + Ce^{-3x} = Ce^{-3x} - \frac{1}{3}$$ —— (4)

จาก $z = \frac{1}{y-2}$ — แทนค่าใน ๒

$\cancel{y-2 = \frac{1}{z}}$

$$\frac{1}{y-2} = Ce^{-3x} - \frac{1}{3}$$

$$y = 2 + \frac{1}{Ce^{-3x} - \frac{1}{3}}$$

ไม่เป็นสมการเชิงเส้น

(3) (1)

$$(y+y^2)\,dx + x\,dy = 0$$

$$x\frac{dy}{dx} + y^2 + y = 0$$

$$\frac{dy}{dx} + \frac{(1+y)y}{x} = 0$$

$Q(x) = 0$

$y \neq 0$

$$(y+y^2)\frac{dx}{dy} + x = 0$$

$$\frac{dx}{dy} + \frac{1}{(y+y^2)}x = 0$$

$Q(y) = 0$

$$\frac{dx}{dy} + \frac{1}{y(1+y)}x = 0$$

ไม่เป็นสมการเชิงเส้น.

(8) 5 10 (1)

$$\boxed{3x^2y^3 + y^4 = c}$$

(4) $6xy\,dx + (4y + 9x^2)\,dy = 0$

เทียบรูปแบบ $M(x,y)\,dx + N(x,y)\,dy = 0$

$$\frac{\partial M}{\partial y} = \frac{\partial (6xy)}{\partial y} = 6x$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(\cancel{4y} + 9x^2) = 18x$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$ ไม่ใช่สมการแม่นตรง (Exact)

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{6x - 18x}{4y + 9x^2} \quad \times$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{18x - 6x}{6xy} = f(y) \quad \checkmark$$

โดยได้ตัวประกอบอินทิเกรต $\mu(x,y) = \frac{1}{y^2}$

$$\frac{6xy}{y^2}\,dx + \frac{(4y + 9x^2)\,dy}{y^2} = 0$$

$$6\frac{x}{y}\,dx + \frac{4}{y}\,dy + 9\frac{x^2}{y^2}\,dy$$

singa $\mu(x,y) = y^2$

10) y^2 gunaon

$$y^2\,6xy\,dx + 4y^2y\,dy + 9x^2y^2\,dy = 0$$

$$6xy^3\,dx + 4y^3\,dy + 9x^2y^2\,dy = 0$$

$$\underline{3y^3\,dx^2} + 4y^3\,dy + \underline{3\,\cancel{y^2}\,x^2\,dy^3} = 0$$

$$3\left[d\,y^3x^2\right] + 4y^3\,dy = 0$$

$$3\int d\,y^3x^2 + 4\int y^3\,dy = \int 0.$$

$$3y^3x^2 + \frac{\cancel{4}\,y^4}{\cancel{4}} = c$$

$$3y^3x^2 + y^4 = c \text{ —— Ans}$$

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w form is exact

$$6xy^3\,dx + (4y^3 + 9x^2y^2)\,dy = 0$$

on $M(x,y)\,dx + N(x,y)\,dy = 0$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}6xy^3 = 6x\,3y^2 = 18xy^2$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(4y^3 + 9x^2y^2) = 9y^2\frac{\partial x^2}{\partial x} = 18y^2x$$

$$(e^y+1)^{-1} + \frac{1}{2}(e^x+1)^{-2} = c$$ (1)

