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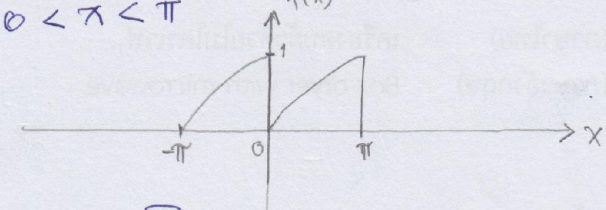
Ex จงหาอนุกรมฟูเรียร์ของฟังก์ชันกำหนดโดย

$$f(x) = \begin{cases} \cos x \,; & -\pi < x < 0 \\ \sin x \,; & 0 < x < \pi \end{cases} \quad \text{เมื่อ } f(x+2\pi) = f(x)$$



Sol ■ หาสัมประสิทธิ์ของ a_0 จะได้

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} \cos x\,dx + \int_{0}^{\pi} \sin x\,dx\right]$$

$$= \frac{1}{\pi}\left[\sin x \Big|_{-\pi}^{0} + -\cos x \Big|_{0}^{\pi}\right] = \frac{1}{\pi}\left[\cancel{\sin(0)}^{0} - \cancel{\sin(-\pi)}^{0} - \cos(\pi) + \cos(0)\right]$$

$$= \frac{1}{\pi}\left[-(-1) + 1\right] = \frac{2}{\pi} \quad \# \# \#$$

■ หาสัมประสิทธิ์ของ a_n จะได้

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} \cos x.\cos nx\,dx + \int_{0}^{\pi} \sin x.\cos nx\,dx\right]$$

⇒ กรณีที่ $n=1$ จะได้

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} \cos x.\cos x\,dx + \int_{0}^{\pi} \sin x\cos x\,dx\right]$$

$$\boxed{\cos^2 x \Rightarrow \frac{1}{2}+\frac{1}{2}\cos 2x \Rightarrow \frac{1}{2}(1+\cos 2x)}$$

$$a_1 = \frac{1}{\pi}\left[\int_{-\pi}^{0} \cos^2 x\,dx + \int_{0}^{\pi} \sin x.\cos x\,dx\right]$$

$$\boxed{\begin{aligned} \sin 2x &= 2\sin x.\cos x \\ \therefore \sin x.\cos x &= \frac{1}{2}\sin 2x \end{aligned}}$$

$$a_1 = \frac{1}{\pi}\left[\int_{-\pi}^{0}\left(\frac{1}{2}+\frac{1}{2}\cos 2x\right)dx + \int_{0}^{\pi}\frac{1}{2}\sin 2x\right]$$

$$a_1 = \frac{1}{\pi}\left[\frac{x}{2} + \frac{1}{4}\sin 2x \Big|_{-\pi}^{0} + -\frac{1}{4}\cos 2x \Big|_{0}^{\pi}\right]$$

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$$a_1 = \frac{1}{\pi}\left[\frac{\pi}{2} + \frac{1}{4}\cancel{\sin 2(0)}^{\,0} - \frac{1}{4}\cancel{\sin 2(-\pi)}^{\,0} - \frac{1}{4}\cancel{\cos 2(\pi)}^{\,1} + \frac{1}{4}\cancel{\cos 2(0)}^{\,1} \right]$$

$$a_1 = \frac{1}{\cancel{\pi}}\left(\frac{\cancel{\pi}}{2} \cancel{-1} \cancel{+1} \right) = \frac{1}{2} \text{ ———— \#\#\#}$$

$\Rightarrow$ กรณีที่ $n \neq 1$ จ.ใจ

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}\cos x\cos nx\,dx + \int_{0}^{\pi}\sin x\cos nx\,dx \right]$$

$$\boxed{\begin{aligned} \cos A\cdot\cos B &= \tfrac{1}{2}\{\cos(A-B)+\cos(A+B)\} \\ \sin A\cdot\cos B &= \tfrac{1}{2}\{\sin(A-B)+\sin(A+B)\} \end{aligned}}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}\frac{1}{2}\{\cos(1-n)x + \cos(1+n)x\}\,dx + \int_{0}^{\pi}\frac{1}{2}\{\sin(1-n)x + \sin(1+n)x\}\,dx \right]$$

