MULTIVARIABLE CALCULUS
Implicit Differentiation

Single Variable	Multivariable
$y = x^2 + \sin x \Rightarrow$ explicit	$z = xy + 2x^2 \cdot \sin y \Rightarrow$ explicit
$xy + \sin(xy) = 4 \Rightarrow$ implicit $\left(\frac{dy}{dx}\right)$	$x^2 + y^2 + z^2 = \sin(yz) \Rightarrow$ implicit took on $\left(\frac{\partial z}{\partial x} \text{ or } \frac{\partial z}{\partial y}\right)$

① Given : $x^2 + y^2 + z^2 = \sin(yz)$

หา $\frac{\partial z}{\partial y} = ?$

→ treat x like a constant !

Sol^n

$$\frac{\partial}{\partial y}(x^2 + y^2 + z^2) = \frac{\partial}{\partial y}(\sin(yz))$$

$$\frac{\partial x^2}{\partial y} + \frac{\partial y^2}{\partial y} + \frac{\partial z^2}{\partial y} = \frac{\partial \sin(yz)}{\partial y}$$

$$0 + 2y + 2z\frac{\partial z}{\partial y} = \cos(yz)\left[z + y \cdot \frac{\partial z}{\partial y}\right]$$

$$2y + 2z\frac{\partial z}{\partial y} = z\cos(yz) + y \cdot \frac{\partial z}{\partial y}\cos(yz)$$

$$2z\frac{\partial z}{\partial y} - y \cdot \frac{\partial z}{\partial y}\cos(yz) = z\cos(yz) - 2y$$

$$\frac{\partial z}{\partial y}\left(2z - y\cos(yz)\right) = z \cdot \cos(yz) - 2y$$

$$\frac{\partial z}{\partial y} = \frac{z \cdot \cos(yz) - 2y}{2z - y \cdot \cos(yz)}$$ ###

Given : $x^2 + y^2 + z^2 = \sin(yz)$

หา $\frac{\partial z}{\partial x} = ?$

$\hookrightarrow$ treat y like a constant !

Sol^n

$$\frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right) = \frac{\partial}{\partial x}\left(\sin(yz)\right)$$

$$\frac{\partial x^2}{\partial x} + \frac{\partial y^2}{\partial x} + \frac{\partial z^2}{\partial x} = \frac{\partial}{\partial x}\sin(yz)$$

$$2x + 0 + 2z\frac{\partial z}{\partial x} = \cos(yz) \cdot \left[y \cdot \frac{\partial z}{\partial x}\right]$$

$$2x + 2z\frac{\partial z}{\partial x} = y \cdot \cos(y \cdot z) \cdot \frac{\partial z}{\partial x}$$

$$2z\frac{\partial z}{\partial x} - y\cos(yz) \cdot \frac{\partial z}{\partial x} = -2x$$

$$\frac{\partial z}{\partial x}\left(2z - y\cos(yz)\right) = -2x$$

$$\therefore \frac{\partial z}{\partial x} = \frac{-2x}{2z - y\cos(yz)} \quad \# \# \#$$

Implicit Differentiation

Explicit

$y = x^2$

$y = \sin x + x$

$\frac{d}{dx}(x^2) = 2x$

Implicit

$x^2 + xy = y$

$xy = x + y$

$$\frac{d}{dx}(y^2) \Rightarrow \frac{d}{dy}\frac{dy}{dx}(y^2)$$

$$= \frac{d}{dy}y^2\frac{dy}{dx} = 2y\frac{dy}{dx}$$

Procedure

Take derivative ; Adding $\frac{dy}{dx}$ where needed

Get rid of $(\;)'s$

Solve for $\frac{dy}{dx}$

Ex Find $\frac{dy}{dx}$

$$x^2 + xy + \cos(y) = 8y$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(\cos y) = \frac{d}{dx}(8y)$$

$$\Rightarrow \quad 2x + \left(1y + x\frac{dy}{dx}\right) + -\sin(y)\frac{dy}{dx} = 8\frac{dy}{dx}$$