(8) $$(e^y+1)^2 e^{-y}\,dx + (e^x+1)^3 e^{-x}\,dy = 0$$

$$\frac{1}{(e^x+1)^3 e^{-x}}\,dx + \frac{1}{(e^y+1)^2 e^{-y}}\,dy = 0$$

$$\frac{e^x}{(e^x+1)^3}\,dx + \frac{1}{(e^y+1)^2}\,e^y\,dy = 0$$

$$\frac{1}{(e^x+1)^3}\,d(e^x+1) + \frac{1}{(e^y+1)^2}\,d(e^y+1) = 0$$

$$\int (e^x+1)^{-3}\,d(e^x+1) + \int (e^y+1)^{-2}\,d(e^y+1) = \rho$$

$$\frac{(e^x+1)^{-2}}{-2} + \frac{(e^y+1)^{-1}}{-1} = \cancel{\rho}\; c$$

$$-\frac{1}{2}(e^x+1)^{-2} - (e^y+1)^{-1} = c$$

-1 [illegible]

$$\frac{1}{2}(e^x+1)^{-2} + (e^y+1)^{-1} = c$$

[illegible] $y\sin^2 x - y^2\cos x = -9$ (2)

(6) [illegible] Exact [illegible]

$$(\sin^2 x - 2y\cos x)\frac{dy}{dx} + 2y\sin x\cos x + y^2\sin x = 0$$

dx [illegible]

$$(\sin^2 x - 2y\cos x)\,dy + (2y\sin x\cos x + y^2\sin x)\,dx = 0$$

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2y\sin x\cos x + y^2\sin x)$$

$$= 2\sin x\cos x + 2y\sin x$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(\sin^2 x - 2y\cos x)$$

$$= 2\sin x\cos x - 2y(-\sin x)$$

$$= 2\sin x\cos x + 2y\sin x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$ → [illegible] Exact.

$$\sin^2 x\,dy - 2y\cos x\,dy + 2y\sin x\cos x\,dx + y^2\sin x\,dx =$$

$$\sin^2 x\, dy \quad - \cos x\, dy^2 - y^2\, d\cos x + 2y \sin x \cos x\, dx$$

$$\sin^2 x\, dy + y\, d\sin^2 x - (d\, y^2 \cos x) = 0$$

$$\int (d\, y \sin^2 x) - \int (d\, y^2 \cos x) = 0$$

$$y \sin^2 x - y^2 \cos x = c$$

2.142 $y = xe^{cx}$

(1)

5) $xy' - y(\ln(y/x)+1) = 0$

$$x\frac{dy}{dx} - y(\ln(y/x)+1) = 0 \quad \text{—— (1)}$$

[illegible] Homogenous

~~$x\,dy = y(\ldots$~~

[illegible] $y = vx$ —— (2)

$$\frac{dy}{dx} = \cancel{\frac{v\,dx}{dx}} + x\frac{dv}{dx}$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{—— (3)}$$

[illegible] (2) (3) [illegible] (1)

$$x\left(v + x\frac{dv}{dx}\right) - vx\left(\ln\left(\frac{v\cancel{x}}{\cancel{x}}\right)+1\right) = 0$$

$$xv + x^2\frac{dv}{dx} - vx(\ln v + 1) = 0$$

$$\cancel{xv} + x^2\frac{dv}{dx} - vx\ln v - \cancel{vx} = 0$$

$$x^2\frac{dv}{dx} - vx\ln v = 0$$

$$x^2\,dv - xv\ln v\,dx = 0$$

$$\frac{1}{v \ln v} dv - \frac{x}{x^2} dx = 0$$

$$\frac{1}{\ln v} d \ln v - \frac{1}{x} dx = 0$$

$$\int (\ln v)^{-1} d \ln v - \int \frac{1}{x} dx = 0$$

$$\cancel{-(\ln v)^{-3}}$$

$$\underline{(\ln v)^{-1+1}}$$

$$\int \frac{1}{\ln v} d \ln v - \int \frac{1}{x} dx = \int 0$$

$$\ln(\ln v) - \ln x = \ln c$$

$$\ln(\ln v) = \ln x + \ln c$$

$$\ln(\ln v) = \ln(cx)$$

take e.

$$\ln v = cx$$

hence

$$v = e^{cx} \quad \text{——} \quad (4)$$

or

$$y = vx$$

$$v = \frac{y}{x} \quad \text{——} \quad (5)$$

[illegible] (5) [illegible] (6)

$$\frac{y}{x} = e^{cx}$$

$$y = x e^{cx}$$