$$= \frac{1}{\pi}\left[\left\{ \frac{\sin(1-n)x}{2(1-n)} + \frac{\sin(1+n)x}{2(1+n)} \right\}\Bigg|_{-\pi}^{0} + \left\{ \frac{-\cos(1-n)x}{2(1-n)} - \frac{\cos(1+n)x}{2(1+n)} \right\}\Bigg|_{0}^{\pi} \right]$$

$$= \frac{1}{\pi}\left[\frac{\cancel{\sin(1-n)(0)}^{\,0} - \cancel{\sin(1-n)(-\pi)}^{\,0}}{2(1-n)} + \frac{\cancel{\sin(1+n)(0)}^{\,0} - \cancel{\sin(1+n)(-\pi)}^{\,0}}{2(1+n)} \right.$$

$$\left. - \frac{\cos(1-n)(\pi) + \cancel{\cos(1-n)(0)}^{\,1}}{2(1-n)} - \frac{\cos(1+n)(\pi) + \cancel{\cos(1+n)(0)}^{\,1}}{2(1+n)} \right]$$

$$= \frac{1}{\pi}\left[\frac{1-\cos(1-n)\pi}{2(1-n)} + \frac{1-\cos(1+n)\pi}{2(1+n)} \right]$$

$$= \frac{1-\cos(1-n)\pi}{2(1-n)\pi} + \frac{1-\cos(1+n)\pi}{2(1+n)\pi}$$

พิจารณา $\cos(1-n)\pi = (-1)^{n-1} = \begin{cases} -1 & ;\ \text{เมื่อ } n = 2, 4, 6, \ldots \\ 1 & ;\ \text{เมื่อ } n = 3, 5, 7, \ldots \end{cases}$

และ $\cos(1+n)\pi = (-1)^{n+1} = \begin{cases} -1 & ;\ \text{เมื่อ } n = 2, 4, 6, \ldots \\ 1 & ;\ \text{เมื่อ } n = 3, 5, 7, \ldots \end{cases}$

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$$a_n = \frac{1-(-1)^{n-1}}{2(n-1)\pi} + \frac{1-(-1)^{n+1}}{2(n+1)\pi}$$

จะได้

$$a_n = \begin{cases} 0 & ; \text{ เมื่อ } n = 3, 5, 7, \ldots \\ \dfrac{2n}{(n^2-1)\pi} & ; \text{ เมื่อ } n = 2, 4, 6, \ldots \end{cases}$$

หาสัมประสิทธิ์ของ b_n จะได้

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}\cos x \sin nx\,dx + \int_{0}^{\pi}\sin x \sin nx\,dx\right]$$

$$\boxed{\sin A \cdot \cos B = \frac{1}{2}\left\{\sin(A-B) + \sin(A+B)\right\}}$$

⇒ กรณี $n=1$ จะได้

$$b_1 = \frac{1}{\pi}\left[\int_{-\pi}^{0}\cos x \sin x\,dx + \int_{0}^{\pi}\sin x \sin x\,dx\right]$$

$$\boxed{\begin{aligned} \sin^2 x &= \frac{1}{2} - \frac{1}{2}\cos 2x \\ \sin x \cos x &= \frac{1}{2}\sin 2x \end{aligned}}$$

$$b_1 = \frac{1}{\pi}\left[\int_{-\pi}^{0}\frac{1}{2}\sin 2x\,dx + \int_{0}^{\pi}\sin^2 x\,dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{4}\cos 2x\Big|_{-\pi}^{0} + \int_{0}^{\pi}\frac{1}{2}\,dx - \int_{0}^{\pi}\frac{1}{2}\cos 2x\,dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{4}\cancelto{1}{\cos 2(0)} + \frac{1}{4}\cancelto{1}{\cos 2(-\pi)} + \frac{x}{2}\Big|_{0}^{\pi} - \frac{1}{4}\sin 2x\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\cancel{\frac{-1}{4}} + \cancel{\frac{1}{4}} + \frac{\pi}{2} - \cancelto{0}{\frac{(0)}{2}} - \frac{1}{4}\cancelto{0}{\sin 2(\pi)} + \frac{1}{4}\cancelto{0}{\sin 2(0)}\right]$$

$$b_1 = \frac{1}{\cancel{\pi}}\left(\frac{\cancel{\pi}}{2}\right) = \frac{1}{2} \quad \text{———} \#\#\#$$