$$2x + y + x\frac{dy}{dx} - \sin(y)\cdot\frac{dy}{dx} = 8\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(x - \sin(y) - 8\right) = -2x - y$$

$$\frac{dy}{dx} = \frac{-2x-y}{x - \sin(y) - 8} \quad \#\#$$

More Implicit Differentiation

$$1 + x = \sin(xy^2) \quad \text{หา} \quad \frac{dy}{dx} = ?$$

Sol^n $$\frac{d}{dx}(1+x) = \frac{d}{dx}\left(\sin(xy^2)\right)$$

$$\cancel{\frac{d}{dx}}^{\,0} + \frac{dx}{dx} = \cos(xy^2)\frac{d}{dx}(xy^2)$$

$$1 = \cos(xy^2)\left[y^2\frac{dx}{dx} + x\frac{dy^2}{dx}\right]$$

$$1 = \cos(xy^2)\left[1 \cdot y^2 + x \cdot 2y \frac{dy}{dx}\right]$$

$$1 = y^2\cos(xy^2) + 2xy\cos(xy^2)\frac{dy}{dx}$$

$$1 - y^2\cos(xy^2) = 2xy\cos(xy^2)\frac{dy}{dx}$$

$$\frac{1 - y^2\cos(xy^2)}{2xy\cos(xy^2)} = \frac{dy}{dx} \quad \# \#$$

Equation the tengent line at (1,1) on curve

$$x^2 + xy + y^2 = 3 \quad \text{และ} \quad \frac{dy}{dx} = ?$$

Sol^n $\frac{d}{dx}(x^2 + xy + y^2) = \frac{d}{dx}3$

$$2x + \left(1 \cdot y + x \cdot 1\frac{dy}{dx}\right) + 2y^2\frac{dy}{dx} = 0$$

$$2x + y + x\frac{dy}{dx} + 2y^2\frac{dy}{dx} = 0$$

$$(x + 2y^2)\frac{dy}{dx} = -2x - y$$

$$\frac{dy}{dx}(x + 2y^2) = -2x - y$$

$$\frac{dy}{dx} = \frac{-2x - y}{x + 2y^2}$$

$$\boxed{y = -x + 2}$$

tangent line : $\Rightarrow$ slop $\quad y - y_1 = m(x - x_1)$

$$y - 1 = m(x - 1) \quad \Rightarrow \boxed{y - 1 = -1(x - 1)}$$

$$\left.\frac{dy}{dx}\right|_{\substack{(1,1)\\x,y}} = \frac{-2(1) - 1}{1 + 2(1)^2} = \frac{-3}{3} = -1$$

MORE IMPLICIT DIFFERENTIATION PROBLEMS !

(Extra Example)

Ex $x^3 + y^3 = xy$

Sol^n

$$\frac{d}{dx}\left(x^3+y^3\right) = \frac{d}{dx}(xy)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 1.y + x\frac{dy}{dx}$$

$$3y^2\frac{dy}{dx} - x\frac{dy}{dx} = -3x^2 + y$$

$$\frac{dy}{dx}\left(3y^2 - x\right) = -3x^2 + y$$

$$\frac{dy}{dx} = \frac{-3x^2+y}{3y^2-x} \quad \text{———— \#\#\#}$$

Ex $x^2y + x.y^2 = 3x$

Sol^n

$$\frac{d}{dx}\left(x^2y + xy^2\right) = \frac{d}{dx}(3x)$$

$$\left(2x.y + x^2.1.\frac{dy}{dx}\right) + \left(1.y^2 + x\,2y\frac{dy}{dx}\right) = 3$$

$$x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} = 3 - 2xy - y^2$$

$$\frac{dy}{dx}\left(x^2 + 2xy\right) = 3 - 2xy - y^2$$

$$\frac{dy}{dx} = \frac{3-2xy-y^2}{x^2+2xy} \quad \text{———— \#\#\#}$$