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⇒ กรณี $n \neq 1$ b_n จะได้

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}\cos x\sin nx\,dx + \int_{0}^{\pi}\sin x\sin nx\,dx\right]$$

$$\boxed{\begin{aligned}\sin A\cos B &= \tfrac{1}{2}\{\sin(A-B)+\sin(A+B)\}\\ \sin A\cdot\sin B &= \tfrac{1}{2}\{\cos(A-B)-\cos(A+B)\}\end{aligned}}$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}\frac{1}{2}\{\sin(n-1)x + \sin(n+1)x\}\,dx + \int_{0}^{\pi}\frac{1}{2}\{\cos(n-1)x - \cos(n+1)x\}\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{-\frac{\cos(n-1)x}{2(n-1)} - \frac{\cos(n+1)x}{2(n+1)}\right\}\Bigg|_{-\pi}^{0} + \left\{\frac{\sin(n-1)x}{2(n-1)} - \frac{\sin(n+1)x}{2(n+1)}\right\}\Bigg|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n-1)(0)}{2(n-1)} - \frac{\cos(n+1)(0)}{2(n+1)} + \frac{\cos(n-1)(-\pi)}{2(n-1)} + \frac{\cos(n+1)(-\pi)}{2(n+1)} + \right.$$

$$\left.\cancel{\frac{\sin(n-1)(\pi)}{2(n-1)}}^{0} - \cancel{\frac{\sin(n+1)(\pi)}{2(n+1)}}^{0} - \cancel{\frac{\sin(n-1)(0)}{2(n-1)}}^{0} + \cancel{\frac{\sin(n+1)(0)}{2(n+1)}}^{0}\right]$$

$$= \frac{1}{\pi}\left[\frac{-1}{2(n-1)} - \frac{1}{2(n+1)} + \frac{\cos(n-1)\pi}{2(n-1)} + \frac{\cos(n+1)\pi}{2(n+1)}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos(n-1)\pi - 1}{2(n-1)} + \frac{\cos(n+1)\pi - 1}{2(n+1)}\right]$$

$$= \frac{\cos(n-1)\pi - 1}{2(n-1)\pi} + \frac{\cos(n+1)\pi - 1}{2(n+1)\pi}$$

พิจารณา $\cos(n-1)\pi = \begin{cases} -1 & ;\ \text{เมื่อ } n = 2, 4, 6, \ldots \\ 1 & ;\ \text{เมื่อ } n = 3, 5, 7, \ldots \end{cases}$ หรือ $= (-1)^{n-1}$

$\cos(n+1)\pi = \begin{cases} -1 & ;\ \text{เมื่อ } n = 2, 4, 6, \ldots \\ 1 & ;\ \text{เมื่อ } n = 3, 5, 7, \ldots \end{cases}$ หรือ $= (-1)^{n+1}$

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$$\therefore b_n = \frac{(-1)^{n-1} - 1}{2(n-1)\pi} + \frac{(-1)^{n+1} - 1}{2(n+1)\pi}$$

$$b_n = \begin{cases} 0 & ; \text{เมื่อ } n = 3, 5, 7, \ldots \\ \dfrac{-2}{(n^2-1)\pi} & ; \text{เมื่อ } n = 2, 4, 6, \ldots \end{cases}$$

∴ อนุกรมฟูเรียร์ของฟังก์ชัน $f(x)$ จะได้จากสูตร

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

$$= \frac{2}{\pi(2)} + \sum_{n=2}^{\infty} \left(\frac{2n}{(n^2-1)\pi} \cos nx + \frac{-2}{(n^2-1)\pi} \sin nx \right)$$

$$+ \frac{2}{\pi} \sum_{n=2}^{\infty} \left(\frac{n}{(n^2-1)} \cos nx - \frac{1}{(n^2-1)} \sin nx \right)$$

แทน a_1, b_1 เมื่อ $n=1$ จะได้

$$f(x) = \frac{1}{\pi} + \frac{\cos x}{2} + \frac{2}{\pi}\left(\frac{2 \cos 2x}{3} + \frac{4 \cos 4x}{15} + \frac{6 \cos 6x}{35} + \ldots \right)$$

$$+ \frac{\sin x}{2} - \frac{2}{\pi}\left(\frac{\sin 2x}{3} + \frac{\sin 4x}{15} + \frac{\sin 6x}{35} + \ldots \right)$$

\# ที่ $x=0$ เป็นจุดที่ $f(x)$ ไม่ต่อเนื่อง จะได้

Will be continuous as soon as !!!